Bifurcations and chaos control in discrete small-world networks*

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An impulsive delayed feedback control strategy to control period-doubling bifurcations and chaos is proposed. The control method is then applied to a discrete small-world network model. Qualitative analyses and simulations show that under a generic condition, the bifurcations and the chaos can be delayed or eliminated completely. In addition, the periodic orbits embedded in the chaotic attractor can be stabilized.

Keywords: bifurcation, chaos, small-world networks, impulsive delayed feedback control

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1. Introduction

A great deal of research interest in the theory and applications of small-world networks has been aroused[1−5] since the pioneering work of Watts and Strogatz.[6] Such small-world networks are usually characterized by two features: high clustering, which is a characteristic of regular networks, and short path length, which is typically observed in random networks. Recently, most of the small-world network models[1,2] are used to study the spreading and response of influences are linear models. The response is immediate as there are no time delays in the models. However, in reality, there are usually time delays in spreading and response. In addition, the nonlinear interactions due to competition and communication congestion of vertices should be modeled to simulate more realistic networks. Thus, in Ref. [5], a general nonlinear delayed differential equation model was formulated for small-world networks. Moreover, bifurcations and chaos caused by the delay and nonlinear interactions were studied in Refs. [5], [7], and [8].

Since the seminal work of Ott *et al.*,[9] there have been many other ideas proposed for the control of bifurcations and chaos.[10−30] Among these, the delayed feedback control (DFC) scheme[18] has gained considerable attention due to its various attractive features. However, as pointed out in Refs. [19] and [20], the DFC scheme has some inherent limitations, i.e., it cannot be applied to the stabilization of some periodic orbits. To overcome the limitations of the DFC scheme, several modifications[21−30] have been proposed. In this paper, we present a new extension. Compared with existing methods, we will show that the proposed extension has the following advantages. First, compared with the existing conditions, a milder condition is needed for the stabilization of the periodic orbits. The condition may be considered as a generic case and we may state that almost all periodic orbits can be stabilized by the proposed extension. Second, the controller only needs to be applied to the original system at the fixed time. It is noticeable that the control interval can be chosen according to our needs. Hence, the implementation of the proposed control strategy is easy and convenient. Last, the design of the controller has some flexibility. To obtain the same control result, the parameters in the controller may have many choices, and we can choose the suitable style of controller according to real situations.

The rest of this paper is organized as follows. The dynamics of the small-world network model is analysed in Section 2. An impulsive delayed feedback control strategy for a discrete nonlinear system is proposed and the stability of the periodic orbits is discussed in Section 3. In Section 4, bifurcations and chaos control of the small-world network are studied, and the simulations are given. Conclusions and a discussion are presented in the last section.

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2. Small-world network model

The nonlinear small-world network model[5] with time delay is described by

$$\frac{\mathrm{d}^d x}{\mathrm{d}t^d} = \xi^d + x(t-\delta) - \mu\xi^d x^2(t-\delta), \tag{1}$$

where $x(t)$ is the total influenced volume at time t, d is the dimension of the network, ξ is the Newman–Watts length scale,[1] δ is the time delay, and μ is a measure of nonlinear interactions in the network.

From the theory of dynamic systems, the dynamic features can be shown more clearly by using the representation in Poincaré plane, which usually transforms a nonlinear differential equation into a nonlinear iterated map. For simplicity, in this paper, we take $\delta = 1$ and $d = 1$. Applying the forward Euler scheme to system (1), we can obtain

$$x_{n+1} = \xi + 2x_n - \mu\xi x_n^2, \tag{2}$$

where $x_{n+1} = x(t)$, and $x_n = x(t-1)$.

In this paper, we choose μ as the bifurcation parameter and fix $\xi = 1.8$ (we can also choose ξ as the bifurcation parameter and fix μ). The bifurcation diagram and Lyapunov exponent diagram are shown in Figs. 1 and 2, respectively.

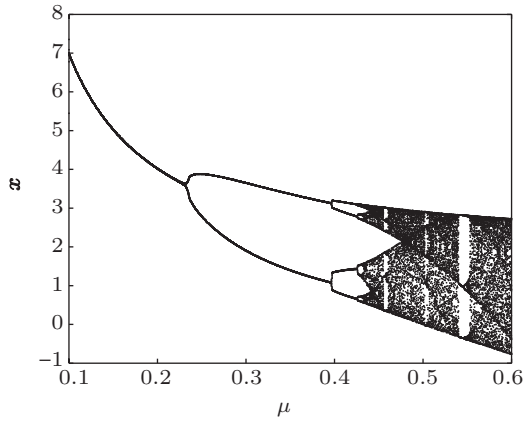


Fig. 1. The bifurcation diagram of system (2).

