The population and decay evolution of a qubit under the time-convolutionless master equation*

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We consider the population and decay of a qubit under the electromagnetic environment. Employing the time-convolutionless master equation, we investigate the Markovian and non-Markovian behaviour of the corresponding perturbation expansion. The Jaynes-Cummings model on resonance is investigated. Some figures clearly show the different evolution behaviours. The reasons are interpreted in the paper.

Keywords: Markovian, non-Markovian, time-convolutionless master equation

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1. Introduction

In recent decades quantum systems[1−3] have been widely investigated due to the remarkable features of quantum mechanics. It is meaningful in both practice and theory. The interaction between a quantum system and the environment is the gate to profound quantum phenomenon, such as quantum entanglement,[4−9] quantum information processing,[10,11] quantum cryptography[12−15] and quantum teleportation.[16] Much work has been done, but the results are not exciting, for the processes usually use the Born–Markov approximation to obtain only the approximate answer. The non-Markovian memory effects have been neglected. For this case, we investigate the non-Markovian effect and compare it with the Markovian situation.

In this paper we present the Hamiltonian for two-level atom interactions with the reservoir for the very first time, to the best of our knowledge. Then we solve the Schrödinger equation to obtain a closed equation. Corresponding to Schmit expansion, the population and decay rate are expressed.

2. The model

We consider a two-level atom interacting with the multi-mode driving electromagnetic field. The total Hamiltonian of the system is given by(in units of $\hbar$)

$$H = H_{\rm A} + H_{\rm E} + V_{\rm I}, \tag{1}$$

where

$$H_{\rm A} = \omega_0 \sigma_+ \sigma_-, \tag{2}$$

$$H_{\rm E} = \sum_k \omega_k a_k^\dagger a_k, \tag{3}$$

$$V_{\rm I} = \sum_k (g_k a_k^\dagger \sigma_- + g_k^* a_k \sigma_+), \tag{4}$$

are the free, environment and interaction Hamiltonians of our model, respectively, ω_0 is the transition frequency, the operators $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$ are the raising and lowering frequencies ω_k, $a_k^\dagger$ is the creation operator and a_k is the annihilation operator, and g_k is the coupling constant.[17,18]

The initial state can be written as

$$|\psi_0\rangle = c_0|0\rangle_{\rm A}|0\rangle_{\rm E} + c_1|1\rangle_{\rm A}|0\rangle_{\rm E} + \sum_k c_k|0\rangle_{\rm A}|k\rangle_{\rm E}, \tag{5}$$

where $|0\rangle_{\rm A}$ is the atomic state and $|0\rangle_{\rm E}$ is the state for the environment. After time t the initial state evolves into

$$|\psi_t\rangle = c_0|0\rangle_{\rm A}|0\rangle_{\rm E} + c_1(t)|1\rangle_{\rm A}|0\rangle_{\rm E} + \sum_k c_k(t)|0\rangle_{\rm A}|k\rangle_{\rm E}. \tag{6}$$

The state $|0\rangle_{\rm E}$ denotes the vacuum state of the environment, $|k\rangle_{\rm E} = b_k^\dagger|0\rangle_{\rm E}$ is the state with one particle

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in mode k. In the interacting picture, the state $|\psi_t\rangle$ of the total system obeys the Schrödinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi_t\rangle = -\,\mathrm{i}\,H_{\mathrm{I}}|\psi_t\rangle, \tag{7}$$

where H_{I} is the interaction picture Hamiltonian

$$H_{\mathrm{I}}(t) = \sigma_+(t)A(t) + \sigma_-(t)A_+(t), \tag{8}$$

where

$$\sigma_\pm(t) = \sigma_\pm\,\mathrm{e}^{\pm\,\mathrm{i}\,\omega_0 t}, \quad A(t) = \sum_k g_k a_k\,\mathrm{e}^{-\,\mathrm{i}\,\omega_k t}. \tag{9}$$

