Role of phase fluctuation and dephasing in the enhancing continuous variable entanglement of a two-photon coherent beat laser

Sintayehu Tesfa†

Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Str. 38, 01187 Dresden, Germany
Physics Department, Dilla University, P. O. Box 419, Dilla, Ethiopia

(Received 23 April 2011; revised manuscript received 29 August 2011)

A steady state analysis of the nonclassical features and statistical properties of the cavity radiation of a two-photon coherent beat laser is presented. Results show that the degree of two-mode squeezing, detectable entanglement and intensity of the cavity radiation can increase with the deviation of the phase fluctuations of the laser employed in preparing the atoms, but decrease with the increasing rate at which the induced coherence superposition decays. Although it is found that varying the phase fluctuations and dephasing can lead to modification in the quantum features and statistical properties of the radiation, it does not alter the similarity in the nature of the degree of entanglement detectable by the criteria following from Duan–Giedke–Cirac–Zoller and logarithmic negativity in a perceivable manner. Since the intensity and quantum features can be readily enhanced, this system is expected to be a viable source of a strong robust entangled (squeezed) light under various conditions. Moreover, comparison of the mean number of photon pairs with intensity difference shows that the chance of inciting a two-photon process can be enhanced by changing the rate of dephasing and phase fluctuations.

Keywords: coherent beat laser, phase fluctuations, dephasing, entanglement

PACS: 42.50.Ar, 42.50.Gy, 42.50.Lc, 03.65.Ud **DOI:** 10.1088/1674-1056/21/1/014204

1. Introduction

The study of the statistical properties, two-mode squeezing and bipartite entanglement of the cavity radiation of a three-level cascade laser of various schemes has received a great deal of attention over the years.[1−11] The nonclassical features arising from the strong correlations are essentially linked to the coherent superposition. The atomic coherence in the cascade configuration, where a direct spontaneous transition between the upper and the lower energy levels is presumed to be electric dipole forbidden, can be induced by either coupling these levels by an external driving mechanism[1−6] or preparing the atoms in an initial superposition of the two.[7,8] For the sake of clarity, the amplification of light when spontaneously emitted photons are correlated by the atomic coherence resulting from the initial preparation and the external pumping is designated earlier as a coherent beat laser.[12] In order to establish the lasing process, the initially prepared atoms are assumed to be injected into the cavity at a constant rate and removed after they decay to the levels that do not directly involve in the lasing process. The coherence beating, on the other hand, can be introduced by pumping the atoms externally while they are traveling across the cavity.

However, in a normal practical setting, it may be impossible to prepare the atoms initially at an arbitrary coherence at will due to the pertinent instability of the employed laser and an inevitable imperfection in the setup.[12−17] This means that the atoms can only be prepared in a partial coherent superposition. Most recently, based on a proposal that the phase fluctuations that emerge in connection to the faulty preparation may enhance nonclassical features via creating indirect spontaneous emission roots,[12] there is an indication that the two-mode squeezing[13] and entanglement[14] can increase with the phase fluctuation especially when there is strong external pumping. In addition, the laser that can be utilized for an external driving has a corresponding instability which leads to fluctuations in the induced coherence. In order to limit the otherwise emerging complications, the effect of phase fluctuations resulting from the external pumping is not considered here, although taking the phase as randomly fluctuating c-number parame-

†Corresponding author. E-mail: sint_tesfa@yahoo.com

© 2012 Chinese Physical Society and IOP Publishing Ltd

ter can lead to a substantial decrease in the degree of entanglement.[15,18,19] Moreover, despite the frequent assumptions that the rate at which the initially prepared coherence decays is the same as the atomic decay rate, the involved quantum fluctuations and the possibility of phase locking encourage us to consider the case when they are actually different.[20,21]

Because of these underlying physical processes, the study of the effects of the phase fluctuations and dephasing on the statistical properties and quantum features of the cavity radiation appears to be very important. Based on the fact that the bipartite entanglement and intensity of the radiation exhibit oscillatory nature when there is no external radiation[14] at the early stages of the lasing process, the main objective in this study is to investigate if the same condition prevails at steady state. Since many optical experiments have been carried out under the steady state condition with promise of applicability over a longer time span, it is envisaged that this study can be very vital in enriching the existing understanding of the entanglement to be detected. With this conviction, the degree of two-mode squeezing is compared with the bipartite entanglement detectable by applying the criteria of the logarithmic negativity. In earlier communications, comparison between the two reveals that the detectable entanglement exhibits a similar trend to the two-mode squeezing for certain parameters.[14,22] Taking this as a motivation, the nature of the relation between the detectable degree of entanglement when different criteria are employed and how the emerging tendency changes with the phase fluctuation and dephasing are analyzed. Moreover, aiming at understanding the strength of the emitted radiation and the efficiency of the process of two-photon emission, the corresponding photon statistics is studied.

In order to achieve these goals, the rate equations of the auto- and cross-correlations are solved at steady state by setting the time derivatives to be zero, although the general solutions of the coupled differential equations were provided elsewhere.[12] This approach significantly reduces the otherwise involving long steps with unnecessary lengthy expressions. By dividing the existing equations into two based on their relationships, and then simultaneously solving the emerging c-number equations, we can obtain the required mean photon numbers of the modes and the correlations between the two radiations. It is found that the phase fluctuation inherent with the imperfect preparation can be applicable in enhancing the detectable entanglement and squeezing, particularly when the atoms are initially prepared in an arbitrary coherent superposition and externally driven by a strong classical radiation.

2. Various correlations

The interaction of a pumped nondegenerate three-level atom with a resonant two-mode cavity radiation can be represented by the rotating-wave approximation and the interaction picture by the Hamiltonian of the form

$$\begin{aligned}\hat{H} =\ & \mathrm{i}g[\hat{a}|a\rangle\langle b| - |b\rangle\langle a|\hat{a}^{\dagger} + \hat{b}|b\rangle\langle c| - |c\rangle\langle b|\hat{b}^{\dagger}] \\ & + \mathrm{i}\frac{\Omega}{2}[|c\rangle\langle a| - |a\rangle\langle c|], \end{aligned} \tag{1}$$

where Ω is a real-positive constant proportional to the amplitude of the driving radiation and g is a coupling constant chosen to be the same for both transitions. Here, $\hat{a}$ and $\hat{b}$ are the annihilation operators that represent the two cavity modes. In the cascade configuration, the transition from the upper energy level $|a\rangle$ to the intermediate level $|b\rangle$ and from level $|b\rangle$ to the lower energy level $|c\rangle$ are taken to be resonant with the cavity radiation, whereas the direct spontaneous transition between $|a\rangle$ and $|c\rangle$ is electric dipole forbidden.

