Impulsive synchronization of a nonlinear coupled complex network with a delay node*

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We investigate the impulsive synchronization of a nonlinear coupled complex network with a delay node. Both delay coupling and non-delay coupling, as well as the symmetrical coupling matrix and the asymmetrical coupling matrix are considered. Based on the comparison theorem of an impulsive differential system, some novel synchronization criteria are derived. Finally, numerical simulations demonstrate the effectiveness of the proposed synchronization criteria.

Keywords: complex network, impulsive control, synchronization, nonlinear coupled

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1. Introduction

Complex networks have been shown to be ubiquitous in the natural world.[1] Among the various behaviours of complex networks, synchronization is a significant and interesting phenomenon.[2] Therefore, the synchronization and control of linearly coupled complex networks has become one of the key issues in various fields, such as mathematics, sociology and biology.[3−9]

However, in many practical problems, it often happens that the states $\boldsymbol{x}^i(t)$ cannot be observed directly. Instead, we can observe only the data $\boldsymbol{h}(\boldsymbol{x}^i) = [h(x_1^i), \ldots, h(x_n^i)]^{\mathrm{T}} \in \boldsymbol{C}(\boldsymbol{R}^n, \boldsymbol{R}^n)$, $i = 1, \ldots, N$, which means that the coupling scheme is nonlinear. The synchronization of nonlinear coupled networks has been addressed intensively.[10−13] In particular, Liu and Chen studied the synchronization of nonlinear coupled dynamical networks with a delayed coupling matrix[10] or with an asymmetrical coupling matrix.[11]

One important consideration in practical networks is the existence of time delays, because obstructions to the transmission of signals are inevitable in a biological neural network, in an epidemiological model, in a communication network or in an electrical power grid, and in either the state variables or the coupling coefficients. There are numerous examples in the real world which are characterized by network models with coupling delays or node delays.[14−16] For example, a distributed computer network, in which time delays in dynamical nodes represent computing time and coupling delays represent communication delays. Ignoring them may lead to design flaws and incorrect analysis conclusions. Even though much is known about the delays at the couplings, delays at the dynamical nodes have almost never been explored. Delayed nodes occur frequently in nature,[17−19] and well-known examples include the chaotic delayed Hopfield neural network and the delayed logistic differential equation. Hence, networks with delay nodes are important systems that warrant further investigation.

Another important consideration in network control and synchronization is related to the design of physically applicable controllers to effectively stabilize or synchronize coupled dynamical systems. However, in some cases, when networked systems cannot endure continuous control nor provide continuous control, continuous control schemes, such as linear feedback control, the adaptive design method, the pinning control method and the backstepping design technique, are not applicable. As is well known, the impulsive control method is uncontinuous, very effective, robust and low-cost, and has been widely applied in many fields, such as space techniques, information science, control systems, communication security, life science and so on. Owing to the effectiveness, robustness

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and low cost of the impulsive control strategy, complex networks with impulsive effects have aroused the interest of many researchers due to their various applications in information science, economic systems and automatical control systems.[20−28] Unfortunately, those complex networks were assumed to be subject to individual influence, i.e., impulsive effects, and complex networks with multiple influences, such as node delay or coupling delay, symmetrical coupling or asymmetrical coupling, nonlinear coupling and impulsive effects have seldom been considered.

Motivated by the above discussion, in this paper, we discuss the impulsive synchronization of a nonlinear coupled complex network with a delay node. Unlike in existing literature, in this paper we design an effective, robust and low-cost control method for a nonlinear coupled network that cannot endure or give continuous control; furthermore, we consider both coupling delay and node delay as well as symmetrical coupling and asymmetrical coupling, and derive some novel criteria which ensure that this complex dynamical network can reach the desired orbit.

The rest of the paper is organized as follows. In Section 2, the model and preliminaries are given. In Section 3, sufficient condition of the impulsive synchronization of nonlinear coupled complex network with delay node is obtained. The numerical simulations for verifying the theoretical result are presented in Section 4. The paper is concluded in Section 5.

2. Model and preliminaries

Consider a nonlinear coupled complex network composed of N identical delay nodes, in which each node is an n-dimensional dynamical unit. The state equations of isolated node is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{f}_1(\boldsymbol{x}(t)) + \boldsymbol{f}_2(\boldsymbol{x}(t-\tau)). \quad (1)$$

The nonlinear coupled or delayed coupled complex network with a delay node and impulsive control input can be described as

$$\begin{cases} \dot{\boldsymbol{y}}^i(t) = \boldsymbol{C}\boldsymbol{y}^i(t) + \boldsymbol{f}_1(\boldsymbol{y}^i(t)) + \boldsymbol{f}_2(\boldsymbol{y}^i(t-\tau)) + \sum\limits_{j=1}^{N} d_{ij}\boldsymbol{\Gamma}\boldsymbol{h}(\boldsymbol{y}^i(t)), \quad t \neq t_k, \\ \Delta\boldsymbol{y}^i = \boldsymbol{y}^i(t_k^+) - \boldsymbol{y}^i(t_k^-) = \boldsymbol{B}_{ik}(\boldsymbol{y}^i(t) - \boldsymbol{x}(t)), \quad t = t_k,\ k = 1, 2, \ldots, \\ \boldsymbol{y}^i(t) = \boldsymbol{\phi}^i(t), t \in [t_0 - \tau, t_0], \quad i = 1, 2, \ldots, N, \end{cases} \quad (2)$$