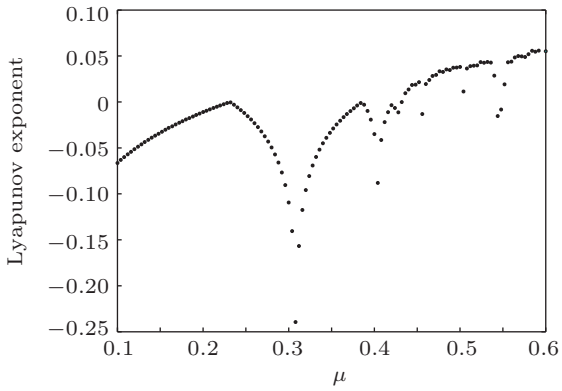


Fig. 2. The Lyapunov exponent diagram of system (2).

From Figs. 1 and 2 we can see the existence of a period-doubling cascade turning to chaos. The main focus of this paper is to design an impulsive delayed feedback control scheme to control the bifurcations and chaos in small-world network model (2).

3. Impulsive delayed feedback control

Consider a discrete nonlinear system

$$x_{n+1} = f(x_n), \tag{3}$$

where $x \in R$ is the state variable, $f : R \to R$ is an appropriate function, which is assumed to be differentiable wherever required. Assume system (3) possesses a periodic T orbit characterized by $\Sigma_T = \{x_1^*, x_2^*, \cdots, x_T^*\}$.

Applying the control scheme to system (3), we have

$$x_{n+1} = f(x_n) + u_n, \tag{4}$$

where u_n is the control input. For the case of $T = 1$, the impulsive delayed feedback control law u_n is described as follows:

$$u_n = \sum_{i=1}^{m-1} \varepsilon_n^i (x_n - x_{n-i}), \tag{5}$$

where m is the impulsive interval to be chosen, and ε_n^i is given by

$$\varepsilon_n^i = \begin{cases} K_i & n|m = 0, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, m-1, \tag{6}$$

where K_i is the constant gain to be determined.

Next, we will discuss the local stability of the periodic 1 orbit x_1^*. Define the error $e_n = x_n - x_1^*$. After linearization of the closed-loop system consisted of Eqs. (4), (5) and (6), we obtain

$$\begin{cases} e_2 = ae_1, \\ e_3 = ae_2 = a^2 e_1, \\ \vdots \\ e_m = a^{m-1} e_1, \\ e_{m+1} = \left[a^m + \sum\limits_{i=1}^{m-1} K_i (a^{m-1} - a^{m-1-i})\right] e_1, \end{cases} \tag{7}$$

where $a = f'(x_1^*)$.

By mathematical induction we can show the following equation holds:

$$e_{km+1} = \left[a^m + \sum_{i=1}^{m-1} K_i (a^{m-1} - a^{m-1-i})\right]^k e_1. \tag{8}$$

From Eq. (7) we can see that Eq. (8) holds for $k = 1$. Assume Eq. (8) holds for $k-1$, that is,

$$e_{(k-1)m+1} = \left[a^m + \sum_{i=1}^{m-1} K_i(a^{m-1} - a^{m-1-i})\right]^{k-1} e_1. \tag{9}$$

Repeating the same procedure, we can obtain

$$\begin{cases} e_{(k-1)m+2} = ae_{(k-1)m+1}, \\ e_{(k-1)m+3} = ae_{(k-1)m+2} = a^2 e_{(k-1)m+1}, \\ \vdots \\ e_{(k-1)m+m} = a^{m-1} e_{(k-1)m+1}, \\ e_{(k-1)m+m+1} = \left[a^m + \sum\limits_{i=1}^{m-1} K_i(a^{m-1} - a^{m-1-i})\right] \\ \qquad \times e_{(k-1)m+1}. \end{cases} \tag{10}$$

Combining Eq. (9), we can conclude that Eq. (8) holds for any k. Then we have the following theorem.

Theorem 1 The closed-loop system consisted of Eqs. (4), (5) and (6) is locally stable at the periodic 1 point x_1^* if and only if

$$\left|a^m + \sum_{i=1}^{m-1} K_i(a^{m-1} - a^{m-1-i})\right| < 1, \tag{11}$$

where $a = f'(x_1^*)$.