The amplitude c_0 is a constant in time since $H_{\mathrm{I}}(T)|0\rangle_{\mathrm{A}}|0\rangle_{\mathrm{E}} = 0$, the amplitudes c_1 and c_2 are time dependent, and the time evolution of these amplitudes can be easily derived from Eq. (7) as

$$\dot{c_1}(t) = -\,\mathrm{i}\sum_k g_k\,\mathrm{e}^{\mathrm{i}(\omega_0-\omega_k)t}c_k(t), \tag{10}$$

$$\dot{c_k}(t) = -\,\mathrm{i}\,g_k^*\,\mathrm{e}^{-\,\mathrm{i}(\omega_0-\omega_k)t}c_1(t). \tag{11}$$

Assuming that $c_k(0) = 0$, i.e., there are no photons in the initial state, we solve Eq. (11) and insert the solution into Eq. (10) to obtain a closed equation for the coefficient $c_1(t)$ as

$$\dot{c_1}(t) = -\int_0^t f(t-t_1)c_1(t)\,\mathrm{d}t_1, \tag{12}$$

where the kernel $f(t) = \int \mathrm{d}\omega J(\omega)\,\mathrm{e}^{\mathrm{i}(\omega_0-\omega)(t-t_1)}$ is related to the spectral density $J(\omega)$ of the reservoir. We now can solve Eq. (12) just by a Laplace transformation and a specific form of the spectral density $J(\omega)$.

For solving the probability amplitudes c_0, c_1 and c_k, we can obtain the time-evolution of the reduced density matrix of the qubit by constructing an exact dynamical map as

$$\begin{aligned}\rho &= \mathrm{tr}_{\mathrm{E}}[|\psi_t\rangle\langle\psi_t|] = \begin{pmatrix} |c_1(t)|^2 & c_0^*c_1(t) \\ c_0c_1^*(t) & |c_1(t)|^2 + \sum_k |c_k(t)|^2 \end{pmatrix} \\ &= \begin{pmatrix} |c_1(t)|^2 & c_0^*c_1(t) \\ c_0c_1^*(t) & 1-|c_1(t)|^2 \end{pmatrix}. \end{aligned} \tag{13}$$

3. Calculation

Differentiating the above expression with respect to time and recalling that $c_k(0) = 0$ means the atom being in the vacuum state, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathrm{A}}(t) = \begin{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}|c_1(t)|^2 & c_0^*\dot{c}_1(t) \\ c_0\dot{c}_1^*(t) & -\dfrac{\mathrm{d}}{\mathrm{d}t}|c_1(t)|^2 \end{pmatrix}. \tag{14}$$

After computation, Eq. (14) can be written as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathrm{A}}(t) &= \mathrm{i}\,S(t)[\sigma_+\sigma_-, \rho_{\mathrm{A}}(t)] + \gamma(t)\Big[\sigma_-\rho_{\mathrm{A}}(t)\sigma_+ \\ &\quad - \frac{1}{2}\sigma_+\sigma_-\rho_{\mathrm{A}}(t) - \frac{1}{2}\rho_{\mathrm{A}}(t)\sigma_+\sigma_-\Big], \end{aligned} \tag{15}$$

where

$$S(t) = -2\Im\frac{\dot{c_1}(t)}{c_1(t)}, \tag{16}$$

$$\gamma(t) = -2\Re\frac{\dot{c_1}(t)}{c_1(t)}. \tag{17}$$

Here, $S(t)$ acts as a time-dependent Lamb shift and $\gamma(t)$ is a time-dependent decay rate. Equation (15) is the form of the time-convolutionless master equation which we are finding. The perturbation expansions of the coefficients $\gamma(t)$ and $S(t)$ in the master equation take the form

$$\gamma(t) = \sum_{n=1}^{\infty}\alpha^{2n}(t), \tag{18}$$

$$S(t) = \sum_{n=1}^{\infty}S^{2n}(t). \tag{19}$$

We can easily obtain the coefficients of these equations by using the same method as that in Ref. [2], the second-order contribution is

$$\gamma_2(t) + \mathrm{i}\,S_2(t) = 2\int_0^t \mathrm{d}t_1 f(t-t_1), \tag{20}$$

and the fourth-order contribution is

$$\begin{aligned}&\gamma_4(t) + \mathrm{i}\,S_4(t) \\ &= 2\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_3 \\ &\quad \times [f(t-t_2)f(t_1-t_3) + f(t_1-t_3)f(t_1-t_2)]. \end{aligned} \tag{21}$$