Due to various quantum effects associated with the preparing procedure, it is assumed that the atoms can only be initially prepared in an imperfect coherent superposition of the upper and lower energy levels described by

$$|\Psi_{\mathrm{A}}(0)\rangle = C_a(0)|a\rangle + C_c(0)\,\mathrm{e}^{\mathrm{i}\varphi}|c\rangle, \tag{2}$$

where $C_a(0)$ and $C_c(0)\,\mathrm{e}^{\mathrm{i}\varphi}$ are probability amplitudes for the atom to be initially in the upper and lower energy levels, respectively, and φ is an arbitrary phase difference between the two states. Note that this term does not include the fluctuations originated from the instability and broadening of the pump laser. In principle, φ can be randomly distributed about a fixed mean phase φ_0, wherein addressing the contribution of every phase change does not seem realistic. Consequently, assuming the phase fluctuations as a Gaussian random process[16−18] and using the deviation of the phase fluctuation instead of the actual phase have led to more tractable situations.[13,14] In this context, employing the property of the Gaussian variables,[23] we can obtain

$$\langle \exp(\pm \mathrm{i}\delta\varphi)\rangle = \exp(-\langle \delta\varphi^2/2\rangle). \tag{3}$$

Moreover, for a Gaussian random process, $\langle\delta\varphi\rangle$ is zero, where $\langle\delta^2\varphi/2\rangle = \theta$ represents the deviation of the phase fluctuations from φ_0, taken as zero for convenience, which is generally designated simply as a phase fluctuation.

In studying the nonclassical features and statistical properties of the generated radiation, deriving various correlations from the master equation is essential. For the system under consideration, the master equation was provided as[12]

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} =& \frac{\kappa}{2}[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}] \\ &+ \frac{AD_+}{2B}\left[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}\right] \\ &+ \frac{1}{2}\left(\frac{AD_-}{B} + \kappa\right)\left[2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}\right] \\ &+ \frac{AE_+}{2B}\left[\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{a}^\dagger\hat{\rho}\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho} + \hat{a}\hat{a}^\dagger\hat{\rho}\right] \\ &+ \frac{AE_-}{2B}\left[\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b} + \hat{\rho}\hat{a}\hat{a}^\dagger\right] \\ &+ \frac{AF_+}{2B}\left[\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} + \hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho}\right] \\ &+ \frac{AF_-}{2B}\left[\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger + \hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\right] \\ &+ \frac{A\zeta'(1+\zeta'\zeta)}{2B}\left[\hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger - \hat{a}\hat{b}\hat{\rho} + \hat{\rho}\hat{a}\hat{b}\right], \quad (4)\end{aligned}$$

where κ is the cavity damping constant,

$$A = \frac{2r_\mathrm{a}g^2}{\gamma^2}, \quad (5)$$

$$B = (4+\zeta^2)(1+\zeta'\zeta), \quad (6)$$

$$D_\pm = 2\zeta'^2 + 2\chi \pm \eta(\zeta'\zeta - 2\chi), \quad (7)$$

$$E_\pm = (2\zeta' + \zeta)\sqrt{1-\eta^2}\,\mathrm{e}^{\pm\mathrm{i}\varphi}, \quad (8)$$

$$F_\pm = 3\eta\zeta' - (2-\zeta'\zeta)\sqrt{1-\eta^2}\,\mathrm{e}^{\pm\mathrm{i}\varphi}, \quad (9)$$

$\zeta = \Omega/\gamma$, $\zeta' = \Omega/\Gamma$ and $\chi = \gamma/\Gamma$. Moreover, γ is the rate of dephasing (the rate at which the coherence superposition decays while the atoms traverse in the cavity), Γ is the spontaneous atomic decay rate, r_a is the rate at which the atoms are injected into the cavity and η is the population inversion.

Making use of the master equation and the fact that $(\mathrm{d}/\mathrm{d}t)\langle\hat{O}(t)\rangle = \mathrm{Tr}\left((\mathrm{d}\hat{\rho}/\mathrm{d}t)\hat{O}\right)$ (where $\hat{O}$ is any operator), it is not difficult to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{a}(t)\rangle = -\left(\frac{\kappa + a_+}{2}\right)\langle\hat{a}(t)\rangle + \frac{b_+}{2}\langle\hat{b}^\dagger(t)\rangle, \quad (10)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{b}(t)\rangle = -\left(\frac{\kappa + a_-}{2}\right)\langle\hat{b}(t)\rangle - \frac{b_-}{2}\langle\hat{a}^\dagger(t)\rangle, \quad (11)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -(\kappa + a_+)\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \frac{b_+}{2}\left[\langle\hat{a}(t)\hat{b}(t)\rangle + \langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle\right] - b_+, \quad (12)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = -(\kappa + a_-)\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle - \frac{b_+}{2}\langle\hat{a}(t)\hat{b}(t)\rangle - \frac{b_-}{2}\langle\hat{a}^\dagger(t)\hat{b}^\dagger(t)\rangle, \quad (13)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{a}(t)\hat{b}(t)\rangle = -\left(\frac{2\kappa + a_+ + a_-}{2}\right)\langle\hat{a}(t)\hat{b}(t)\rangle - \frac{b_-}{2}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \frac{b_+}{2}\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle - \frac{b_-}{2}, \quad (14)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{a}^2(t)\rangle = -(\kappa + a_+)\langle\hat{a}^2(t)\rangle + b_+\langle\hat{a}(t)\hat{b}^\dagger(t)\rangle, \quad (15)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{b}^2(t)\rangle = -(\kappa + a_-)\langle\hat{b}^2(t)\rangle - b_-\langle\hat{a}(t)\hat{b}^\dagger(t)\rangle, \quad (16)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{a}^\dagger(t)\hat{b}(t)\rangle = -(\kappa + a_- + a_+)\langle\hat{a}^\dagger(t)\hat{b}(t)\rangle + \frac{b_+}{2}\langle\hat{b}^2(t)\rangle - \frac{b_-}{2}\langle\hat{a}^{\dagger 2}(t)\rangle, \quad (17)$$

where

$$a_\pm = \frac{A}{B}\left[(2\zeta' + \zeta)\Theta + \eta(2\chi - \zeta'\zeta) \mp 2(\zeta'^2 + \chi)\right], \quad (18)$$

$$b_\pm = \frac{A}{B}\left[3\eta\zeta' + (\zeta'\zeta - 2)\Theta \pm \zeta'(1+\zeta'\zeta)\right], \quad (19)$$

in which

$$\Theta = \mathrm{e}^{-\theta}\sqrt{1-\eta^2}. \quad (20)$$

Note that the above expressions are c-number equations. Since the operators are already put in a normal order, it would be mathematically more appealing using the corresponding c-number equations associated with the normal order in the forthcoming discussion.