Remark 1 From Eq. (11) we can see that when we stabilize the periodic 1 point x_1^*, there are almost no restrictions to the value of a. Hence, the limitations[18−30] of the existing control strategies are overcome in the control scheme described by Eqs. (5) and (6). We can state that almost all periodic orbits can be stabilized with this scheme. On the other hand, the impulsive interval m and constant gains $K_1, K_2, \ldots, K_{m-1}$ can be chosen flexibly according to our requirement. The more previous data are used in Eqs. (5) and (6), the smaller the gains may be. When m is suitably large, the implementation of the control scheme becomes easy and convenient.

Now, let us consider the case of $T > 1$. To stabilize the periodic T orbit of system (3), we propose the following control law:

$$u_n = \sum_{i=1}^{m-1} \varepsilon_n^i (x_{n-T+1} - x_{n-(i+1)T+1}), \tag{12}$$

where m is the impulsive interval to be chosen, and ε_n^i is given by

$$\varepsilon_n^i = \begin{cases} K_i & n|(mT) = 0, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, m-1, \tag{13}$$

where K_i is the constant gain to be determined.

Now we begin to study the relation of the control laws given by Eqs. (12), (13) and (5), (6). Define the T-iterate map F as $F = f^T$. Then periodic T orbits of f are periodic 1 orbits of F, that is, $F(x_i^*) = x_i^*$, $i = 1, 2, \ldots, T$. Denote

$$z_j = x_{(j-1)T+1}, \quad j = 1, 2, \ldots \tag{14}$$

If $j|m \neq 0$, that is, $jT|(mT) \neq 0$, then

$$x_{jT+1} = f(x_{jT}) = f^2(x_{jT-1}) = \cdots = f^T(x_{(j-1)T+1}), \tag{15}$$

which is equivalent to

$$z_{j+1} = F(z_j). \tag{16}$$

Otherwise, if $j|m = 0$, that is, $jT|(mT) = 0$, then

$$x_{jT+1} = f^T(x_{(j-1)T+1}) + \sum_{i=1}^{m-1} K_i(x_{(j-1)T+1} - x_{jT-(i+1)T+1}), \tag{17}$$

which is equivalent to

$$z_{j+1} = F(z_j) + \sum_{i=1}^{m-1} K_i(z_j - z_{j-i}). \tag{18}$$

Combining Eqs. (16) and (18), we can obtain

$$z_{j+1} = F(z_j) + u_j, \tag{19}$$

where u_j is described by

$$u_j = \sum_{i=1}^{m-1} \varepsilon_j^i (z_j - z_{j-i}), \tag{20}$$

where ε_j^i is given by

$$\varepsilon_j^i = \begin{cases} K_i & j|m = 0, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, m-1. \tag{21}$$

Hence, the impulsive delayed feedback control scheme given by Eqs. (12) and (13) is equivalent to the scheme given by Eqs. (5) and (6) in terms of the variable z defined in Eq. (14).

Next, we will study the local stability of the periodic T orbit of system (3). As mentioned above, x_i^* $(i = 1, 2, \ldots, T)$ is a fixed point of F. Hence, x_i^* is stable if and only if $\left|a^m + \sum\limits_{i=1}^{m-1} K_i(a^{m-1} - a^{m-1-i})\right| < 1$, where $a = F'(x_i^*)$. Since $F = f^T$, we have $a = \prod\limits_{j=1}^{T} a_j$, where $a_j = f'(x_j^*)$. Summarize the above statements, we have the following theorem.

Theorem 2 The closed-loop system consisted of Eqs. (4), (12) and (13) is locally stable at the periodic T point $\Sigma_T = \{x_1^*, x_2^*, \cdots, x_T^*\}$ if and only if

$$\left|a^m + \sum_{i=1}^{m-1} K_i(a^{m-1} - a^{m-1-i})\right| < 1, \tag{22}$$

where $a = \prod\limits_{j=1}^{T} a_j$, $a_j = f'(x_j^*)$.

4. Control effects analyses and simulations

4.1. Bifurcation control in small-world networks

In this section, we will show that the controlled small-world network model can maintain its stable dynamical behaviours in a larger range of parameter values. Therefore, the period-doubling route to chaos is at least delayed to greater parameter values.