We define the real functions $\phi(t)$ and $\varphi(t)$ to simplify our expression

$$2f(t) = \phi(t) + \mathrm{i}\,\varphi(t). \tag{22}$$

In terms of these functions, the decay rate $\gamma(t)$ and the energy shift $S(t)$ determining the Born approximation are

$$\gamma_2(t) = \int_0^t \mathrm{d}t_1\phi(t-t_1), \tag{23}$$

$$S_2(t) = \int_0^t \mathrm{d}t_1\varphi(t-t_1), \tag{24}$$

$$\begin{aligned}\gamma_4(t) &= \frac{1}{2}\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_3 \\ &\quad \times [\phi(t-t_2)\phi(t_1-t_3) + \phi(t-t_3)\phi(t_1-t_2)\end{aligned}$$

$$
\begin{aligned}
&- \varphi(t-t_2)\varphi(t_1-t_3) \\
&+ \varphi(t-t_3)\varphi(t_1-t_2)], \qquad (25)\\
S_4(t) = &\frac{1}{2}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_3 \\
&\times [\varphi(t-t_2)\phi(t_1-t_3) + \phi(t-t_3)\phi(t_1-t_2) \\
&- \varphi(t-t_2)\phi(t_1-t_3) \\
&+ \phi(t-t_3)\varphi(t_1-t_2)]. \qquad (26)
\end{aligned}
$$

4. Analysis

In order to compute the exact probability amplitude $c_1(t)$, we take a Lorentzian spectral density of the reservoir

$$J(\omega) = \frac{1}{2\pi}\frac{\nu_0\lambda^2}{(\omega_0-\delta-\omega)^2+\lambda^2}, \qquad (27)$$

whose centre is detuned from the transition frequency ω_0 of the two-level atom by an amount δ. Here, we just investigate the resonant situation for $\delta = 0$. The parameter λ defines the spectral width of the coupling which is associated with the reservoir correction time by the relation $\tau_{\mathrm{E}} = \lambda^{-1}$, and the parameter ν_0 is related to the relaxation time scale by the relation $\tau_{\mathrm{R}} = \gamma_0^{-1}$. Using the spectral density Eq. (22), we solve the function $f(t)$ as

$$f(t) = \frac{1}{2}\gamma_0\lambda\,\mathrm{e}^{-\lambda|t|}. \qquad (28)$$

By inserting $f(t)$ into Eq. (12), the probability amplitude $c_1(t)$ and the population of the excited state $\rho_{11}(t)$ is easily obtained as

$$
\begin{aligned}
c_1(t) &= c_1(0)\,\mathrm{e}^{-\frac{\lambda t}{2}}\left[\cosh\left(\frac{dt}{2}\right)\right. \\
&\quad \left. + \frac{\lambda}{d}\sinh\left(\frac{dt}{2}\right)\right], \qquad (29)\\
\rho_{11}(t) &= \rho_{11}(0)\,\mathrm{e}^{-\lambda t}\left[\cosh\left(\frac{dt}{2}\right)\right. \\
&\quad \left. + \frac{\lambda}{d}\sinh\left(\frac{dt}{2}\right)\right]^2, \qquad (30)
\end{aligned}
$$

where $d = \sqrt{\lambda^2 - 2\gamma_0\lambda}$. Combining Eq. (16) with Eq. (17), we can also solve $S(t)$ and $\gamma(t)$ as

$$S(t) = 0, \qquad (31)$$

$$\gamma(t) = \frac{2\gamma_0\lambda\sinh(dt/2)}{d\cosh(dt/2) + \lambda\sinh(dt/2)}. \qquad (32)$$