It is a common practice that the various correlations of interest can be determined at steady state by setting the time derivatives of the preceding expressions to be zero. Assuming the cavity to be initially in a two-mode vacuum state and the noise force at a time is not statistically related to the cavity mode variables at earlier times, one can find

$$\langle\alpha(t)\rangle = \langle\beta(t)\rangle = \langle\alpha^2(t)\rangle = \langle\beta^2(t)\rangle = \langle\alpha^*(t)\beta(t)\rangle = 0, \tag{21}$$

$$\langle\alpha^*(t)\alpha(t)\rangle = A\left[\frac{L(1-p)^2 + Mq_+(1-p)}{8B\mu_+}\right] + A\left[\frac{L(1+p)^2 - Mq_+(1+p)}{8B\mu_-}\right] + A\left[\frac{L(1-p^2) + Mq_+p}{2B(\mu_+ + \mu_-)}\right], \tag{22}$$

$$\langle\beta^*(t)\beta(t)\rangle = A\left[\frac{Lq_-^2 + Mq_-(1+p)}{8B\mu_+}\right] + A\left[\frac{Lq_-^2 - Mq_-(1-p)}{8B\mu_-}\right] - A\left[\frac{Lq_-^2 + Mq_-p}{2B(\mu_+ + \mu_-)}\right], \tag{23}$$

$$\langle\alpha(t)\beta(t)\rangle = A\left[\frac{2Lq_-(1-p) + M(1-p^2+q_-q_+)}{16B\mu_+}\right] - A\left[\frac{Lq_-(1+p) - M(1-p^2+q_-q_+)}{16B\mu_-}\right] + A\left[\frac{2Lq_-p + M(1+p^2-q_-q_+)}{4B(\mu_+ + \mu_-)}\right]. \tag{24}$$

Here,

$$L = 2\zeta'^2 + 2\chi - (2\zeta' + \zeta)\Theta - \eta(2\chi - \zeta'\zeta), \tag{25}$$

$$M = \zeta'(1 + \zeta'\zeta) + (2 - \zeta'\zeta)\Theta - 3\eta\zeta', \tag{26}$$

$$\mu_\pm = \frac{\kappa}{2} + \frac{A}{2B}\big[(2\zeta' + \zeta)\Theta + \eta\big(2\chi - \zeta'\zeta\big) \pm Z\big], \tag{27}$$

$$p = \frac{2\big[\zeta'^2 + \chi\big]}{Z}, \tag{28}$$

$$q_\pm = \frac{-\zeta'(1 + \zeta'\zeta) \mp \big[3\eta\zeta - (2 - \zeta'\zeta)\Theta\big]}{Z}, \tag{29}$$

where

$$Z = \sqrt{\zeta'^2(1 + \zeta\zeta')^2 + 4(\zeta'^2 + \chi)^2 - [3\eta\zeta' - (2 - \zeta'\zeta)\Theta]^2}.$$

It is possible to assert from a straightforward algebra that the steady state solution would be physically plausible as long as $\mu_\pm \geq 0$. Hence, since μ_+ is always positive, the analysis in this study is limited to the case when $\mu_- > 0$.

3. Quantum features

3.1. Two-mode quadrature squeezing

It has been experimentally demonstrated that a three-level laser of various scheme can be a source of two-mode squeezed light.[24–30] Specifically, the competition between the coherence induced by pumping and atomic configuration in a similar cascade scheme (coherence beating) in the framework of four-wave mixing[24] and phase locking[25] lead to a strong two-mode squeezing. Moreover, a recent theoretical work on the time evolution of the two-mode squeezing (when the atoms are partially prepared with 50% probability to be in the upper energy level initially) showed that dephasing and phase fluctuation can enhance the degree of squeezing under certain conditions.[13] Taking this as a motivation and following a similar approach, the nature of the two-mode quadrature squeezing at steady state is studied for an arbitrary initial preparation. It is expected that this consideration provides a far more general picture of the situation.

Nowadays, it is a common experience describing a two-mode cavity radiation by an operator

$$\hat{c} = \frac{1}{\sqrt{2}}\big(\hat{a} + \hat{b}\big), \tag{30}$$

where $\hat{a}$ and $\hat{b}$ represent the separate modes. Then, the pertinent quadrature operators that can be applied in studying the two-mode squeezing are defined as

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{31}$$

and

$$\hat{c}_- = \mathrm{i}(\hat{c}^\dagger - \hat{c}), \tag{32}$$

where the corresponding variances can be readily expressed in terms of the c-number variables associated with the normal ordering for the system as

$$\Delta c_\pm^2 = 1 + \langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle \pm \left[\langle\alpha^*(t)\beta^*(t)\rangle + \langle\alpha(t)\beta(t)\rangle\right]. \tag{33}$$

In what follows, the nature of the two-mode squeezing is analyzed for various cases applying

Eqs. (22)–(29) and (33). It is expected that the spontaneous emission during cascade transitions establishes more correlations in time (supported by the injection process), which can ultimately overcome the thermal fluctuations and hence lead to a strong two-mode squeezing at steady state. Aiming at circumventing the unpleasantness corresponding to the number of involved parameters, the values of the linear gain coefficient and amplitude of the driving radiation are fixed at $A = 2$ and $\Omega = 10\Gamma$, respectively. Note that these values are chosen in accordance with the pertinent steady state condition.

The results in Figs. 1 and 2 show that a two-photon coherent beat laser can be a source of two-mode squeezed light under a broad range of conditions as might anticipated from the form of the master equation. Figure 1 particlularly indicates that the degree of two-mode squeezing increases with the decreasing rate at which the coherent superposition decays. This demonstrates that if we lock the upper and lower energy levels without significantly compromising the spontaneous emission while the atoms traverse in the cavity, there would be a good chance for enhancing the squeezing. Moreover, we can find that other than shifting Δc_-^2 up or down, varying the rate of dephasing does not fundamentally alter the general nature of the squeezing for the parameters under consideration. Since the degree of squeezing depends on the linear gain coefficient and amplitude of the driving radiation,[7,31] it would be reasonable to expect changes in the above trend while the involved parameters are manipulated.[14] For instance, at the early stages of the lasing operation, the dependence of the squeezing on the dephasing was found to be different in the weak and strong driving limits.[13]

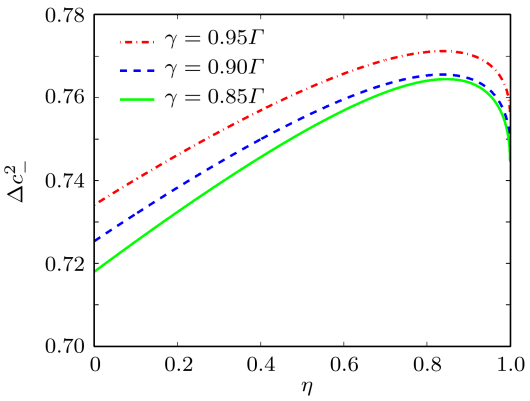


Fig. 1. (colour online) The variance of the minus quadrature (Δc_-^2) of the cavity radiation for $\kappa = 0.5$, $\theta = 0.1$, $A = 2$, $\Omega = 10\Gamma$ and different values of γ/Γ.