or

$$\begin{cases} \dot{\boldsymbol{y}}^i(t) = \boldsymbol{C}\boldsymbol{y}^i(t) + \boldsymbol{f}_1(\boldsymbol{y}^i(t)) + \boldsymbol{f}_2(\boldsymbol{y}^i(t-\tau)) + \sum\limits_{j=1}^{N} d_{ij}\boldsymbol{\Gamma}\boldsymbol{h}(\boldsymbol{y}^i(t-\tau_1)), \quad t \neq t_k, \\ \Delta\boldsymbol{y}^i = \boldsymbol{y}^i(t_k^+) - \boldsymbol{y}^i(t_k^-) = \boldsymbol{B}_{ik}(\boldsymbol{y}^i(t) - \boldsymbol{x}(t)), \quad t = t_k,\ k = 1, 2, \ldots, \\ \boldsymbol{y}^i(t) = \boldsymbol{\phi}^i(t), t \in [t_0 - \tau, t_0], \quad i = 1, 2, \ldots, N, \end{cases} \quad (3)$$

where $\boldsymbol{y}^i = (y_1^i, \ldots, y_n^i)^{\mathrm{T}} \in \boldsymbol{R}^n$ is the state vector of node i, $\boldsymbol{f}_j (j = 1, 2)$: $\boldsymbol{R}^n \times [0, \infty] \to \boldsymbol{R}^n$ are both continuous vector functions, $\boldsymbol{h}(\boldsymbol{y}^i(t)) = [h(y_1^i(t)), ..., h(y_n^i(t))]^{\mathrm{T}}$ is a nonlinear function, $\tau \geq 0$ is the node delay, $\tau_1 \geq 0$ is the coupled delay, $\boldsymbol{C}$ is the constant matrix, $\boldsymbol{D} = (d_{ij}) \in R^{N\times N}$ is the coupling matrix, which represents the coupling strength and the underlying topology for the non-delayed configuration or delayed one, and satisfies $d_{ij} \geq 0$ for $i \neq j$, $d_{ii} = -\sum\limits_{j=1, j\neq i}^{N} d_{ij}$, and $\boldsymbol{\Gamma} \in R^{n\times n}$ is an positive definite diagonal matrix which describes the individual couplings between node i and node j. The discrete set $\{t_k\}$ satisfies

$$0 \leq t_0 < t_1 < \ldots < t_k < \ldots, t_k \to \infty (t \to \infty),$$

and $\boldsymbol{y}^i(t_k^+) = \lim_{t\to t_k^+} \boldsymbol{y}^i(t)$, $\boldsymbol{y}^i(t_k^-) = \lim_{t\to t_k^-} \boldsymbol{y}^i(t)$. Moreover, any solution of Eqs. (2) and (3) are left continuous at each $t = t_k$, i.e., $\boldsymbol{y}^i(t_k^-) = \boldsymbol{y}^i(t_k)$.

Let $\boldsymbol{C}([-\tau, 0], \boldsymbol{R}^n)$ be the Banach space of continuous function mapping the interval $[-\tau, 0]$ onto $\boldsymbol{R}^n$ with the norm $\|\boldsymbol{\phi}\| = \sup_{-\tau\leq\theta\leq 0}\|\boldsymbol{\phi}(\theta)\|$. For the functional differential Eq. (2) or (3), its initial conditions are given by $\boldsymbol{y}^i(t) = \boldsymbol{\phi}^i(t) \in \boldsymbol{C}([-\tau, 0], \boldsymbol{R}^n)$. We always assume that Eq. (2) or (3) has a unique solution with respect to the initial conditions.

We introduce some definitions, hypothesis and lemmas which are to be used in the paper.

Definition 1 Matrix $\boldsymbol{G} = (g_{ij}) \in R^{N\times N}$ is said to belong to $\boldsymbol{A}_1$, if

(i) $g_{ij} = g_{ji} \geq 0 (i \neq j)$, $g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij}$, for $i = 1, 2, \ldots, N$;

(ii) real parts of eigenvalues of $\boldsymbol{G}$ are all negative except an eigenvalue 0 with multiplicity 1.

Hypothesis 1 $\boldsymbol{f}_i(x) \in \boldsymbol{C}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ satisfies

$$\|\boldsymbol{f}_i(\boldsymbol{x}_1) - \boldsymbol{f}_i(\boldsymbol{y}_1)\| \leq L_i\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|$$

for some $L_i > 0$, $i = 1, 2$, all $\boldsymbol{x}_1$, $\boldsymbol{y}_1 \in \boldsymbol{R}^n$.

Hypothesis 2 The nonlinear function $h(\cdot)$ satisfies $\theta_1 \leq \frac{h(z_1)-h(z_2)}{z_1-z_2} \leq \theta_2$ for some $\theta_1 > 0, \theta_2 > 0$, all $z_1, z_2 \in R(z_1 \neq z_2)$.