First, we will discuss the stabilization of the fixed point. By simple calculation, we can find that system (2) has a positive fixed point $x_1^* = \frac{1}{2\mu\xi}(1 + \sqrt{1+4\mu\xi^2})$. The stability condition of system (2) at x_1^* is $|2 - 2\mu\xi x_1^*| < 1$, then the stability range of μ is $0 < \mu < \frac{3}{4}\xi^{-2}$.

The controlled small-world network model can be described by

$$x_{n+1} = \xi + 2x_n - \mu\xi x_n^2 + u_n, \tag{23}$$

where u_n is given in Eqs. (5) and (6). The controlled model has the same fixed point with the original system (2).

By Theorem 1, the stability range of the fixed point x_1^* can be obtained from (11). For $m = 2$, x_1^* is locally stable if and only if

$$\left|a^2 + K_1(a-1)\right| < 1, \tag{24}$$

where $a = 1 - \sqrt{1+4\mu\xi^2}$, and K_1 is the constant gain. Then the stability range of μ is

$$0 < \mu < \frac{1}{4\xi^2}(3 + 4K_1 + K_1^2). \tag{25}$$

Note that (24) is equivalent to

$$-(1+a^2) < K_1(a-1) < 1 - a^2, \tag{26}$$

where $a = 1 - \sqrt{1+4\mu\xi^2} < 1$. Then the range of K_1 for stabilization is

$$-(a+1) < K_1 < \frac{a^2+1}{1-a}. \tag{27}$$

If x_1^* is unstable, then we have $|a| > 1$, so $a < 1$ implies $a < -1$. In this case, K_1 is positive. Hence, in (25), the stability range of μ is enlarged.

For $m > 2$, as the analytical form of the stability range of parameter μ is difficult to obtain, we will discuss it numerically later.

Next, we will consider the stabilization of the periodic 2 point. After calculation, we can find that system (2) has a periodic 2 orbit $\Sigma_2 = \{x_1^*, x_2^*\}$, where $x_{1,2}^* = \frac{1}{2\mu\xi}(3 \pm \sqrt{-3+4\mu\xi^2})$. The stability condition of system (2) at Σ_2 is $|(2 - 2\mu\xi x_1^*)(2 - 2\mu\xi x_2^*)| < 1$, then the stability range of μ is $\frac{3}{4}\xi^{-2} < \mu < \frac{5}{4}\xi^{-2}$.

Choose $m = 2$, the control strategy expressed in Eqs. (12) and (13) can be rewritten by

$$u_n = \varepsilon_n^1(x_{n-1} - x_{n-3}), \tag{28}$$

where

$$\varepsilon_n^1 = \begin{cases} K_1 & n|4 = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

By (22) in Theorem 2, $\Sigma_2 = \{x_1^*, x_2^*\}$ is locally stable if and only if

$$\left|a^2 + K_1(a-1)\right| < 1, \tag{30}$$

where $a = a_1 a_2$, $a_1 = 2 - 2\mu\xi x_1^*$, $a_2 = 2 - 2\mu\xi x_2^*$. Then the stability range of μ is

$$\frac{3}{4}\xi^{-2} < \mu < \frac{5 + K_1}{4\xi^2}. \tag{31}$$

Similarly, we can prove that K_1 is positive, then the stability range of μ is enlarged. For the $4, 8, 16, \ldots, 2^n$ period orbits, we can use the similar method to obtain the stability range of parameter μ.

The bifurcations control results are shown in Fig. 3. Compared with Fig. 1, Figs. 3(a) and 3(b) indicate that the first period-doubling bifurcation of the small-world network is delayed or eliminated under the control scheme, Fig. 3(c) shows that the second period-doubling bifurcation is delayed.