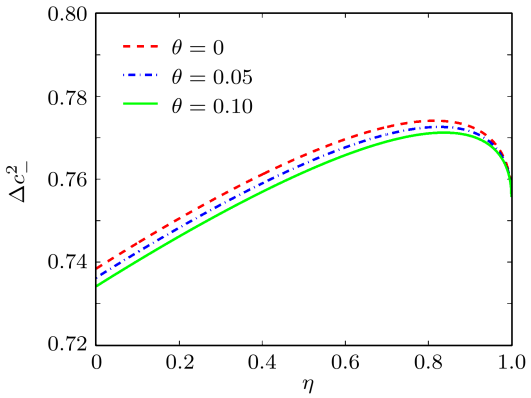


Fig. 2. (colour online) The variance of the minus quadrature (Δc_-^2) of the cavity radiation for $\kappa = 0.5$, $\gamma = 0.95\Gamma$, $A = 2$, $\Omega = 10\Gamma$ and different values of θ.

In addition, Figure 2 clearly shows that the degree of two-mode squeezing increases with the deviation of the phase fluctuations. A similar nature of squeezing was demonstrated at the early stage of the lasing process when there is a strong external driving radiation.[13] It is also possible to learn from Eq. (20) that, when $\eta = 1$, the degree of squeezing would be independent of the phase fluctuation, since $\eta = 1$ corresponds to the absence of coherent superposition at the beginning. This asserts that the existence of a strong correlation near $\eta = 1$ can be fairly attributed to the external driving mechanism and dephasing. In-depth analysis reveals that a similar tendency of squeezing exists as long as the rate at which the atoms are injected into the cavity is kept to be small (A is small). Therefore, varying the degree of phase fluctuation can shift Δc_-^2, but it does not change the overall trend of the squeezing in a perceivable manner, at least for the parameters under consideration.

Furthermore, the degree of two-mode squeezing is found to be higher when the atoms are initially prepared in the maximum coherent superposition ($\eta = 0$) despite the assumed strength of the external driving radiation. It is established based on earlier studies that the appearance of the squeezing for $\eta = 0$ is attributed to the coherence induced by the external driving mechanism, without disregarding the contribution of dephasing and phase fluctuation. Moreover, comparison of the results presented in Figs. 1 and 2 reveals that by varying the rate at which the coherent superposition decays and fluctuations of the laser deployed for the initial preparation of the atoms along with the way the atoms are prepared, it is possible to optimize the generated squeezing. Even though the values of A and Ω/Γ are fixed in this work for convenience, it is not difficult to envisage that the degree

of squeezing can be further manipulated upon varying the rate at which the atoms are injected into the cavity and the amplitude of the driving radiation.[31]

On the other hand, based on the fact that for the nondegenerate three-level cascade laser, the sum of the variance of the Einstein–Podelsky–Rosen (EPR) type quadrature operators[32] is related to the variance of the quadrature operators by (details can be found for instance in Refs. [7] and [33])

$$\Delta u^2 + \Delta v^2 = 2\Delta c_-^2,$$

it is obvious that the discussion provided earlier for the two-mode squeezing equally holds for a bipartite entanglement detectable using the criterion from Duan–Giedke–Cirac–Zoller (DGCZ).[34] With this understanding, it is possible to conclude that the generated bipartite entanglement can be enhanced by decreasing the rate at which the coherent superposition decays and by slightly increasing the deviation of the fluctuations associated with the laser deployed for preparing the atoms. Note that the actual optimum value can be obtained by choosing the proper parameters involved.

3.2. Continuous variable entanglement

Nowadays, many criteria can be applicable in detecting a bipartite entanglement.[35−41] In many instances, different criteria give different conclusions, since fundamentally they are derived based on different contexts and also by measuring different aspects of the system.[22] One of the inseparability criteria for the continuous variable bipartite system corresponds to a logarithmic negativity ($E_{\rm N}$)[42−44] that can be defined as[44−46]

$$E_{\rm N} = \max[0, -\log_2 V_{\rm s}], \tag{34}$$

where $V_{\rm s}$ is the smallest eigenvalue of the symplectic matrix.[44,45] The inseparability condition pertaining to this approach (which is usually dubbed as logarithmic negativity) is based on the well established fact that for the composite state to be separable to its constitute, the product density operator should have a positive partial transpose.

By solving the eigenvalue equations for the symplectic spectrum of the covariance matrix of the partially transposed density operator, the form of the smallest eigenvalue turns out to be[42]

$$V_{\rm s} = \left[\frac{\xi - \sqrt{\xi^2 - 4{\rm det}\Xi}}{2}\right]^{1/2}, \tag{35}$$

where

$$\xi = {\rm det}\sigma_{\rm A} + {\rm det}\sigma_{\rm B} - 2{\rm det}\sigma_{\rm AB}, \tag{36}$$

in which for the system under consideration[22,47]

$${\rm det}\sigma_{\rm A} = 1 + 4\langle\alpha^*(t)\alpha(t)\rangle[\langle\alpha^*(t)\alpha(t)\rangle + 1], \tag{37}$$

$${\rm det}\sigma_{\rm B} = 1 + 4\langle\beta^*(t)\beta(t)\rangle[\langle\beta^*(t)\beta(t)\rangle + 1], \tag{38}$$

$${\rm det}\sigma_{\rm AB} = -4\langle\alpha(t)\beta(t)\rangle^2, \tag{39}$$

$$\begin{aligned}{\rm det}\Xi = 16\Bigg[&\frac{1}{4} + \frac{\langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle}{2} \\ &+ \langle\alpha^*(t)\alpha(t)\rangle\langle\beta^*(t)\beta(t)\rangle - \langle\alpha(t)\beta(t)\rangle^2\Bigg]^2.\end{aligned} \tag{40}$$

It is well known that the two-mode Gaussian product state would be entangled provided that $E_{\rm N} = \log_2 V_{\rm s}$, which implies that $\log_2 V_{\rm s}$ should be negative. This in other word means that

$$V_{\rm s} < 1 \tag{41}$$

stands for the condition of detecting entanglement.

Applying criterion 41, a substantially robust bipartite entanglement for a similar scheme at steady state and early stages of the lasing operation were separately reported.[14,22] In particular, when the atoms are initially prepared with a maximum partial coherence, the degree of detectable entanglement exhibits fluctuations with time when there is no external pumping.[14] Following a similar approach, the degree of detectable entanglement is analyzed for various values of the population inversion, rate of dephasing and deviation of phase fluctuations.

It can be seen from Figs. 3, 4 and 5 that the generated radiation exhibits a robust bipartite entanglement. Figure 3 shows that the degree of detectable entanglement decreases with the increasing rate at which the coherent superposition decays. Even though it is not included in this work, it has been demonstrated earlier that in the absence of the external pumping, there is no entanglement for $\eta = 1$.[22] However, in relation to the strength of the external driving radiation ($\Omega = 10\Gamma$), the degree of entanglement is significantly large when the atoms are initially prepared at the lower energy level. The phase fluctuation has no effect on the outcome. This result is not surprising since substantial correlation of the two radiations was predicted earlier when there was a strong external driving and $\eta = 1$.[1] Note that although the degree of detectable entanglement generally decreases with increasing γ/Γ, the general nature of the entanglement

does not change in a perceivable manner while the rate of dephasing is narrowly varied. In addition, when the coherent superposition is induced by the initial preparation ($\eta \neq 1$) and there is a strong external pumping ($\Omega > \Gamma$), the existence of the coherence beating decreases the nonclassical correlation except for certain particular cases.