Lemma 1[10] Suppose $\boldsymbol{D} \in \boldsymbol{A}_1$ is symmetrical, then for any two vectors $\boldsymbol{u} = (u_1, ..., u_N)^{\mathrm{T}}, \boldsymbol{v} = (v_1, ..., v_N)^{\mathrm{T}}$, we have

$$\boldsymbol{u}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{v} = -\sum_{j>i}^{N} d_{ij}(u_i - u_j)(v_i - v_j).$$

Lemma 2 (The Shur complements) Let $\boldsymbol{Q}$ and $\boldsymbol{R}$ be two symmetric matrices, then

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^{\mathrm{T}} & \boldsymbol{R} \end{bmatrix} < 0$$

will be equivalent to

$$\boldsymbol{R} < 0;\ \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^{\mathrm{T}} < 0.$$

3. Impulsive synchronization with nonlinear coupling

Let the synchronization errors $\boldsymbol{e}^i(t) = \boldsymbol{y}^i(t) - \boldsymbol{x}(t)$, then we will have the following error dynamical equations:

$$\begin{aligned} \dot{\boldsymbol{e}}^i(t) =& \boldsymbol{C}\boldsymbol{y}^i(t) - \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{f}_1(\boldsymbol{y}^i(t)) - \boldsymbol{f}_1(\boldsymbol{x}(t)) \\ &+ \boldsymbol{f}_2(\boldsymbol{y}^i(t-\tau)) - \boldsymbol{f}_2(\boldsymbol{x}(t-\tau)) \\ &+ \sum_{j=1}^{N} d_{ij}\boldsymbol{\Gamma}\boldsymbol{h}(\boldsymbol{e}^j(t)),\ i = 1, 2, \ldots, N, t \neq t_k, \\ \Delta\boldsymbol{e}^i =& \boldsymbol{B}_{ik}(\boldsymbol{y}^i(t) - \boldsymbol{x}(t)),\ t = t_k, k = 1, 2, \ldots, \end{aligned} \tag{4}$$

or

$$\begin{aligned} \dot{\boldsymbol{e}}^i(t) =& \boldsymbol{C}\boldsymbol{y}^i(t) - \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{f}_1(\boldsymbol{y}^i(t)) - \boldsymbol{f}_1(\boldsymbol{x}(t)) \\ &+ \boldsymbol{f}_2(\boldsymbol{y}^i(t-\tau)) - \boldsymbol{f}_2(\boldsymbol{x}(t-\tau)) \\ &+ \sum_{j=1}^{N} d_{ij}\boldsymbol{\Gamma}\boldsymbol{h}(\boldsymbol{e}^j(t-\tau_1)),\ i = 1, 2, \ldots, N, t \neq t_k, \\ \Delta\boldsymbol{e}^i =& \boldsymbol{B}_{ik}(\boldsymbol{y}^i(t) - \boldsymbol{x}(t)),\ t = t_k, k = 1, 2, \ldots, \end{aligned} \tag{5}$$

Then, the objective of this paper is to design an impulsive controller $\boldsymbol{B}_{ik}$ and the impulsive distances $\Delta_k = t_k - t_{k-1} < \infty\,(k = 1, 2, \ldots, N)$ such that the state of networks (2) or (3) synchronizes with the state of isolated node (1), that is, $\lim_{t\to\infty} \|\boldsymbol{y}^i(t) - \boldsymbol{x}(t)\| = 0$, where $\boldsymbol{x}(t)$ is a solution of the isolated node (1).

3.1. Coupling terms without delay

Suppose that Hypothesis 1 and Hypothesis 2 hold, $\boldsymbol{D} \in \boldsymbol{A}_1$, λ_1, λ_2, β_k are the largest eigenvalues of

$$\boldsymbol{C}, \boldsymbol{D} \otimes \boldsymbol{\Gamma}, (\boldsymbol{I} + \boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I} + \boldsymbol{B}_{ik})(k = 1, 2, \ldots),$$

respectively. In what follows, it will be shown that if there exists a constant $\xi > 1$ such that

$$(2\lambda_1 + 2L_1 + L_2^2 + 1 + 2\theta_1\lambda_2)\Delta_k + \ln(\xi\beta_k) \leq 0, \quad k = 1, 2, \ldots,$$

then network (2) globally asymptotically synchronizes to the given trajectory $\boldsymbol{x}(t)$ of isolated node (1).

We choose a Lyapunov function candidate as

$$V(\boldsymbol{e}(t)) = \sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + \sum_{i=1}^{N}\int_{t-\tau}^{t}(\boldsymbol{e}^i(s))^{\mathrm{T}}\boldsymbol{e}^i(s)\,\mathrm{d}s.$$

Obviously,

$$\begin{aligned} \sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) \leq& V(\boldsymbol{e}(t)) \\ \leq& \sum_{i=1}^{N}\Big[(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + \tau\sum_{i=1}^{N}\|\boldsymbol{e}^i\|^2\Big], \end{aligned}$$

where $\|\boldsymbol{e}^i\| = \sup_{t-\tau\leq s\leq t}\|\boldsymbol{e}^i(s)\|$.