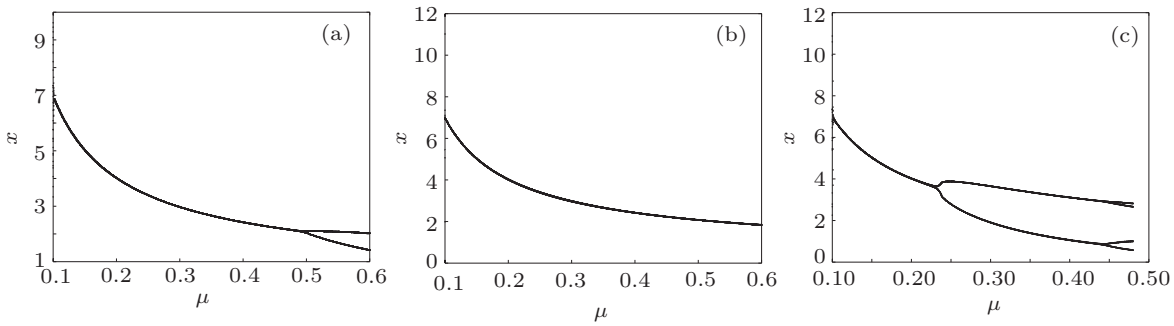


Fig. 3. The results of bifurcation control. (a) The result of controlling the first period-doubling bifurcation with $m = 2$ and $K_1 = 0.72$. (b) The result of controlling the first period-doubling bifurcation with $m = 3$, $K_1 = 0.9$, and $K_2 = 0.5$. (c) The result of controlling the second period-doubling bifurcation with $m = 2$ and $K_1 = 0.75$.

The above research indicates that the parameter range to guarantee the stability of the periodic orbits can be enlarged under the control scheme. Thus, the period-doubling route to chaos is at least delayed.

4.2. Chaos control in small-world networks

In this section, we will show that some unstable periodic orbits that are embedded in the chaotic attractor of system (2) can be stabilized by the impulsive delayed feedback control. In addition, a chaotic trajectory in the small-world network model will settle to a periodic orbit upon activation of the control law.

Figure 4 shows the results of controlling the chaotic orbits to the periodic orbits with $\mu = 0.47$. From Figs. 4(a) and 4(b), we can see that the system is stabilized in the same periodic 2 orbit by choosing different control parameters. figure 5 shows the results of controlling a periodic orbit to the other periodic orbits with $\mu = 0.42$. Figures 5(a) and 5(b) show that the system can always be stabilized in the same periodic 1 orbits under different groups of control parameters.

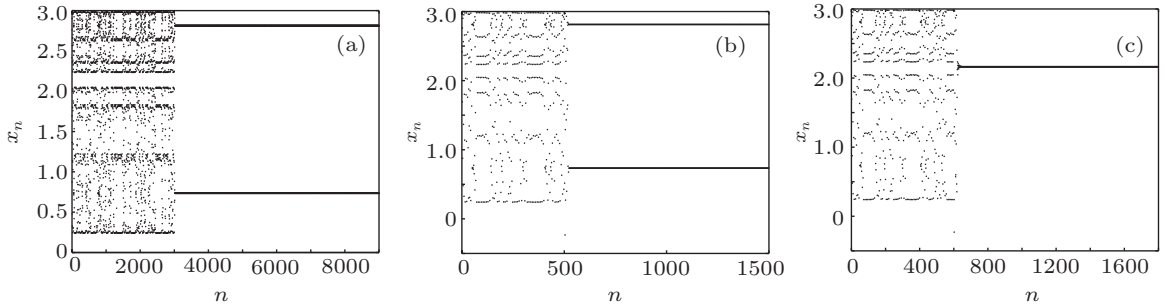


Fig. 4. The results of controlling the chaotic orbits to the periodic 2 and 1 orbits: (a) $m = 2$ and $K_1 = 1.5$; (b) $m = 3$, $K_1 = 1.5$, and $K_2 = 0$; (c) $m = 3$, $K_1 = 0.9$, and $K_2 = 0$.

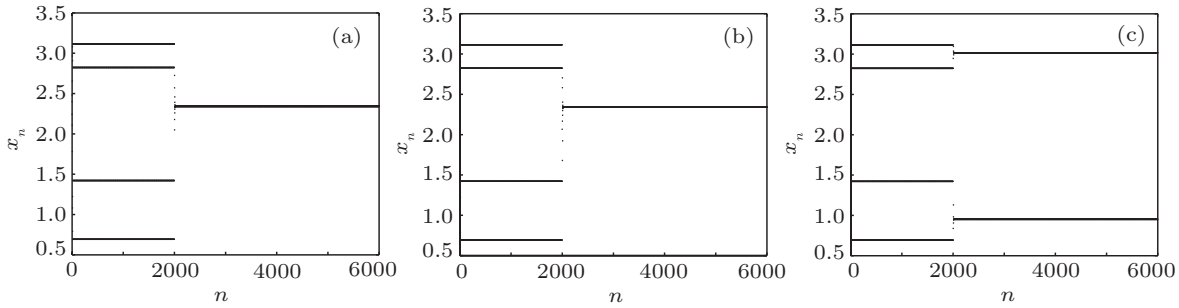


Fig. 5. The results of controlling a periodic 4 orbit to the periodic 1 and 2 orbits: (a) $m = 2$ and $K_1 = 0.8$; (b) $m = 3$, $K_1 = 0.8$, and $K_2 = 0.1$; (c) $m = 2$ and $K_1 = 0.8$.

