Energy and momentum of general spherically symmetric frames on the regularizing teleparallelism

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(Received 12 June 2011; revised manuscript received 21 July 2011)

In the context of the covariant teleparallel framework, we use the 2-form translational momentum to compute the total energy of two general spherically symmetric frames. The first one is characterized by an arbitrary function $H(r)$, which preserves the spherical symmetry and reproduces all the previous solutions, while the other one is characterized by a parameter ξ which ensures the vanishing of the axial of trace of the torsion. We calculate the total energy by using two procedures, i.e., when the Weitzenböck connection $\Gamma_\alpha{}^\beta$ is trivial, and show how $H(r)$ and ξ play the role of an inertia that leads the total energy to be unphysical. Therefore, we take into account $\Gamma_\alpha{}^\beta$ and show that although the spacetimes we use contain an arbitrary function and one parameter, they have no effect on the form of the total energy and momentum as it should be.

Keywords: gravitation, teleparallel gravity, energy–momentum, conserved currents

PACS: 04.20.–q, 04.20.Cv, 04.20.Jb, 04.50.Kd **DOI:** 10.1088/1674-1056/21/1/010401

1. Introduction

In the gravitational field, particles with different masses and compositions acquire the same acceleration and follow the same path. Such universality referred to as the universality of free fall and is the most fundamental characteristic of gravitational interaction.[1,2] This universality is usually identified as the weak equivalence principle (WEP), which establishes the equality between inertial (passive) and gravitational masses. In order to move along the same trajectory (path), the motion of different particles must be independent of their masses, which have to be canceled out from the equations of motion. Since this cancelation can only be made when the inertial and gravitational masses coincide, this coincidence will naturally imply universality. The most well-known gravitational theory is Einstein's general relativity (GR), which is fundamentally based on the WEP. This theory, up to now, is consistent with observational data. The geometry which Einstein GR is based on is the Riemannian geometry with unique metric and unique connection. However, due to the geometric structure and the WEP of the gravitational theory, the problem of energy remained unsolved for a long time. Using the Lagrange–Noether approach, one can derive the conserved currents that arise from the invariance of the classical action under transformations of fields. However, in Riemannian geometry, one cannot find symmetries that can be used to generate the conserved energy-momentum currents. One can only speak about the energy of asymptotically flat spacetime. More details of how this problem has been tackled can be found in Refs. [3]–[7] and references therein.

From the gauge-theoretic point of view, teleparallel gravity is distinguished from the other metric-affine gravity (MAG) models[8−12] by reducing the affine symmetry group to the translation subgroup. In geometric language, teleparallelism arises from the general metric-affine spacetime after we impose two constraints, by putting curvature and nonmetricity equal to zero. In this sense, teleparallelism can be treated, both physically and geometrically, as a particular case of the general MAG theory.

Teleparallel theories are interesting for several reasons: first of all, GR can be viewed as a particular theory of teleparallelism and, thus, teleparallelism could be considered, at the very least, as a different point of view that can lead to the same results.[13] Second, in this framework, one can define an energy-momentum complex for the gravitational field that is a true tensor under general coordinate transformations, but not under local Lorentz transformation. This is

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the reason why teleparallelism was reconsidered by Møller when he was studying the problem of defining an energy-momentum tensor for the gravitational field.[14,15] The idea was taken over by Pellegrini and Plebański, who constructed the general Lagrangian for these theories.[16] The third reason why these theories are interesting is that they can be seen as gauge theories of the translation group (not the full Poincare group), so they give an alternative interpretation of GR.[17,18−23]

An important difference between Einstein's theory and teleparallel theories is that it is possible to distinguish gravitational field from inertia.[24] Since inertia is in the realm of the pseudotensor behaviour of the usual expressions for the gravitational energy-momentum density, it turns out to be possible in teleparallel gravity to write down a tensorial expression for such a density.[25] With the purpose of giving a deeper insight into the covariant teleparallel formalism, as well as to test how it works, Lucas *et al.*[26] have computed the total energy of two important examples, namely, the Schwarzschild and Kerr solutions. Recently, Obukhov *et al.*[27] have computed the energy and momentum transported by exact plane gravitational-wave solutions of Einstein equations using the teleparallel equivalent formulation of Einstein's theory. It is our aim to generalize the calculations performed by Lucas *et al.*[26] using general spherically symmetric frames, one with an arbitrary function $H(r)$ and the other with a parameter ξ,[28,29] and to reanalyse the calculation of the total energy. We will calculate the total energy and momentum associated with these coframes using two different procedures: one is when $\Gamma_\alpha{}^\beta = 0$ and the other is when we take $\Gamma_\alpha{}^\beta$ into account (we will use the same notation given in Ref. [26]).

In Section 2, we use the language of exterior forms to give an outline of the teleparallel approach. A brief review of the covariant formalism for the gravitational energy-momentum which is described by the pair $(\vartheta^\alpha, \Gamma_\alpha{}^\beta)$ is given. In Section 3, we show that due to an inconvenient choice of a reference system, the traditional computation of the total energies of the generally spherically symmetric spacetimes are unphysical. Using the covariant formalism, the Weitzenböck connection acts as a regularizing tool that separates the inertial contribution and always provides the physically meaningful result. The final section is devoted to the main results and discussion.