Figure 1 describes the population of the excited state. The non-Markovian effects are clearly shown. In the strong coupling region($\tau_{\mathrm{R}} = 0.1\tau_{\mathrm{B}}$), the memory effects are much more obvious than in the sub-strong coupling region ($\tau_{\mathrm{R}} = \tau_{\mathrm{B}}$). However, after a long-time evolution, the revival phenomenon eventually disappears which manifests that the memory effect is just a short time-keeping phenomenon. At the same time, this is enough for us to investigate the characteristic of exchanging information from the atom to the environment.

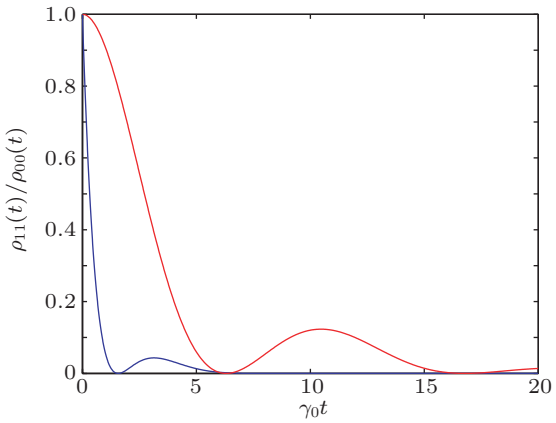


Fig. 1. (colour online) The population of the excited state for $\tau_{\mathrm{R}} = 0.1\tau_{\mathrm{B}}$ as the strong coupling which is shown in the red line, and for $\tau_{\mathrm{R}} = \tau_{\mathrm{B}}$ shown in the blue line.

We show the time-dependent decay rate $\gamma(t)$ for $\tau_{\mathrm{R}} = 10\tau_{\mathrm{B}}$ and $\tau_{\mathrm{R}} = 5\tau_{\mathrm{B}}$ in Fig. 2. There is only a Markovian effect here. Because the corresponding time is very short, the decay rate becomes linear after a short time. The difference of the two curves is the eventual stable value. The picture also shows that the parameter d is not the kernel factor which affects the evolution.

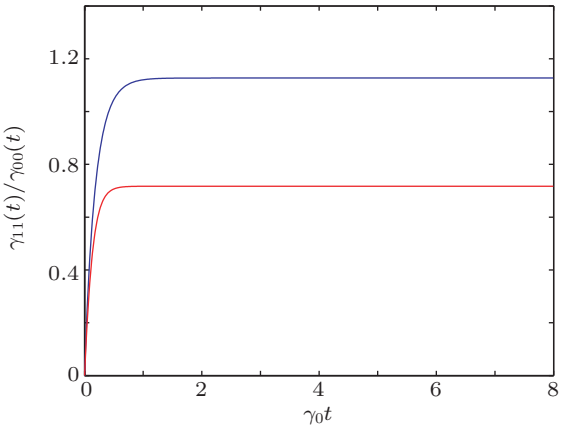


Fig. 2. (colour online) The decay rate of the excited state. The red line is for $\tau_{\mathrm{R}} = 10\tau_{\mathrm{B}}$ and the blue line is for $\tau_{\mathrm{R}} = 5\tau_{\mathrm{B}}$.

5. Conclusion

In this paper, we have considered a two-level atom interacting with a reservoir. The time-convolutionless master equation is used here. We have analysed

the population of the excited state and the decay rate in Markovian, weak Markovian, and strong non-Markovian regimes without detunings. We have contrasted the Markovian and non-Markovian curves, and the non-Markovian effects are shown and explained. There are also many other works to do such as our model with detuning. In addition, our model is meaningful in the quantum information transition.

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