For $t \neq t_k$, by Lemma 1, Hypothesis 1 and Hypothesis 2, the derivative of $V(\boldsymbol{e}(t))$ along the trajectories of Eq. (4) is

$$\begin{aligned} \dot{V}(\boldsymbol{e}(t)) =& 2\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\Big[\boldsymbol{C}\boldsymbol{y}^i(t) - \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{f}_1(\boldsymbol{y}^i(t)) - \boldsymbol{f}_1(\boldsymbol{x}(t)) + \boldsymbol{f}_2(\boldsymbol{y}^i(t-\tau)) - \boldsymbol{f}_2(\boldsymbol{x}(t-\tau)) \\ &+ \sum_{j=1}^{N} d_{ij}\boldsymbol{\Gamma}\boldsymbol{h}(\boldsymbol{e}^j(t))\Big] + \sum_{i=1}^{N}\big[(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) - (\boldsymbol{e}^i(t-\tau))^{\mathrm{T}}\boldsymbol{e}^i(t-\tau)\big] \\ \leq& 2\sum_{i=1}^{N}\big[(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{C}\boldsymbol{e}^i(t) + L_1(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + L_2\|(\boldsymbol{e}^i(t))^{\mathrm{T}}\|\|\boldsymbol{e}^i(t-\tau)\|\big] \\ &+ 2(\boldsymbol{e}(t))^{\mathrm{T}}(\boldsymbol{D}\otimes\boldsymbol{\Gamma})\boldsymbol{H}(\boldsymbol{e}(t)) + \sum_{i=1}^{N}\big[(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) - (\boldsymbol{e}^i(t-\tau))^{\mathrm{T}}\boldsymbol{e}^i(t-\tau)\big] \\ \leq& (2\lambda_1(\boldsymbol{C}) + 2L_1 + L_2^2 + 1)\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + 2(\boldsymbol{e}(t))^{\mathrm{T}}\bar{\boldsymbol{D}}\boldsymbol{H}(\boldsymbol{e}(t)) \end{aligned}$$

$$
\begin{aligned}
&= (2\lambda_1 + 2L_1 + L_2^2 + 1)\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) - 2\sum_{j>i} d_{ij}\boldsymbol{\Gamma}[\boldsymbol{e}^i(t) - \boldsymbol{e}^j(t)]^{\mathrm{T}}\big[\boldsymbol{h}(\boldsymbol{y}^i(t)) - \boldsymbol{h}(\boldsymbol{y}^j(t))\big] \\
&\leq (2\lambda_1 + 2L_1 + L_2^2 + 1)\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) - 2\theta_1\sum_{j>i} d_{ij}\boldsymbol{\Gamma}[\boldsymbol{e}^i(t) - \boldsymbol{e}^j(t)]^{\mathrm{T}}[\boldsymbol{y}^i(t) - \boldsymbol{y}^j(t)] \\
&= (2\lambda_1 + 2L_1 + L_2^2 + 1)\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + 2\theta_1\boldsymbol{e}(t)^{\mathrm{T}}\bar{\boldsymbol{D}}\boldsymbol{e}(t) \\
&\leq [2\lambda_1 + 2L_1 + L_2^2 + 1 + 2\theta_1\lambda_2(\bar{\boldsymbol{D}})]\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) \\
&\leq (2\lambda_1 + 2L_1 + L_2^2 + 1 + 2\theta_1\lambda_2)\Big(\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + \sum_{i=1}^{N}\int_{t-\tau}^{t}(\boldsymbol{e}^i(s))^{\mathrm{T}}\boldsymbol{e}^i(s)\mathrm{d}s\Big) \\
&\leq (2\lambda_1 + 2L_1 + L_2^2 + 1 + 2\theta_1\lambda_2)V(\boldsymbol{e}(t)),
\end{aligned}
$$

where $\bar{\boldsymbol{D}} = \boldsymbol{D}\otimes\boldsymbol{\Gamma}$ and $\boldsymbol{h}(\boldsymbol{e}^i) = \boldsymbol{h}(\boldsymbol{y}^i) - \boldsymbol{h}(\boldsymbol{x})$. When $t = t_k$, we have

$$
\begin{aligned}
V(\boldsymbol{e}(t_k^+)) &= \sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})\boldsymbol{e}^i(t) + \sum_{i=1}^{N}\int_{t-\tau}^{t}(\boldsymbol{e}^i(s))^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})\boldsymbol{e}^i(s)\mathrm{d}s \\
&\leq \lambda_{\max}[(\boldsymbol{I}+\boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})]\Big(\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + \sum_{i=1}^{N}\int_{t-\tau}^{t}(\boldsymbol{e}^i(s))^{\mathrm{T}}\boldsymbol{e}^i(s)\mathrm{d}s\Big) \\
&\leq \beta_k V(\boldsymbol{e}(t_k)),
\end{aligned}
\tag{6}
$$

where $\beta_k = \lambda_{\max}[(\boldsymbol{I}+\boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})]$. Then, we can obtain the following comparison system:

$$
\begin{aligned}
&\dot{w}(t) = (2\lambda_1 + 2L_1 + L_2^2 + 1 + 2\theta_1\lambda_2)w(t),\ t \neq t_k, \\
&w(t_k^+) = \beta_k w(t_k),\ t = t_k, \\
&w(t_0^+) = w(0) \geq 0.
\end{aligned}
\tag{7}
$$

Since $2\lambda_1 + 2L_1 + L_2^2 + 1 + 2\theta_1\lambda_2 \geq 0$ and $(2\lambda_1 + 2L_1 + L_2^2 + 1 + 2\theta_1\lambda_2)\Delta_k + \ln(\xi\beta_k) \leq 0$, $k = 1, 2, \ldots$, the stability conditions in comparison theorem are satisfied. Then, the error system (4) is asymptotically stable.