2. The teleparallel approach

2.1. Notation

We use the Latin indices $i, j, \ldots$ for local holonomic spacetime coordinates and the Greek indices α, β, ... label (co)frame components. Particular frame components are denoted by hats, $\hat{0}$,$\hat{1}$, etc. As usual, the exterior product is denoted by $\wedge$, while the interior product of a vector ξ and a p-form Ψ is denoted by $\xi\rfloor\Psi$. The vector basis dual to the frame 1-forms ϑ^α is denoted by e_α and they satisfy $e_\alpha\rfloor\vartheta^\beta = \delta^\beta_\alpha$. Using local coordinates x^i, we have $\vartheta^\alpha = h^\alpha_i \mathrm{d}x^i$ and $e_\alpha = h^i_\alpha \partial_i$, where h^α_i and h^i_α are the covariant and contravariant components of the tetrad field. We define the volume 4-form by $\eta \overset{\text{def.}}{=} \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}$. Furthermore, with the help of the interior product, we define

$$\eta_\alpha \overset{\text{def.}}{=} e_\alpha\rfloor\eta = \frac{1}{3!}\,\epsilon_{\alpha\beta\gamma\delta}\,\vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta,$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\epsilon_{0123} = 1$. Furthermore,

$$\eta_{\alpha\beta} \overset{\text{def.}}{=} e_\beta\rfloor\eta_\alpha = \frac{1}{2!}\epsilon_{\alpha\beta\gamma\delta}\,\vartheta^\gamma \wedge \vartheta^\delta,$$
$$\eta_{\alpha\beta\gamma} \overset{\text{def.}}{=} e_\gamma\rfloor\eta_{\alpha\beta} = \frac{1}{1!}\epsilon_{\alpha\beta\gamma\delta}\,\vartheta^\delta,$$

which are bases for 3-, 2- and 1-forms respectively. Finally,

$$\eta_{\alpha\beta\mu\nu} \overset{\text{def.}}{=} e_\nu\rfloor\eta_{\alpha\beta\mu} = e_\nu\rfloor e_\mu\rfloor e_\beta\rfloor e_\alpha\rfloor\eta,$$

is the Levi-Civita tensor density. The η-forms satisfy the useful identities

$$\vartheta^\beta \wedge \eta_\alpha = \delta^\beta_\alpha\eta, \quad \vartheta^\beta \wedge \eta_{\mu\nu} = \delta^\beta_\nu\eta_\mu - \delta^\beta_\mu\eta_\nu,$$
$$\vartheta^\beta \wedge \eta_{\alpha\mu\nu} = \delta^\beta_\alpha\eta_{\mu\nu} + \delta^\beta_\mu\eta_{\nu\alpha} + \delta^\beta_\nu\eta_{\alpha\mu},$$
$$\vartheta^\beta \wedge \eta_{\alpha\gamma\mu\nu} = \delta^\beta_\nu\eta_{\alpha\gamma\mu} - \delta^\beta_\mu\eta_{\alpha\gamma\nu} + \delta^\beta_\gamma\eta_{\alpha\mu\nu} - \delta^\beta_\alpha\eta_{\gamma\mu\nu}. \quad (1)$$

The line element $\mathrm{d}s^2 \overset{\text{def.}}{=} g_{\alpha\beta}\vartheta^\alpha \bigotimes \vartheta^\beta$ is defined by the spacetime metric $g_{\alpha\beta}$.

2.2. Brief review of teleparallel gravity

Teleparallel geometry can be viewed as a gauge theory of translation.[17−34] The coframe ϑ^α plays the role of the gauge translational potential of the gravitational field. GR can be reformulated as the teleparallel theory. Geometrically, teleparallel gravity can be considered as a special case of the metric-affine gravity in which ϑ^α and the local Lorentz connection are subject to the distant parallelism constraint $R_\alpha{}^\beta = 0$.[35−44] In this geometry the torsion 2-form

$$T^\alpha = D\vartheta^\alpha = \mathrm{d}\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \frac{1}{2}T_{\mu\nu}{}^\alpha\vartheta^\mu \wedge \vartheta^\nu$$

$$= \frac{1}{2}T_{ij}{}^{\alpha}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \tag{2}$$

arises as the gravitational gauge field strength, with $\Gamma_{\alpha}{}^{\beta}$ being the Weitzenböck 1-form connection, d the exterior derivative and D the exterior covariant derivative. The torsion T^{α} can be decomposed into three irreducible pieces,[26] the tensor part, the trace and the axial trace given, respectively, by

$$T^{\alpha} \overset{\text{def.}}{=} T^{\alpha} - {}^{(2)}T^{\alpha} - {}^{(3)}T^{\alpha}, \tag{3a}$$

with

$$T^{\alpha} \overset{\text{def.}}{=} \frac{1}{3}\vartheta^{\alpha} \wedge T, \tag{3b}$$

where $T = \left(e_{\beta}\rfloor T^{\beta}\right)$, $e_{\alpha}\rfloor T = T_{\mu\alpha}{}^{\mu}$, vector of trace of torsion; and

$$T^{\alpha} \overset{\text{def.}}{=} \frac{1}{3}e^{\alpha}\rfloor P, \tag{3c}$$

where $P = \left(\vartheta^{\beta} \wedge T_{\beta}\right)$, $e_{\alpha}\rfloor P = T^{\mu\nu\lambda}\eta_{\mu\nu\lambda\alpha}$, axial of trace of torsion.