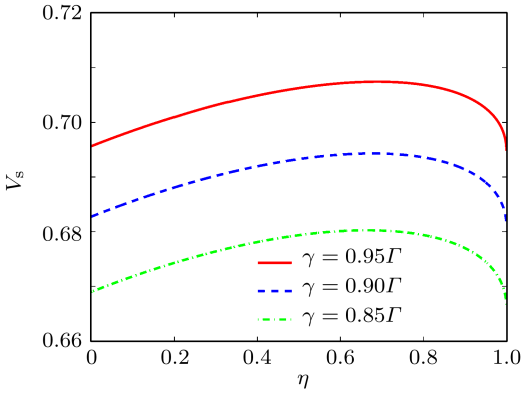


Fig. 3. (colour online) The smallest eigenvalue V_s of the cavity radiation for $\kappa = 0.5$, $\theta = 0.1$, $A = 2$, $\Omega = 10\Gamma$ and different values of γ/Γ.

Figures 4 and 5 show that other than near $\eta = 1$, the degree of detectable entanglement increases with the deviation of the phase fluctuations. As discussed elsewhere, for the time-dependent case, this situation remains true as long as there is a strong external driving radiation.[14] Specifically, the effect of the phase fluctuation is more prominent when the atoms are initially prepared in a maximum coherent superposition. At this juncture, the phase fluctuation weakens the expected strong coherence superposition resulting from the initial preparation and the external driving mechanism. In the process, it serves in lessening the emerging locking of the upper and lower energy levels and in a way facilitates the cascade spontaneous emission, which is basically the source of the nonclassical correlations in such a system. However, it is also possible to realize that, upon comparing the results given in Figs. 4 and 5, if one manages to decrease the rate at which the coherence decays while the atoms traverse in the cavity slightly without compromising the emission process, the degree of detectable entanglement increases by a perceivable amount especially near $\eta = 0$.

Furthermore, in recent years, comparative studies on various approaches of detecting entanglement indicate that there is an overlap of the predictions.[22] Researchers try to detect entanglement using different criteria to enrich the understanding of the pertinent underlying mechanism, which may allow to ease the practical challenges by providing an alternatives. In what follows, the degree of entanglement that can be detected applying the criteria obtained from the logarithmic negativity is compared with the sum of the EPR-type quadrature operators for selected parameters (note that $\Delta u^2 + \Delta v^2$ is multiplied by half so that the comparison can be straightforward).

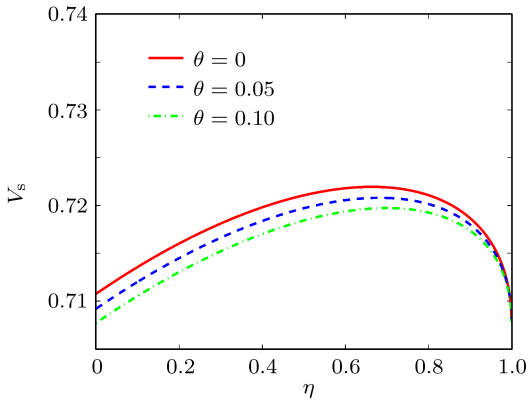


Fig. 4. (colour online) The smallest eigenvalue V_s of the cavity radiation for $\kappa = 0.5$, $\gamma = \Gamma$, $A = 2$, $\Omega = 10\Gamma$ and different values of θ.

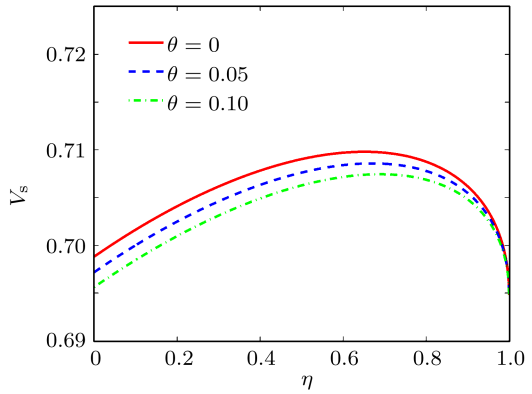


Fig. 5. (colour online) The smallest eigenvalue V_s of the cavity radiation for $\kappa = 0.5$, $\gamma = 0.95\Gamma$, $A = 2$, $\Omega = 10\Gamma$ and different values of θ.

Figure 6 shows that the degree of entanglement predicted by the logarithmic negativity is significantly higher than the corresponding DGCZ, which is in agreement with earlier work.[14] For the parameters under consideration, the tendency of the detectable entanglement is similar. However, critical scrutiny reveals that the difference between the two is smaller for $\eta = 0$, in connection to the fact that the two criteria predicted maximum entanglement for different values of η. That is, for parameters used, the criterion obtained from DGCZ predicts that the entanglement would be maximum for $\eta = 0$, whereas the logarithmic negativity when $\eta = 1$. This kind of disparity can be

linked to the context from which they are derived and the physical quantities they are referring to. Detailed discussion on the disparity between the prediction of the two criteria can be found in Ref. [22].

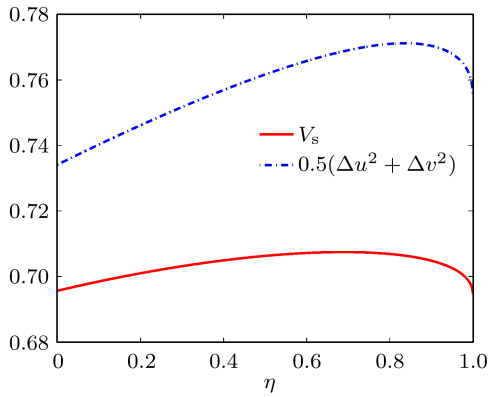


Fig. 6. (colour online) The smallest eigenvalue V_s and the variance of the minus quadrature operators (Δc_-^2) of the cavity radiation for $\kappa = 0.5$, $\gamma = 0.95\Gamma$, $A = 2$, $\Omega = 10\Gamma$ and $\theta = 0.1$.

4. Photon statistics

Despite the already existing detailed studies on the statistical properties of the cavity radiation of a three-level cascade laser,[7,13,33] it still seems necessary to investigate the changes in the mean photon number, Mandel's Q-factor and normalized intensity-difference fluctuations at steady state in relation to dephasing and phase fluctuations.