The above simulations show that we can control the small-world network model (2) into different desired periodic orbits by choosing the appropriate forms of controllers. Moreover, we can see that the same control effect can be obtained by choosing different groups of control parameters, which shows that the implementation of the proposed control method is flexible.

5. Conclusion and discussion

In this paper, we have proposed an impulsive delayed feedback control strategy to control the period-doubling bifurcations and chaos in a small-world network model. The period-doubling bifurcations can be delayed or eliminated completely. Therefore, the period-doubling route to chaos is at least delayed to greater parameter values. When doing this, there may be many choices for the form of the controller, and we can choose the suitable control scheme according to our requirement.

The proposed method can be extended to higher dimensional systems, which is outside of the scope of this paper. An interesting difference will be found because the multiplication of matrices is not exchangeable in general. Further application of this control

strategy in complex networks will be our future research topic due to its easy implementation.

References

[1] Newman M E J and Watts D J 1999 *Phys. Rev.* E **60** 7332

[2] Moukarzel C F 1999 *Phys. Rev.* E **60** 6263

[3] Newman M E J, Morre C and Watts D J 2000 *Phys. Rev. Lett.* **84** 3201

[4] Watts D J 1999 *Small-Worlds: The Dynamics of Networks Between Order and Randomness* (Princeton, NJ: Princeton University Press)

[5] Yang X S 2001 *Phys. Rev.* E **63** 046206

[6] Watts D J and Strogatz S H 1998 *Nature* **393** 440

[7] Li C G and Chen G R 2004 *Chaos, Solitons Fractals* **20** 353

[8] Chen Z, Zhao D H and Ruan J 2007 *Chin. Ann. Math.* **20** 453

[9] Ott E, Grebogi C and Yorke J A 1990 *Phys. Rev. Lett.* **64** 1196

[10] Luo X S, Chen G R, Wang B H, Fang J Q, Zou Y L and Quan H J 2003 *Acta Phys. Sin.* **52** 790 (in Chinese)

[11] Zhang R, Weng J Q, Luo X S and Fang J Q 2006 *Chin. Phys.* **15** 1226

[12] Zou Y L, Luo X S and Chen G R 2006 *Chin. Phys.* **15** 1719

[13] Liu Y Z, Jiang C S, Lin C S and Jiang Y M 2007 *Chin. Phys.* **16** 660

[14] Wang F Z, Chen Z Q, Wu W J and Yuan Z Z 2007 *Chin. Phys.* **16** 3238

[15] Li R H, Xu W and Li S 2007 *Chin. Phys.* **16** 1591

[16] Liu F, Guan Z H and Wang H 2008 *Chin. Phys.* B **17** 2405

[17] Tan P A, Zhang B and Qiu D Y 2010 *Acta Phys. Sin.* **59** 5299 (in Chinese)

[18] Pyragas K 1992 *Phys. Lett.* A **170** 421

[19] Ushio T 1996 *IEEE Trans. Circ. Syst.* I **43** 815

[20] Morgül Ö 2003 *Phys. Lett.* A **314** 278

[21] Socolar J E, Sukow D W and Gauthier D J 1994 *Phys. Rev.* E **50** 3245

[22] Kittel A, Parisi J and Pyragas K 1995 *Phys. Lett.* A **198** 433

[23] Pyragas K 1995 *Phys. Lett.* A **206** 323

[24] Pyragas K 2001 *Phys. Rev. Lett.* **86** 2265

[25] Schuster H G and Stemmler M B 1997 *Phys. Rev.* E **56** 6410

[26] Morgül Ö 2006 *Int. J. Bifurcation Chaos* **16** 311

[27] Morgül Ö 2007 *Int. J. Bifurcation Chaos* **17** 4431

[28] Wu S H, Hao J H and Xu H B 2010 *Chin. Phys.* B **19** 020509

[29] Morgül Ö 2009 *Int. J. Bifurcation Chaos* **19** 365

[30] Lin W, Ma H F, Feng J F and Chen G R 2010 *Phys. Rev.* E **82** 046214