The Lagrangian of the teleparallel equivalent of GR has the form

$$V = -\frac{1}{2\kappa}T^{\alpha} \wedge^{*} \left({}^{(1)}T_{\alpha} - 2^{(2)}T_{\alpha} - \frac{1}{2}{}^{(3)}T_{\alpha}\right), \tag{4}$$

where $\kappa = 8\pi G/c^3$, G is the Newton gravitational constant, c is the speed of light and $*$ denotes the Hodge duality in the metric $g_{\alpha\beta}$ which is assumed to be flat Minkowski metric $g_{\alpha\beta} = o_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$ used to raise and lower local frame (Greek) indices.

The variation of the total action with respect to the coframe gives the field equations in the form

$$DH_{\alpha} - E_{\alpha} = \Sigma_{\alpha}, \tag{5}$$

where

$$\Sigma_{\alpha} \overset{\text{def.}}{=} \frac{\delta L_{\text{mattter}}}{\delta\vartheta^{\alpha}}$$

is the canonical energy-momentum tensor 3-form of matter considered to be the source. In accordance with the general Lagrange–Noether scheme,[11,31] one derives from (4) the translational momentum 2-form and the canonical energy-momentum 3-form of the gravitational field

$$H_{\alpha} \overset{\text{def.}}{=} -\frac{\partial V}{\partial T^{\alpha}} = \frac{1}{\kappa} * \left({}^{(1)}T_{\alpha} - 2^{(2)}T_{\alpha} - \frac{1}{2}{}^{(3)}T_{\alpha}\right), \tag{6}$$

$$E_{\alpha} \overset{\text{def.}}{=} \frac{\partial V}{\partial\vartheta^{\alpha}} = e_{\alpha}\rfloor V + \left(e_{\alpha}\rfloor T^{\beta}\right) \wedge H_{\beta}. \tag{7}$$

Due to geometric identities,[45] the Lagrangian (4) can be recast as

$$V = -\frac{1}{2}T^{\alpha} \wedge H_{\alpha}. \tag{8}$$

The presence of the connection field $\Gamma_{\alpha}{}^{\beta}$ plays an important regularizing role due to the following reasons:

1) The theory becomes explicitly covariant under local Lorentz transformations of the coframe, i.e., the Lagrangian (4) is invariant under the change of variables

$$\begin{aligned} &\vartheta'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\vartheta^{\beta}, \\ &\Gamma'^{\,\beta}_{\alpha} = \Lambda^{\mu}{}_{\alpha}\Gamma_{\mu}{}^{\nu}(\Lambda^{-1})^{\beta}{}_{\nu} - (\Lambda^{-1})^{\beta}{}_{\gamma}\,\mathrm{d}\Lambda^{\gamma}{}_{\alpha}. \end{aligned} \tag{9}$$

Due to the non-covariant transformation law of $\Gamma_{\alpha}{}^{\beta}$, see Eq. (9), if a connection vanishes in a given frame, it will not vanish in any other frame related to the first by a local Lorentz transformation.

2) $\Gamma_{\alpha}{}^{\beta}$ plays an important role in the teleparallel framework. This role represents the inertial effects which arise from the choice of the reference system.[25] The contributions of this inertial object in many cases lead to unphysical results for the total energy of the system. Therefore, the role of the teleparallel connection is to separate the inertial contribution from the truly gravitational one. Since the teleparallel curvature is zero, the connection is a pure gauge, that is

$$\Gamma_{\alpha}{}^{\beta} = (\Lambda^{-1})^{\beta}{}_{\gamma}\,\mathrm{d}\Lambda^{\gamma}{}_{\alpha}. \tag{10}$$

The Weitzenböck connection always has the form of Eq. (10). The translational momentum has the form[26]

$$\widetilde{H}_{\alpha} = \frac{1}{2\kappa}\widetilde{\Gamma}^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}, \quad \Gamma_{\alpha}{}^{\beta} \overset{\text{def.}}{=} \widetilde{\Gamma}_{\alpha}{}^{\beta} - K_{\alpha}{}^{\beta}, \tag{11}$$

where $\widetilde{\Gamma}_{\alpha}{}^{\beta}$ is the purely Riemannian connection and $K^{\mu\nu}$ is the contortion 1-form which is related to the torsion through

$$T^{\alpha} = K^{\alpha}{}_{\beta} \wedge \vartheta^{\beta}. \tag{12}$$

3. Total energy of spherically symmetric spacetimes with an arbitrary function $H(r)$ and a parameter ξ

In this section, we are going to show how the Weitzenböck connection $\Gamma_{\alpha}{}^{\beta}$ acts as an inertia and contributes to the physical quantities like energy, momentum etc., when it is trivial. In addition, when this connection is not trivial, we show how it separates the inertia from the physical quantities. We will show this by studying two different spherically symmetric

spacetimes which are related by local Lorentz transformation. This study is carried out for the spherically symmetric cases only.