4.1. Mean photon number

In order to learn about the brightness of the generated light, it is necessary to study the mean number of photon pairs describing the two-mode cavity radiation that can be defined as

$$\bar{N} = \langle \hat{c}^\dagger(t)\hat{c}(t)\rangle, \tag{42}$$

where $\hat{c}(t)$ is the annihilation operator given by Eq. (30). On the basis that the operators in Eq. (42) are already put in the normal order, it is possible to rewrite it in terms of c-number variables associated with the normal ordering as

$$\bar{N} = \frac{1}{2}\big[\langle \alpha^*(t)\alpha(t)\rangle + \langle \beta^*(t)\beta(t)\rangle\big]. \tag{43}$$

Since $\langle \alpha^*(t)\alpha(t)\rangle$ and $\langle \beta^*(t)\beta(t)\rangle$ represent the mean photon numbers in mode a and mode b, respectively, $\bar{N}$ can be interpreted as the mean number of photon pairs. Therefore, the mean photon number describing the cavity radiation can be taken as $2\bar{N}$. Earlier study for a similar scheme when the atoms are initially prepared in a partial maximum coherence revealed that the mean photon number steadily increases with time.[14] This suggests that at steady state a much more strong light is expected. For convenience, the parameters in this analysis are fixed to be the same as the corresponding values in the preceding discussion except for θ, which is slightly changed for the purpose of clarity.

The result presented in Fig. 7 indicates that the mean photon number increases with the decreasing value of γ/Γ, since more atoms are expected to participate in the spontaneous emission process. However, Figure 8 shows that the mean photon number increases with the deviation of the phase fluctuations.

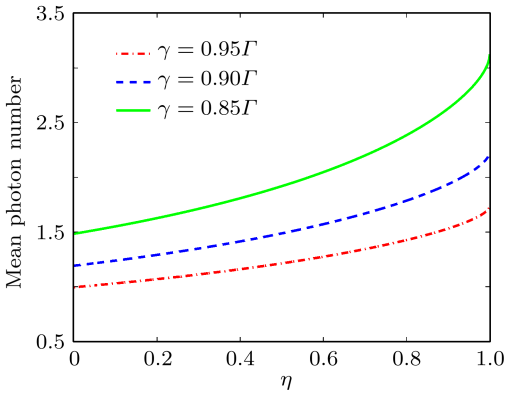


Fig. 7. (colour online) The mean photon number $2\bar{N}$ of the cavity radiation for $\kappa = 0.5$, $\theta = 0$, $A = 2$, $\Omega = 10\Gamma$ and different values of γ/Γ.

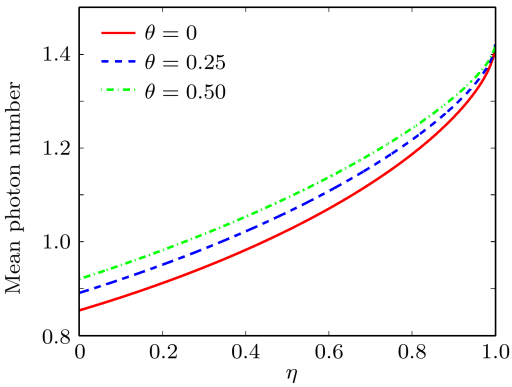


Fig. 8. (colour online) The mean photon number $2\bar{N}$ of the cavity radiation for $\kappa = 0.5$, $\gamma = \Gamma$, $A = 2$, $\Omega = 10\Gamma$ and different values of θ.

In both cases, the mean photon number increases with η. Thus, the more the atoms are initially prepared in the lower energy level, the stronger the intensity of the radiation would be. This outcome bears meaning when viewed in relation to the strength of the external pumping radiation that inhibits a direct spontaneous

emission severely due to coherence beating when the coherent superposition is initially prepared. Moreover, comparing the results obtained here with the discussion on entanglement and two-mode squeezing, we find that the mean photon number is significantly higher for the case in which there is a strong nonclassical correlation. Hence, based on the possibility of increasing the intensity by a proper selection of the involved parameters, this may lead to the conclusion that the system under consideration can be a source of bright entangled (squeezed) light.

4.2. Mandel's Q-factor

It is a common experience that a nonclassical photon number correlation can be studied applying the measure of the departure of the photon statistics from the Poisson character.[48,49] This measure of departure can be represented by the Mandel's Q-factor defined as

$$Q = \frac{\langle(\Delta\hat{n})^2\rangle - \langle\hat{n}\rangle}{\langle\hat{n}\rangle}, \tag{44}$$

which can be expressed in terms of c-number variables associated with the normal ordering as[22]

$$Q = \bar{N} + \frac{\langle\alpha(t)\beta(t)\rangle^2}{\bar{N}}. \tag{45}$$

It is well known that the negativity of the Mandel's parameter refers to a sub-Poissonian character of the photon statistics that essentially refers to a nonclassical property. Since the mean number of photon pairs and $\langle\alpha(t)\beta(t)\rangle^2$ are positive, the Mandel's Q-factor in this case is definitely greater than 0. This ensures that the generated radiation demonstrates super-Poissonian photon statistics while exhibiting nonclassical properties such as squeezing and entanglement. Detailed analysis shows that this behavior would not be affected in an essential way by changing the rate of dephasing and phase fluctuation. The situation in which a given state can exhibit nonclassical behavior even when the Q-factor is positive has been reported earlier.[22,50]

4.3. Normalized intensity-difference fluctuations

The intensity-difference fluctuations allow us to investigate how the difference between the mean photon numbers of the two radiations deviates from each other. This study is based on the assumption that there is a difference between the mean photon number of the two radiations due to the disparity of the absorption-emission mechanism among the involved atomic energy levels. Earlier report, in this direction, showed that the variance of the intensity-difference per mean photon number (normalized intensity-difference fluctuations) lies between 0 and 1 when the atoms are assumed to be initially prepared in an arbitrary coherent superposition.[7] In this study, the main task is to check whether this outcome can be affected by changing the rate of dephasing and phase fluctuation or not.

Therefore, the variance of the intensity-difference can be defined as

$$\Delta I_{\rm D}^2 = \langle\hat{I}_{\rm D}^2\rangle - \langle\hat{I}_{\rm D}\rangle^2, \tag{46}$$

where the difference of the intensity is given by

$$\hat{I}_{\rm D} = \hat{a}^\dagger(t)\hat{a}(t) - \hat{b}^\dagger(t)\hat{b}(t). \tag{47}$$

Hence, making use of Eq. (47), it is possible to express Eq. (46) in terms of c-number variables associated with the normal order as[7]

$$\begin{aligned}\Delta I_{\rm D}^2 =\ & \langle\alpha^*(t)\alpha(t)\rangle[1+\langle\alpha^*(t)\alpha(t)\rangle] \\ & + \langle\beta^*(t)\beta(t)\rangle[1+\langle\beta^*(t)\beta(t)\rangle] \\ & - 2\langle\alpha(t)\beta(t)\rangle^2. \end{aligned}\tag{48}$$

For simplicity, the normalized intensity-difference fluctuations is defined as

$$\Delta I_{\rm ND}^2 = \frac{\Delta I_{\rm D}^2}{2\bar{N}}. \tag{49}$$

Figures 9 and 10 show that the normalized intensity-difference fluctuations lie between 0 and 1. We can also see that the normalized intensity-difference fluctuations increase with the decreasing rate of dephasing and deviation of the phase fluctuations. Critical observation indicates that the overall tendency follows the same pattern as the squeezing and entanglement. Although the author is not clear about this so far, the form of Eq. (48) and the observed pattern are suggestive of some sort of a nonclassical behavior. With the comparative advantage of measuring intensity difference fluctuations between two lights, it is hoped that such a study can aid in devising other approaches for detecting nonclassical features. Note that the effects of the rate of dephasing and phase fluctuations on squeezing, entanglement and normalized intensity-difference fluctuations are not generally the same.