In practice, for convenience, the gain matrices $\boldsymbol{B}_{ik}$ are always selected as a constant matrix and the impulsive distances Δ_k are each set to be a positive constant.

If the coupling matrix $\boldsymbol{D}'$ is asymmetrical, according to Lemma 2, we have

$$
\begin{aligned}
&2(\boldsymbol{e}(t))^{\mathrm{T}}\,\bar{\boldsymbol{D}}'\boldsymbol{H}(\boldsymbol{e}(t)) \\
&\quad\leq (\boldsymbol{e}(t))^{\mathrm{T}}\bar{\boldsymbol{D}}'\bar{\boldsymbol{D}}'^{\mathrm{T}}\boldsymbol{e}(t) + \boldsymbol{H}^{\mathrm{T}}(\boldsymbol{e}(t))\boldsymbol{H}(\boldsymbol{e}(t)) \\
&\quad\leq (\boldsymbol{e}(t))^{\mathrm{T}}\bar{\boldsymbol{D}}'\bar{\boldsymbol{D}}'^{\mathrm{T}}\boldsymbol{e}(t) + \theta_2^2\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t),
\end{aligned}
$$

then network (2) globally asymptotically synchronizes to the given trajectory $\boldsymbol{x}(t)$ of isolated node (1) if there exists a constant $\xi > 1$ such that

$$
(2\lambda_1 + 2L_1 + L_2^2 + 1 + \theta_2^2 + \lambda_3)\Delta_k + \ln(\xi\beta_k) \leq 0,\quad k = 1, 2, \ldots,
$$

where λ_3 is the largest eigenvalue of $(\boldsymbol{D}'\otimes\boldsymbol{\Gamma})(\boldsymbol{D}'\otimes\boldsymbol{\Gamma})^{\mathrm{T}}$.

3.2. Coupling terms with delay

In what follows, we will show if there exists a delay in coupling term and draw the corresponding conclusion. Namely, if there exists a constant $\xi > 1$ such that

$$
(2\lambda_1 + 2L_1 + L_2^2 + 2 + \theta_2^2\lambda_4)\Delta_k + \ln(\xi\beta_k) \leq 0,\quad k = 1, 2, \ldots,
$$

then network (3) globally asymptotically synchronizes to the given trajectory $\boldsymbol{x}(t)$ of isolated node (1), where λ_4 is the largest eigenvalue of $(\boldsymbol{D}\otimes\boldsymbol{\Gamma})(\boldsymbol{D}\otimes\boldsymbol{\Gamma})^{\mathrm{T}}$.

We choose a Lyapunov function candidate as

$$
\begin{aligned}
V(\boldsymbol{e}(t)) = \sum_{i=1}^{N}\Big[&(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + \int_{t-\tau}^{t}(\boldsymbol{e}^i(s))^{\mathrm{T}}\boldsymbol{e}^i(s)\mathrm{d}s \\
&+ \int_{t-\tau_1}^{t}(\boldsymbol{e}^i(s))^{\mathrm{T}}\boldsymbol{e}^i(s)\mathrm{d}s\Big].
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
&\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) \leq V(\boldsymbol{e}(t)) \\
&\qquad\leq \sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + \sum_{i=1}^{N}(\tau+\tau_1)\|\boldsymbol{e}^i\|^2,
\end{aligned}
$$

where $\|\boldsymbol{e}^i\| = \sup_{t-\tau_0\leq s\leq t}\|\boldsymbol{e}^i(s)\|$ and $\tau_0 = \max\{\tau, \tau_1\}$.

For $t \neq t_k$, by Lemma 2, Hypothesis 1 and Hypothesis 2, the derivative of $V(\boldsymbol{e}(t))$ along the trajectories of Eq. (5) is