3.1. Case I: total energy and spatial momentum of spherically symmetric spacetime with an arbitrary function $H(r)$

Using the spherical local coordinates (t, r, θ, ϕ), the Reissner Nordström solution is described by the coframe components

$$\overset{AF}{\vartheta}{}^{\alpha} = \left(\Lambda_1{}^{\alpha}{}_{\gamma}\right)\left(\Lambda_2{}^{\gamma}{}_{\delta}\right)\vartheta_R{}^{\delta}, \tag{13}$$

where the coframe $\vartheta_R{}^{\delta}$ has the form

$$\vartheta_R{}^{\hat{0}} = \frac{1}{\alpha}\mathrm{d}t, \quad \vartheta_R{}^{\hat{1}} = \alpha\mathrm{d}r,$$
$$\vartheta_R{}^{\hat{2}} = r\mathrm{d}\theta, \quad \vartheta_R{}^{\hat{3}} = r\sin\theta\mathrm{d}\phi,$$
$$\alpha \overset{\text{def.}}{=} \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-\frac{1}{2}}, \tag{14}$$

where m and q are the mass and charge parameters, respectively. $\left(\Lambda_1{}^{\alpha}{}_{\gamma}\right)$ and $\left(\Lambda_2{}^{\gamma}{}_{\delta}\right)$ are the local Lorentz transformations that are defined respectively as

$$\left(\Lambda_1{}^{\alpha}{}_{\gamma}\right) \overset{\text{def.}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix}, \tag{15}$$

$$\left(\Lambda_2{}^{\gamma}{}_{\delta}\right) \overset{\text{def.}}{=} \begin{pmatrix} \beta & H\sin\theta\cos\phi & H\sin\theta\sin\phi & H\cos\theta \\ -H\sin\theta\cos\phi & -1+(1-\beta)\sin^2\theta\cos^2\phi & (1-\beta)\sin^2\theta\sin\phi\cos\phi & (1-\beta)\sin\theta\cos\theta\cos\phi \\ -H\sin\theta\sin\phi & (1-\beta)\sin^2\theta\sin\phi\cos\phi & -1+(1-\beta)\sin^2\theta\sin^2\phi & (1-\beta)\sin\theta\cos\theta\sin\phi \\ -H\cos\theta & (1-\beta)\sin\theta\cos\theta\cos\phi & (1-\beta)\sin\theta\cos\theta\sin\phi & -1+(1-\beta)\cos^2\theta \end{pmatrix}, \tag{16}$$

where $\beta = \sqrt{H^2+1}$, $H \equiv H(r)$ is an arbitrary function of radial coordinate r which preserves spherical symmetry. Shirafuji *et al.*[46] studied the general spherically symmetric tetrad derived in Ref. [47] and divided it into two sets:

Set I In this set, the axial trace of the torsion vanishes identically. The spacetime in this case contains an arbitrary function which preserves the spherical symmetry and generates all the previous solution.[46]

Set II In this set, the axial trace of the torsion did not vanish identically. The authors of Ref. [46] have derived the condition that makes the axial trace of torsion vanish and have obtained a spacetime with an arbitrary parameter.

The coframe given by Eq. (13) with the arbitrary function is similar to Set I. All the previous solutions derived in Refs. [28] and [48] can be derived when $H(r)$ takes the form

$$H(r) = \left(\frac{r^2B'^2 - 2rB' + \frac{2m}{r} - \frac{q^2}{r^2}}{1 - \frac{2m}{r} + \frac{q^2}{r^2}}\right)^{1/2}, \tag{17}$$

with $B(r)$ representing an arbitrary function which reproduce all previous solutions derived in Refs. [26], [49] and $B' = \mathrm{d}B(R)/\mathrm{d}R$. In this paper, we will replace the constants m and q appearing in Eq. (16) by another constants c_1 and c_2. The motivation for this change is due to the fact that the parameter m appeared in the local Lorentz transformation given by (4.14) presented in Ref. [26] (which represents an inertia) contributes to the physical quantities. It is our aim to show in an explicate way how this inertia contributes to the physical quantity. This is not clear in general, since when Lucas *et al.*[26] calculated the energy of a freely falling tetrad using trivial Weitzenböck connection, they obtained a trivial value for the energy. The reason of this triviality of energy is related to the fact that the gravitational mass appearing in the coframe (4.1) cancels the constant m appearing in the local Lorentz transformation (4.14) of Ref. [26].

If we take tetrad (13), as well as the non-vanishing components of Riemannian connection of coframe (13), i.e., $\widetilde{\Gamma}^{\beta\gamma}$ (The Riemannian connection components of coframe (13) are very lengthy), and substitute

it into Eq. (11), we finally obtain

$$\widetilde{H}_{\hat{0}} = \frac{r\sin\theta}{8\pi}\Big\{[1-\sin^2\theta\cos^2\phi - \cos\theta\sin\phi\,(1+\sin\theta\cos\phi)] \times(\beta-1)-\frac{2\beta}{\alpha}\Big\}(\mathrm{d}\theta\wedge\mathrm{d}\phi)+\cdots. \quad (18)$$

Computation of the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S=\{r=R,\theta,\phi\}$ gives the form

$$\widetilde{E}=\int_{\partial S}\widetilde{H}_{\hat{0}}=\frac{R}{3}\left[2+\beta\left(1-\frac{3}{\alpha}\right)\right]. \quad (19)$$

Using Eq. (18), we discuss the following cases:

1) When $H(R)=0$, then Eq. (16) gives $\beta=1$ and from Eq. (18) we obtain the energy in the form

$$\widetilde{E}\cong m-\frac{q^2}{2R}.$$

In this case the local Lorentz transformation given by Eq. (16) will be identical to the Minkowski metric, $o_{\alpha\beta}=\mathrm{diag}(+1,-1,-1,-1)$. We call the tetrad formulating from this case the proper tetrad.