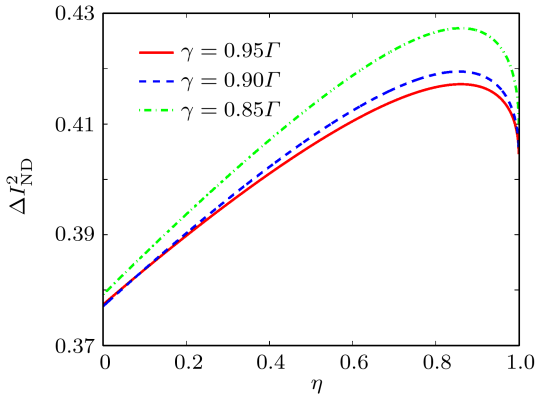


Fig. 9. (colour online) The normalized intensity-difference fluctuations $\Delta I^2_{\rm ND}$ of the cavity radiation for $\kappa = 0.5$, $\theta = 0$, $A = 2$, $\Omega = 10\Gamma$ and different values of $\gamma = \Gamma$.

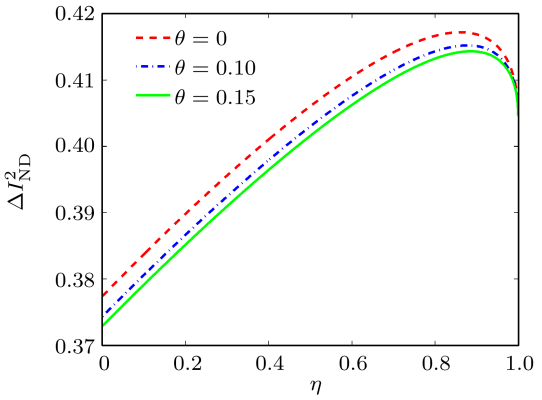


Fig. 10. (colour online) The normalized intensity-difference fluctuations $\Delta I^2_{\rm ND}$ of the cavity radiation for $\kappa = 0.5$, $\gamma = \Gamma$, $A = 2$, $\Omega = 10\Gamma$ and different values of θ.

On the other hand, the extent to which the photons are emitted in a successive manner, which is usually interpreted as a two-photon process, can be evaluated by comparing the mean number of photon pairs with the mean intensity difference. Based on the mean intensity difference, $\langle \hat{I}_{\rm D} \rangle = \langle \alpha^*(t)\alpha(t) \rangle - \langle \beta^*(t)\beta(t) \rangle$ and, the mean number of photon pairs, $\bar{N} = [\langle \alpha^*(t)\alpha(t) \rangle + \langle \beta^*(t)\beta(t) \rangle]/2$, it is not difficult to see from Fig. 11 that the mean photon number in mode a is greater than mode b. This implies that the mean number of photon pairs of radiation emitted when the atom decays from the upper energy level to the intermediate is larger than that when it goes from the intermediate to the lower energy level. The fact that the mean number of photon pairs is greater than the mean intensity difference implies that $[\langle \alpha^*(t)\alpha(t) \rangle]/3 < \langle \beta^*(t)\beta(t) \rangle < \langle \alpha^*(t)\alpha(t) \rangle$. This indicates that contrary to the earlier report, the mean number of photons in mode b can be quite close to that in mode a. This, on the other hand, can be interpreted as there is a significant probability for photons with different frequencies to be emitted at the same time. In other words, the external driving while supported by dephasing and phase fluctuation can provide a chance for establishing a two-photon process which can be useful when a twin beam source is required. Therefore, it is possible to deduce that the nonclassical features and the photon statistics observed in this work can be associated with the two-photon process in the amplification of the correlated emission.

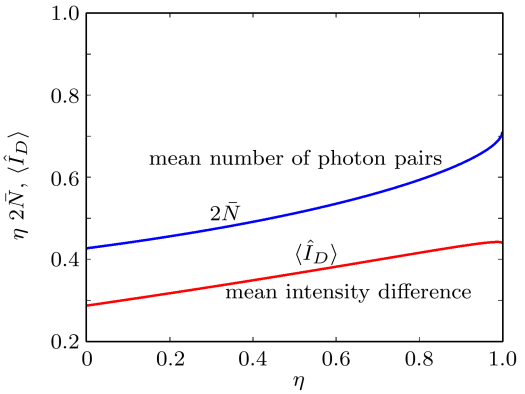


Fig. 11. (colour online) The mean photon number $2\bar{N}$ and mean intensity difference $\langle \hat{I}_D \rangle$ of the cavity radiation for $\kappa = 0.5$, $\gamma = 0.95\Gamma$, $A = 2$, $\Omega = 10\Gamma$ and $\theta = 0.1$.

5. Conclusion

In this study, the effects of the phase fluctuation and dephasing on the quantum features and statistical properties of the cavity radiation of a two-photon coherent beat laser is presented. It is found that the degree of two-mode squeezing, detectable bipartite entanglement, intensity and fluctuations of the intensity-difference significantly rely on the rate of dephasing and deviation of the phase fluctuations. Even in a small range of change of dephasing from the usual assumption ($\gamma = \Gamma$), the improvement in the degree of squeezing and detectable entanglement is obvious, especially when the atoms are initially prepared in a maximum coherent superposition ($\eta = 0$). Moreover, slightly disturbing the initial preparation of the coherent superposition with the aid of the associated phase fluctuations leads to a significant modification in the nonclassical features and photon statistics.

In connection to the strong external driving mechanism, the degree of two-mode squeezing and detectable entanglement of the cavity radiation increases with the deviation of the phase fluctuations, which can be observed when the atoms are initially prepared with a certain coherent superposition ($\eta \neq 1$). It is shown that the effect of phase fluctuation and

dephasing would be significant when the atoms can be initially prepared in a maximum coherent superposition. The other important aspect of including the effect of phase fluctuation is witnessed by the increment of the mean photon number with the deviation of the phase fluctuations especially near $\eta = 0$. These results can be associated with the fact that the change of the phase fluctuation affects the actually available coherence locking of the involved energy levels which is induced by preparation and driving.