$$\begin{aligned}
\dot{V}(\boldsymbol{e}(t)) =& 2\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\Big[\boldsymbol{C}\boldsymbol{y}^i(t) - \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{f}_1(\boldsymbol{y}^i(t)) - \boldsymbol{f}_1(\boldsymbol{x}(t)) + \boldsymbol{f}_2(\boldsymbol{y}^i(t-\tau)) - \boldsymbol{f}_2(\boldsymbol{x}(t-\tau)) \\
&+ \sum_{j=1}^{N} d_{ij}\boldsymbol{\Gamma}\boldsymbol{h}(\boldsymbol{y}^j(t-\tau_1))\Big] + \sum_{i=1}^{N}\Big[2(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) - (\boldsymbol{e}^i(t-\tau))^{\mathrm{T}}\boldsymbol{e}^i(t-\tau) - (\boldsymbol{e}^i(t-\tau_1))^{\mathrm{T}}\boldsymbol{e}^i(t-\tau_1)\Big] \\
\leq& 2\sum_{i=1}^{N}\Big[(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{C}\boldsymbol{e}^i(t) + L_1(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + L_2\|(\boldsymbol{e}^i(t))^{\mathrm{T}}\|\|\boldsymbol{e}^i(t-\tau)\|\Big] + 2(\boldsymbol{e}(t))^{\mathrm{T}}(\boldsymbol{D}\otimes\boldsymbol{\Gamma})\boldsymbol{H}(\boldsymbol{e}(t-\tau_1)) \\
&+ \sum_{i=1}^{N}\Big[2(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) - (\boldsymbol{e}^i(t-\tau))^{\mathrm{T}}\boldsymbol{e}^i(t-\tau) - (\boldsymbol{e}^i(t-\tau_1))^{\mathrm{T}}\boldsymbol{e}^i(t-\tau_1)\Big] \\
\leq& (2\lambda_1(\boldsymbol{C}) + 2L_1 + L_2^2 + 2)\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + 2(\boldsymbol{e}(t))^{\mathrm{T}}\bar{\boldsymbol{D}}\boldsymbol{H}(\boldsymbol{e}(t-\tau_1)) - \sum_{i=1}^{N}(\boldsymbol{e}^i(t-\tau_1))^{\mathrm{T}}\boldsymbol{e}^i(t-\tau_1) \\
=& (2\lambda_1 + 2L_1 + L_2^2 + 2)\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + \theta_2^2(\boldsymbol{e}(t))^{\mathrm{T}}\bar{\boldsymbol{D}}\bar{\boldsymbol{D}}^{\mathrm{T}}\boldsymbol{e}(t) \\
&+ \frac{1}{\theta_2^2}\boldsymbol{H}^{\mathrm{T}}(\boldsymbol{e}(t-\tau_1))\boldsymbol{H}(\boldsymbol{e}(t-\tau_1)) - \sum_{i=1}^{N}(\boldsymbol{e}^i(t-\tau_1))^{\mathrm{T}}\boldsymbol{e}^i(t-\tau_1) \\
\leq& (2\lambda_1 + 2L_1 + L_2^2 + 2 + \theta_2^2\lambda_4(\bar{\boldsymbol{D}}\bar{\boldsymbol{D}}^{\mathrm{T}}))\sum_{i=1}^{N}(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) \\
\leq& (2\lambda_1 + 2L_1 + 2L_2^2 + 2 + \theta_2^2\lambda_4)\sum_{i=1}^{N}\Big[(\boldsymbol{e}^i(t))^{\mathrm{T}}\boldsymbol{e}^i(t) + \int_{t-\tau}^{t}(\boldsymbol{e}^i(s))^{\mathrm{T}}\boldsymbol{e}^i(s)\mathrm{d}s + \int_{t-\tau_1}^{t}(\boldsymbol{e}^i(s))^{\mathrm{T}}\boldsymbol{e}^i(s)\mathrm{d}s\Big] \\
\leq& (2\lambda_1 + 2L_1 + L_2^2 + 2 + \theta_2^2\lambda_4)V(\boldsymbol{e}(t)).
\end{aligned}$$

Then error system given by Eq. (5) can be shown to be asymptotically stable using the above-mentioned method. It is worth noting that the coupling matrix $\boldsymbol{D}$ is not necessarily symmetrical.

4. Numerical results

In this section, numerical simulations are presented to verify the theoretical result obtained in the previous sections. We consider chaotic delayed Hopfield neural networks as nodes of the network, in which each node is described by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{x}(t-\tau)), \tag{8}$$

with $\boldsymbol{x}(t) = (x_1(t), x_2(t))^{\mathrm{T}} \in \boldsymbol{R}^2$, $\boldsymbol{f}(\boldsymbol{x}(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^{\mathrm{T}} \in \boldsymbol{R}^2$ and

$$\boldsymbol{C} = \begin{bmatrix} -1.0 & 0 \\ 0 & -1.0 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 3.0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}.$$

We show that a nonlinear coupled complex network with 10 nodes described by

$$\begin{cases} \dot{\boldsymbol{y}}^i(t) = \boldsymbol{C}\boldsymbol{y}^i(t) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{y}^i(t)) + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{y}^i(t-\tau)) + \sum\limits_{j=1}^{10} d_{ij}\boldsymbol{\Gamma}\boldsymbol{h}(\boldsymbol{y}^j(t)), \ t \neq t_k, \\ \Delta\boldsymbol{y}^i = \boldsymbol{y}^i(t_k^+) - \boldsymbol{y}^i(t_k^-) = \boldsymbol{B}_{ik}(\boldsymbol{y}^i(t) - \boldsymbol{x}(t)), \ t = t_k, \\ \boldsymbol{y}^i(t) = \boldsymbol{\phi}^i(t), \ t \in [t_0 - \tau, t_0], \ i = 1, \ldots, 10, \end{cases} \tag{9}$$