2) When

$$H(R)=\left(\frac{R^2B'^{\,2}-2RB'+\dfrac{2c_1}{R}-\dfrac{{c_2}^2}{R^2}}{1-\dfrac{2c_1}{R}+\dfrac{{c_2}^2}{R^2}}\right)^{1/2}, \quad (20)$$

we obtain

$$\widetilde{E}=\frac{R}{3}\left[2+\beta_1\left(1-\frac{3}{\alpha}\right)\right], \quad (21)$$

where

$$\beta_1=\frac{(1-RB')}{\sqrt{1-\dfrac{2c_1}{R}+\dfrac{{c_2}^2}{R^2}}},$$

and $B=B(R)$ is an arbitrary function. This case can be divided into the following subcases:

① When $B(R)=0$, or $B(R)=$ constant, and $c_1=c_2=0$, then from Eq. (21), $\beta_1=1$ and we obtain

$$\widetilde{E}=m-\frac{q^2}{2R},$$

which is the energy of Reissner Nordström.[28,48] We call this case the proper tetrad for Reissner Nordström since the local Lorentz transformation of Eq. (16) will have the Minkowski metric.

② When $B(R)=0$ or $B(R)=$ constant, and $c_1\neq c_2\neq 0$, then we obtain

$$\widetilde{E}=\frac{R}{3}\left[2+\beta_2\left(1-\frac{3}{\alpha}\right)\right], \quad (22)$$

where

$$\beta_2=\frac{1}{\sqrt{1-\dfrac{2c_1}{r}+\dfrac{{c_2}^2}{r^2}}}.$$

This choice makes the total energy different from Reissner Nordström energy.[28,48] Equation (22) shows clearly how the constants c_1 and c_2 share the physical quantity.

③ When $B(R)=0$ or $B(R)=$ constant, $c_1=m$ and $c_2=q$, then from Eq. (21) $\beta_1=\alpha$, we obtain

$$\widetilde{E}=\frac{1}{3}\left(m-\frac{q^2}{2R}\right),$$

which is also not the form of Reissner Nordström energy.[28,48]

④ When

$$B(R)=\int\frac{1-\sqrt{1-\frac{2c_1}{R}+\frac{{c_2}^2}{R^2}}}{R}\mathrm{d}R,$$

then $\beta_1=1$ and we obtain

$$E=m-\frac{q^2}{2R}.$$

The tetrad, resulting from the choice of this case, is called the proper tetrad because $(\Lambda_2{}^{\mu}{}_{\nu})$ will have the form of Minkowski metric.

Equation (19) informs us that the total energy in general depends on the arbitrary function $H(r)$ which keeps spherical symmetry. By giving a specific value of $H(r)$ as we discuss above, we obtain different values of the energy. To overcome this problem, we are going to use the regularization framework based on the covariance property, i.e., we will take into account the Weitzenböck connection $\Gamma^{\alpha}{}_{\beta}$. The non-vanishing components of $\Gamma^{\alpha}{}_{\beta}$ have the form

$$\Gamma_{\hat{1}}{}^{\hat{0}} = \sqrt{\beta-1}\Big\{\frac{1}{\beta\sqrt{\beta-1}}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)(\sqrt{\beta-1})[\cos^2\phi\sin^2\theta-\sin\phi\cos\theta(1-\sin\theta\cos\phi)]\,\mathrm{d}r + [\sin\phi\cos\phi\cos2\theta+\sin\theta(\sin\phi+2\cos\theta\cos^2\phi)]\,\mathrm{d}\theta - [2\sin\phi\cos\phi\sin^2\theta+\cos\theta(\cos\phi-2\sin\theta\cos^2\phi+\sin\theta)]\,\mathrm{d}\phi\Big\},$$