To study the trend of the detectable entanglement from different perspectives, the degree of entanglement that can be predicted applying the logarithmic negativity is compared with the criterion of DGCZ. It is shown that the general tendency is more or less similar, although the actual values are not the same. The criterion obtained from logarithmic negativity is violated more strongly than the corresponding DGCZ criterion. Although the change in the parameters under consideration leads to a substantial modification in the result, it generally does not alter this trend much for a similar scheme.[22] On the other hand, the mean number of photon pairs and the mean of intensity-difference are compared. It turns out that the mean photon number of the radiation emitted when the atoms spontaneously decay from upper to intermediate energy level is greater than that when they go from intermediate to the lower energy level. Nevertheless, the two mean photon numbers are closer to each other in this case than the earlier report.[31] This indicates that the chance for obtaining two-photon process is highly likely, which makes this system a viable candidate when generation of a twin beam is required.

This study demonstrates that by varying the rate at which the coherent superposition decays and the degree of phase fluctuations along with the initially prepared coherent superposition, the rate at which the atoms are injected into the cavity and the amplitude of the external driving radiation, it is possible to generate a strong entangled (squeezed) light. Therefore, it is worth emphasizing that the degree of detectable entanglement is larger when the intensity of the radiation is also stronger. With the possibility of manipulating the involved parameters, it is hoped that the nondegenerate three-level cascade laser can be a viable candidate for studying nontrivial nonclassical correlations. Moreover, the similarity between the nature of normalized intensity-difference fluctuations on one hand and the squeezing and entanglement on the other suggests the possibility of detecting the latter via straightforward intensity measurement. This study shows that the inclusion of the effect of phase fluctuation and the rate at which the coherent superposition decays provides a more flexible scenario in the practical utilization of the system.

Acknowledgement

I thank the Max Planck Institute for the Physics of Complex Systems for allowing me to visit and use their facility in carrying out this research and Dilla University for granting the leave of absence. I also acknowledge the valuable comments of Klaus Hornberger.

References

[1] Tesfa S 2008 *J. Phys.* B: *At. Mol. Opt. Phys.* **41** 145501
[2] Ansari N A, Banacloche J G and Zubairy M S 1990 *Phys. Rev.* A **41** 5179
[3] Ansari N A 1992 *Phys. Rev.* A **46** 1560
[4] Tan H T, Zhu S Y and Zubairy M S 2005 *Phys. Rev.* A **72** 022305
[5] Xiong H, Scully M O and Zubairy M S 2005 *Phys. Rev. Lett.* **94** 023601
[6] Kiffner M, Zubairy M S, Evers J and Keitel C H 2007 *Phys. Rev.* A **75** 033816
[7] Tesfa 2006 *Phys. Rev.* A **74** 043816
[8] Scully M O, Wodkiewicz K, Zubairy M S, ergou J, Lu N and Meyer ter Van J 1988 *Phys. Rev. lett.* **60** 1832
[9] Anwar J and Zubairy M S 1994 *Phys. Rev.* A **49** 481
[10] An S and Sargent III M 1989 *Phys. Rev.* A **39** 1841
[11] Ansari N A 1993 *Phys. Rev.* A **48** 4686
[12] Tesfa S 2010 *Phys. Rev.* A **82** 053835
[13] Tesfa S 2011 *Phys. Rev.* A **83** 023809
[14] Tesfa S 2011 *arXiv*: 1101.3234v1 [quant-ph]
[15] Qamar S, Al-Amri M and Zubairy M S 2009 *Phys. Rev.* A **79** 013831
[16] Qamar S, Qamar S and Zubairy M S 2010 *Opt. Commun.* **283** 781
[17] Majeed M and Zubairy M S 1991 *Phys. Rev.* A **44** 4688
[18] Qamar S, Ghafoor F, Hillery M and Zubairy M S 2008 *Phys. Rev.* A **77** 062308
[19] Qamar S, Xiong H and Zubairy M S 2007 *Phys. Rev.* A **75** 062305
[20] Tesfa S 2009 *Phys. Rev.* A **79** 033810
[21] Tesfa S 2009 *Phys. Rev.* A **79** 063815
[22] Tesfa S 2009 *J. Phys.* B: *At. Mol. Opt. Phys.* **42** 215506
[23] Barnett S M and Badmore P M 1997 *Methods in Theoretical Quantum Optics* (Oxford University Press, New York)
[24] Malcuit M S, Gauthier D J and Boyd R W 1985 *Phys. Rev. Lett.* **55** 1086
[25] Steiner I and Toschek P E 1995 *Phys. Rev. Lett.* **74** 4639

[26] Winters M P, Hall J L and Toschek P E 1990 *Phys. Rev. Lett.* **65** 3116
[27] Kolchin P, Du S, Belthangady C, Yin G Y and Harris S E 2006 *Phys. Rev. Lett.* **97** 113602
[28] McCormick C F, Marino A M, Boyer V and Lett P D 2008 *Phys. Rev.* A **78** 043816
[29] Wu H and Xiao M 2009 *Phys. Rev.* A **80** 063415
[30] Shi W X, Hu X M and Wang F 2009 *J. Phys.* B: *At. Mol. Opt. Phys.* **42** 165506
[31] Tesfa S 2008 *J. Phys.* B: *At. Mol. Opt. Phys.* **41** 055503
[32] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[33] Tesfa S 2008 *Phys. Rev.* A **77** 013815
[34] Duan L M, Giedke G, Cirac J I and Zoller P 2000 *Phys. Rev. Lett.* **84** 2722
[35] Horodecki M, Horodcki P and Horodecki R 1996 *Phys. Lett.* A **223** 1
[36] Simon R 2000 *Phys. Rev. Lett.* **84** 2726
[37] Giovannetti V, Mancini S, Vitali D and Tombesi P 2003 *Phys. Rev.* A **67** 022320
[38] Shchukin E and Vogel W 2005 *Phys. Rev. Lett.* **95** 230502
[39] Serafini A 2006 *Phys. Rev. Lett.* **96** 110402
[40] Hillery M and Zubairy M S 2006 *Phys. Rev. Lett.* **96** 050503
[41] Nha H and Zubairy M S 2008 *Phys. Rev. Lett.* **101** 130402
[42] Vidal G and Wener R F 2002 *Phys. Rev.* A **65** 032314
[43] Laurat J, Keller G, Oliviera-Huguenim Augusto J, Fabre C, Coudreau T, Serafini A, Adesso G and Illuminati F 2005 *J. Opt.* B: *Quantum Semiclass. Opt.* **7** 577
[44] Adesso G, Serafini A and Illuminati F 2004 *Phys. Rev.* A **70** 022318
[45] Fiurasek J and Cerf N J 2004 *Phys. Rev. Lett.* **93** 063601
[46] Ma Y H, Mu Q X, Yang G H and Zhou L 2008 *J. Phys.* B: *At. Mol. Opt. Phys.* **41** 215502
[47] Fujikawa K 2009 *Phys. Rev.* A **79** 032334
[48] Mandel L 1979 *Opt. Lett.* **4** 205
[49] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (New York: Cambridge University Press)
[50] Agrwal G S and Tara K 1992 *Phys. Rev.* A **46** 485