or

$$\begin{cases} \dot{\boldsymbol{y}}(t) = \boldsymbol{C}\boldsymbol{y}^i(t) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{y}^i(t)) + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{y}^i(t-\tau)) + \sum_{j=1}^{10} d_{ij}\boldsymbol{\Gamma}\boldsymbol{h}(\boldsymbol{y}^j(t-\tau_1)), \ t \neq t_k, \\ \Delta\boldsymbol{y}^i = \boldsymbol{y}^i(t_k^+) - \boldsymbol{y}^i(t_k^-) = \boldsymbol{B}_{ik}(\boldsymbol{y}^i(t) - \boldsymbol{y}(t)), \ t = t_k, \\ \boldsymbol{y}^i(t) = \boldsymbol{\phi}^i(t), \ t \in [t_0 - \tau, t_0], \ i = 1, \ldots, 10, \end{cases} \tag{10}$$

will reach a specified trajectory $\boldsymbol{x}(t)$ of the corresponding chaotic delayed Hopfield neural network. Without loss of generality, the coupled structure is chosen as two-way cricoid coupling or one-way cricoid coupling, i.e., the coupling matrix is

$$\boldsymbol{D} = \begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ddots & -2 \end{bmatrix}_{10\times 10},$$

or

$$\boldsymbol{D}' = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}_{10\times 10}.$$

In simulation, we choose $\boldsymbol{\Gamma} = \boldsymbol{I}$, $\boldsymbol{B}_{ik} = \boldsymbol{B}' = \mathrm{diag}\{\delta, \delta, \delta\}$, $\delta = -0.3$, $\xi = 1.1$, and the nonlinear function $h(z) = 3z + \sin(z)$ which satisfies $\theta_1 \leq (h(z_1) - h(z_2))/(z_1 - z_2) \leq \theta_2$ with $\theta_1 = 2$ and $\theta_2 = 4$. After simple calculations, we obtain $L_1 = 6.0989$, $L_2 = 2.5226$, $\lambda_1 = -1$, $\lambda_2 = 0$, and $\beta = (1+\delta)^2 = 0.49$, therefore, the impulsive interval Δ can be estimated as follows:

$$0 < \Delta < -\frac{\ln\xi + \ln\beta}{2\lambda_1 + 2L_1 + L_2^2 + 1 + \lambda_2} = 0.0352.$$

Let impulsive interval $\Delta = 0.02$ and node delay $\tau = 1$, the state variables $\boldsymbol{y}^i(t)\,(i = 1, \ldots, 10)$ of nonlinear coupled complex network with non-delay coupling, the state $x_j(t)\,(j = 1, 2)$ of isolated node and network error

$$E(t) = \sqrt{\sum_{i=1}^{10}(y_1^i(t) - x_1(t))^2 + (y_2^i(t) - x_2(t))^2}$$

are displayed in Fig. 1. It is obvious that $E(t)$ converges to zero, and the nonlinear coupled complex network achieves synchronization globally. When the coupling matrix $\boldsymbol{D}'$ is asymmetrical, then the simulation results are plotted in Fig. 2. Clearly, the network with symmetrical coupling will achieve synchronization faster than the network with asymmetrical coupling given the same impulsive distance $\Delta = 0.02$. Figure 3 shows that the evolutions of the states $\boldsymbol{y}^i(t)\,(i = 1, \cdots, 10)$ of nonlinear coupled complex network with coupling delay $\tau_1 = 0.1$ and symmetrical coupling matrix $\boldsymbol{D}$, the state $x_j(t)\,(j = 1, 2)$ of isolated node and network error $E(t)$. Figure 4 illustrates the evolutions of the states $\boldsymbol{y}^i(t)\,(i = 1, \cdots, 10)$ of nonlinear coupled complex network with coupling delay $\tau_1 = 0.1$ and asymmetrical coupling matrix $\boldsymbol{D}'$, the state $x_j(t)\,(j = 1, 2)$ of isolated node and network error $E(t)$.

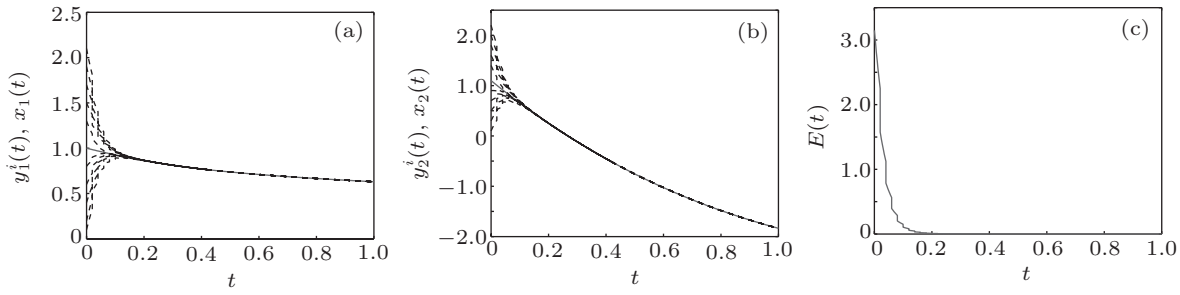


Fig. 1. (a) $y_1^i(t)\,(i = 1, \ldots, 10)$ (dashed) and $x_1(t)$ (solid), (b) $y_2^i(t)\,(i = 1, \ldots, 10)$ (dashed) and $x_2(t)$ (solid), (c) synchronization error $E(t)$ of networks (9) with non-delay coupling, symmetrical coupling matrix $\boldsymbol{D}$ and $\Delta = 0.02$.