$$
\begin{aligned}
\Gamma_{\hat{2}}{}^{\hat{0}} &= \sqrt{\beta-1}\Big\{\frac{1}{\beta\sqrt{\beta-1}}\Big(\frac{\mathrm{d}}{\mathrm{d}r}\Big)(\sqrt{\beta-1})(\sin\phi\cos\phi\sin^2\theta+\sin\theta\cos\theta\sin^2\phi+\cos\theta\cos\phi)\,\mathrm{d}r\\
&\quad+[\cos\theta\sin\phi(2\sin\theta\cos\phi-1)-\sin^2\theta+2\cos^2\phi\sin^2\theta]\,\mathrm{d}\theta+[\sin\theta\cos\phi(2\cos\theta\sin\phi-1)\\
&\quad+\cos 2\theta-\cos 2\theta\cos^2\phi]\,\mathrm{d}\phi\Big\},\\
\Gamma_{\hat{3}}{}^{\hat{0}} &= \sqrt{\beta-1}\Big[\frac{1}{\beta\sqrt{\beta-1}}\Big(\frac{\mathrm{d}}{\mathrm{d}r}\Big)(\sqrt{\beta-1})\sin\theta(\cos\phi\cos\theta-\sin\phi\sin\theta)\,\mathrm{d}r+(\cos\phi\cos 2\theta-\sin 2\theta\sin\phi)\,\mathrm{d}\theta\\
&\quad-(\cos\phi\sin^2\theta+\cos\theta\sin\theta\sin\phi)\,\mathrm{d}\phi\Big].\\
\Gamma_{\hat{1}}{}^{\hat{2}} &= (\beta-1)\{[\sin\theta\cos\phi(1+\cos^2\theta)+\sin\phi\cos^3\theta]\,\mathrm{d}\theta-[\cos^2\phi+2\cos^2\theta-3\cos^2\phi\cos^2\theta+2\cos^2\phi\cos^4\theta\\
&\quad+\sin\phi\cos\theta\sin^2\theta-\cos^4\theta-\sin\theta\cos\phi\cos\theta(\cos\theta-2\sin\phi\sin^2\theta)]\,\mathrm{d}\phi\},\\
\Gamma_{\hat{1}}{}^{\hat{3}} &= (\beta-1)\{[\sin^2\theta\cos\phi-\sin\theta\sin^2\phi+\cos^2\theta(\sin\phi\cos\phi\cos\theta-\sin\theta\sin^2\phi)]\,\mathrm{d}\theta-[4\cos\phi\cos^2\theta\\
&\quad+\sin\theta\cos^2\theta-3\cos^3\phi\cos^2\theta-2\sin^2\phi\cos\phi\cos^4\theta+\sin^3\theta\sin\phi\cos 2\phi\cos\theta-\cos\phi(1+\sin^2\phi)]\,\mathrm{d}\phi\},\\
\Gamma_{\hat{2}}{}^{\hat{3}} &= (\beta-1)\{[\sin^2\theta\sin\phi-\cos\phi(\cos\phi\cos^3\theta-\sin\theta\sin\phi\cos^2\theta-\sin\theta\sin\phi)]\,\mathrm{d}\theta\\
&\quad+\sin^2\phi(\cos^2\phi\sin\phi\cos 2\theta+\sin\theta\cos\theta\cos\phi\cos 2\phi+\cos\theta+\sin\phi)\,\mathrm{d}\phi\}. \qquad (23)
\end{aligned}
$$

Using coframe (13), $\widetilde{\Gamma}_{\hat{\alpha}}^{\hat{\beta}}$ as well as $\Gamma_{\hat{\alpha}}{}^{\hat{\beta}}$ given by Eq. (22) we finally obtain the superpotential

$$H_{\hat{0}}=\frac{r\beta\sin\theta(1-1/\alpha)}{4\pi}(\mathrm{d}\theta\wedge\mathrm{d}\phi)+\cdots. \qquad (24)$$

If we compute the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S=\{r=R,\theta,\phi\}$, with $R\to\infty$, we obtain

$$E^*\cong m-\frac{q^2-2m(\cdots)}{2R}+O(1/R^2). \qquad (25)$$

Let us now turn our attention to calculate the spatial momentum associated with coframe (13). The non-vanishing components needed to calculate these physical quantities $\widetilde{H}_{\hat{\alpha}}=H_{\hat{\alpha}}$ $(\hat{\alpha}=1,2,3)$ have the form

$$
\begin{aligned}
\widetilde{H}_{\hat{1}}=H_{\hat{1}} &\cong \frac{rH(r)(1-\frac{1}{\alpha})\sin\theta[\sin\phi\cos\phi(1-\sin\theta\cos\theta)-\cos^2\phi\sin^2\phi]}{4\pi}(\mathrm{d}\theta\wedge\mathrm{d}\phi)+\cdots,\\
\widetilde{H}_{\hat{2}}=H_{\hat{2}} &\cong \frac{rH(r)(1-\frac{1}{\alpha})\sin\theta[\cos\theta\cos\phi(\sin\theta\cos\phi-1)-\cos\phi\sin\phi\sin^2\theta-\cos\theta\sin\theta]}{4\pi}(\mathrm{d}\theta\wedge\mathrm{d}\phi)+\cdots,\\
\widetilde{H}_{\hat{3}}=H_{\hat{3}} &\cong \frac{rH(r)(1-\frac{1}{\alpha})\sin^2\theta(\cos\theta\cos\phi-\sin\theta\sin\phi)}{4\pi}(\mathrm{d}\theta\wedge\mathrm{d}\phi)+\cdots. \qquad (26)
\end{aligned}
$$

Using Eq. (25) in Eq. (11), we finally obtain the spatial momentum in the form

$$P_1=\int_{\partial S}\widetilde{H}_{\hat{1}}=RH(R)(1-\frac{1}{\alpha_1})\cong m\ H(R)=O\Big(\frac{1}{R}\Big), \qquad (27)$$

if $H(R)\sim O\Big(\frac{1}{R}\Big)$, $P_2=P_3=0$, which is also in agreement with the previous results.[48]