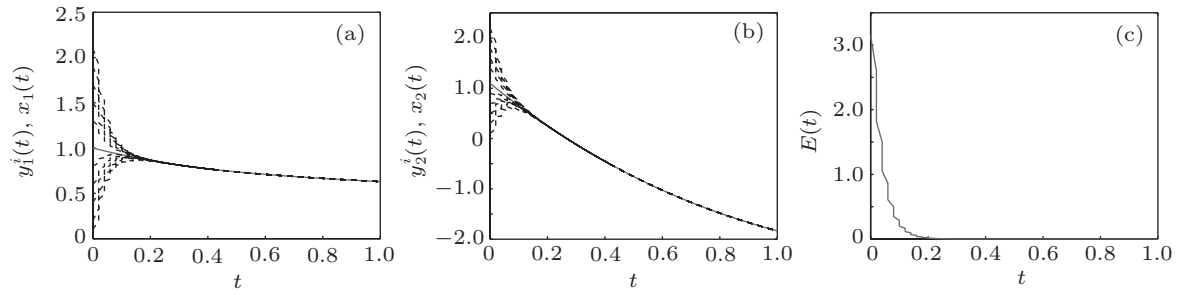


Fig. 2. (a) $y_1^i(t)\,(i = 1, \ldots, 10)$ (dashed) and $x_1(t)$ (solid), (b) $y_2^i(t)\,(i = 1, \ldots, 10)$ (dashed) and $x_2(t)$ (solid), (c) synchronization error $E(t)$ of networks (9) with non-delay coupling, asymmetrical coupling matrix $\boldsymbol{D}'$ and $\Delta = 0.02$.

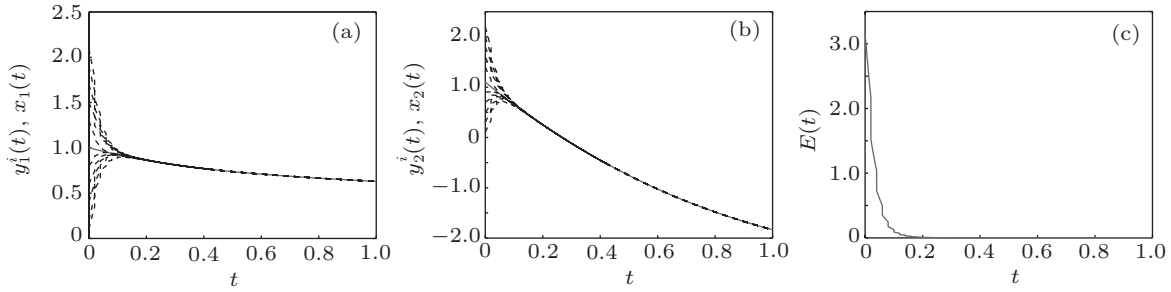


Fig. 3. (a) $y_1^i(t)\,(i = 1, \ldots, 10)$ (dashed) and $x_1(t)$ (solid), (b) $y_2^i(t)\,(i = 1, \ldots, 10)$ (dashed) and $x_2(t)$ (solid), (c) synchronization error $E(t)$ of networks (10) with delay coupling, symmetrical coupling matrix $\boldsymbol{D}$ and $\Delta = 0.02$.

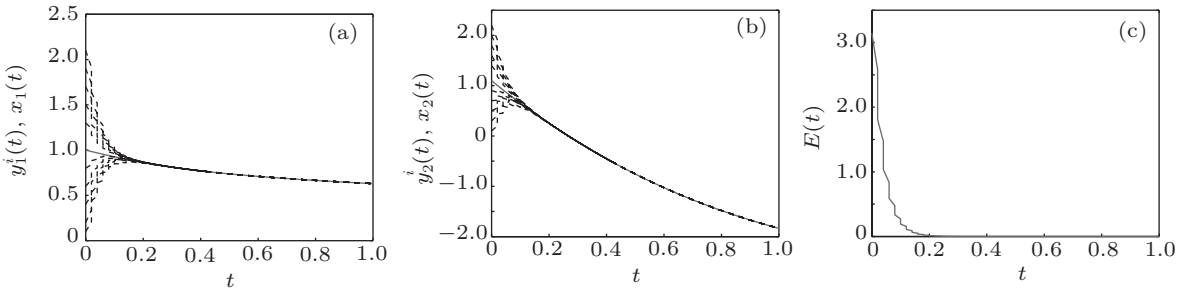


Fig. 4. (a) $y_1^i(t)\,(i = 1, \ldots, 10)$ (dashed) and $x_1(t)$ (solid), (b) $y_2^i(t)\,(i = 1, \ldots, 10)$ (dashed) and $x_2(t)$ (solid), (c) synchronization error $E(t)$ of networks (10) with delay coupling, asymmetrical coupling matrix $\boldsymbol{D}'$ and $\Delta = 0.02$.

5. Conclusion

This paper deals with the problem of the impulsive synchronization of a nonlinear coupled complex networks with a delay node. On the basis of the comparison theorem of impulsive differential system, some novel synchronization criteria are derived in the cases of delay coupling and non-delay coupling regardless of whether coupling matrix is symmetrical or not. Finally, numerical simulations are presented to verify the effectiveness of the proposed synchronization criteria.

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