3.2. Case II: total energy and spatial momentum of spherically symmetric spacetime with an arbitrary parameter ξ

Using the spherical local coordinates (t,r,θ,ϕ), the Reissner Nordström solution is described by the coframe components

$$\overset{P}{\vartheta}{}^{\alpha}=(\Lambda_1{}^{\alpha}{}_{\gamma})(\Lambda_3{}^{\gamma}{}_{\delta})\vartheta_R{}^{\delta}, \qquad (28)$$

where the coframe $\vartheta_R{}^{\delta}$ has the form (13), $(\Lambda_1{}^{\gamma}{}_{\delta})$ is given by Eq. (14) and $(\Lambda_3{}^{\gamma}{}_{\delta})$ is the local Lorentz transformation which has the form

$$(\Lambda_3{}^{\gamma}{}_{\delta}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\dfrac{4f^4-\xi^2}{4f^4+\xi^2} & \dfrac{4\xi f^2}{4f^4+\xi^2} \\ 0 & 0 & -\dfrac{4\xi f^2}{4f^4+\xi^2} & -\dfrac{4f^4-\xi^2}{4f^4+\xi^2} \end{pmatrix}, \quad (29)$$

where $f \equiv f(r) = \ln(r - m + \sqrt{r^2 - 2mr + q^2})$, and ξ is an arbitrary parameter. The coframe given by Eq. (28) with the arbitrary parameter ξ is similar to set II discussed in Refs. [46] and [48]. The non-vanishing components of the Riemannian connection of coframe (28), i.e., $\widetilde{\Gamma}_{\hat{\alpha}}^{\hat{\beta}}$ have the form

$$\begin{aligned}
\widetilde{\Gamma}_{\hat{1}}^{\hat{0}} &= \sin\theta\cos\phi\alpha\alpha'\,\mathrm{d}t, \quad \widetilde{\Gamma}_{\hat{2}}^{\hat{0}} = \sin\theta\sin\phi\alpha\alpha'\,\mathrm{d}t, \\
\widetilde{\Gamma}_{\hat{3}}^{\hat{0}} &= \cos\theta\alpha\alpha'\,\mathrm{d}t, \\
\widetilde{\Gamma}_{\hat{1}}^{\hat{2}} &= \frac{1}{4f^4+\xi^2}\{8ff'\xi\cos\theta\,\mathrm{d}r + 4f^2\xi\alpha\sin\theta\,\mathrm{d}\theta \\
&\quad + \sin^2\phi[4f^4(1-\alpha) + \xi^2(1+\alpha)]\,\mathrm{d}\phi\}, \\
\widetilde{\Gamma}_{\hat{1}}^{\hat{3}} &= -\frac{1}{4f^4+\xi^2}\Big(8ff'\xi\sin\theta\sin\phi\,\mathrm{d}r \\
&\quad + \{\cos\phi[4f^4(1-\alpha) + \xi^2(1+\alpha)] \\
&\quad - 4f^2\xi\alpha\cos\theta\sin\phi\}\,\mathrm{d}\theta \\
&\quad + \sin\theta\{\cos\theta\sin\phi[4f^4(1-\alpha) \\
&\quad + \xi^2(1+\alpha)] + 4f^2\xi\alpha\cos\phi\}\,\mathrm{d}\phi\Big), \\
\widetilde{\Gamma}_{\hat{2}}^{\hat{3}} &= \frac{1}{4f^4+\xi^2}\Big(8ff'\xi\sin\theta\cos\phi\,\mathrm{d}r \\
&\quad - \{\sin\phi[4f^4(1-\alpha) + \xi^2(1+\alpha)] \\
&\quad + 4f^2\xi\alpha\cos\theta\cos\phi\}\,\mathrm{d}\theta \\
&\quad - \sin\theta\{\cos\theta\cos\phi[4f^4(1-\alpha) \\
&\quad + \xi^2(1+\alpha)] - 4f^2\xi\alpha\sin\phi\}\,\mathrm{d}\phi\Big), \qquad (30)
\end{aligned}$$

where

$$f' = \frac{\mathrm{d}f(r)}{\mathrm{d}r}, \alpha' = \frac{\mathrm{d}\alpha(r)}{\mathrm{d}r}.$$

If we use Eq. (28), as well as the non-vanishing components of the Riemannian connection given by Eq. (30), and substitute it into Eq. (11), we finally obtain

$$\widetilde{H}_{\hat{0}} = \frac{2\sin\theta}{\kappa}\left[r\frac{4f^4(r)-\xi^2}{4f^4(r)+\xi^2} - \frac{1}{\alpha}\right](\mathrm{d}\theta\wedge\mathrm{d}\phi) + \cdots. \quad (31)$$

Computation of the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S = \{r = R, \theta, \phi\}$, has the form

$$\widetilde{E} = \int_{\partial S}\widetilde{H}_{\hat{0}} = R\frac{4f^4(R)-\xi^2}{4f^4(R)+\xi^2} - \frac{1}{\alpha}. \quad (32)$$

From Eq. (32), we can deduce that if the parameter $\xi = 0$, then we recover the correct energy of the spacetime of Reissner Nordström.[48] This calculation also shows clearly that the parameter ξ acts as an inertial quantity.

Now we are going to calculate the Weitzenböck connection to use its components in the regularized form. The non-vanishing components of this connection have the form

$$\begin{aligned}
\Gamma_{\hat{1}}{}^{\hat{2}} &= -\frac{1}{4f^4+\xi^2}(8ff'\xi\cos\theta\,\mathrm{d}r \\
&\quad + 4f^2\xi\sin\theta\,\mathrm{d}\theta + 2\sin^2\phi\xi^2\,\mathrm{d}\phi), \\
\Gamma_{\hat{1}}{}^{\hat{3}} &= \frac{1}{4f^4+\xi^2}[8ff'\xi\sin\theta\sin\phi\,\mathrm{d}r \\
&\quad - 2\xi(2f^2\cos\theta\sin\phi - \xi\cos\phi)\,\mathrm{d}\theta \\
&\quad - 2\xi\sin\theta(2f^2\cos\phi + \xi\cos\theta\sin\phi)\,\mathrm{d}\phi], \\
\Gamma_{\hat{2}}{}^{\hat{3}} &= -\frac{1}{4f^4+\xi^2}[8ff'\xi\sin\theta\cos\phi\,\mathrm{d}r \\
&\quad - 2\xi(2f^2\cos\theta\cos\phi + \xi\sin\phi)\,\mathrm{d}\theta \\
&\quad + 2\xi\sin\theta(2f^2\sin\phi - \xi\cos\theta\cos\phi)\,\mathrm{d}\phi]. \qquad (33)
\end{aligned}$$

Using the regularization framework based on the covariance property, i.e., taking into account the Weitzenböck connection $\Gamma^{\alpha}{}_{\beta}$ in Eq. (11) and calculating the necessary components, we finally obtain the superpotential

$$H_{\hat{0}} = \frac{\sin\theta}{4\pi}\left[r - \frac{1}{\alpha}\right](\mathrm{d}\theta\wedge\mathrm{d}\phi) + \cdots. \quad (34)$$

If we compute the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S = \{r = R, \theta, \phi\}$, with $R \leftarrow \infty$, we obtain

$$E = R\left(1 - \frac{1}{\alpha}\right) \cong m - \frac{q^2}{2R} + O(1/R^2). \quad (35)$$

The components needed to calculate the spatial momentum in this case are trivial and so the spatial momentum will be identically vanishing.

A comparison between the results of the arbitrary function which reproduces Reissner Nordström spacetime and its energy using different translational momentums are given in Table 1.

Table 1. comparison between the results of the arbitrary function $H(r)$ which keeps spherical symmetry and the other constants which create Reissner Nordström spacetime and its energy using different translational momenta.

Spacetime	Arbitrary function $H(r)$	The constants c_1 and c_2	Translational momentum	Energy and reference
Schwarzschild	$H(r)=0$, $q=0$ in Eq. (14)	$c_1=0$, $c_2=0$	Møller within the framework of tetrad spacetime	$E=m$, Ref. [46]
	$H(r)\neq 0$, $q=0$ in Eq. (32)	$c_1=m$, $c_2=0$	Møller Within the framework of tetrad spacetime	$E\neq m$, Ref. [46]
	$H(r)\neq 0$, $q=0$ in Eq. (14)	$c_1\neq 0$, $c_2=0$	Equation (11) with trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$	$E\neq m$, Eq. (19)
Reissner Nordström	$H(r)=0$	$c_1=0$, $c_2=0$	Møller within the framework of tetrad spacetime	$E=m-\frac{q^2}{2r}$, Ref. [48]
	$H(r)\neq 0$	$c_1=m$, $c_2=q$	Møller within the framework of tetrad spacetime	$E\neq m-\frac{q^2}{2r}$, Ref. [48]
	$H(r)\neq 0$	$c_1\neq 0$, $c_2\neq 0$	Equation (11) with non trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}\neq 0$	$E=m-\frac{q^2}{2r}$, Eq. (25)

4. Discussion and conclusion

In this study, we have used two spherically symmetric frames, one with arbitrary function $H(r)$, which preserves spherical symmetry and reproduces all previous solution,[28,29] and the other with an arbitrary parameter ξ. These frames are similar to the frames studied in Refs. [46] and [48] and are related to each other by local Lorentz transformation which is complicated and includes both the arbitrary function and the arbitrary parameter. The above two frames create the same spacetime which is the Reissner Nordström metric that is characterized by the two constants m, which is related to the mass of the system, and q is the electric charge.

We have calculated the energy associated with the above two frames using two different procedures. In the first one, we took a trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$ and show that the two forms of energies related to the above two frames depend on the two constants c_1, c_2, the arbitrary function $H(r)$ and the arbitrary parameter ξ. The formulas of energies given by Eqs. (19) and (32) show that the arbitrary function $H(r)$ and the parameter ξ play the role of inertia which make the total energy always unphysical. The choices that make the two formulas of energies always physical are $H(r)=0$ for the first frame and $\xi=0$ for the second one. The vanishing of the arbitrary function H and the parameter ξ make the local Lorentz transformations given by Eqs. (16) and (29) identical with the Minkowski metric. This means that local the Lorentz transformation which is responsible for the change of frame makes the total conserved energy-momentum $\widetilde{P}_{\alpha}$ corresponding to $\widetilde{H}_{\alpha}$ not transform covariantly.

In the second procedure, we have used the tensorial expression for the energy-momentum density of gravity, in which $\Gamma^{\alpha}{}_{\beta}\neq 0$, and calculated the total energy associated with the above two coframes. Using this procedure, we have obtained a physical quantity of the energy which is well known in the literature without the arbitrary function characterized the first frame and from the arbitrary parameter characterized the second one. This means that the Weitzenböck connection acts as a regularizing tool which separates the inertial energy-momentum density, leaving the tensorial, physical energy-momentum density of the system untouched. Finally, we calculate the spatial momentum of the two frames using the above two procedures and show that they gave the desired results.

Acknowledgements

The author would like to thank Professor M. I. Wanas of Cairo University, and Dr. Adel Awad and

Miss M. Kamal of Ain Shams University, for their stimulating discussions.

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