Partial evolution based local adiabatic quantum search*

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Recently, Zhang and Lu provided a quantum search algorithm based on partial adiabatic evolution, which beats the time bound of local adiabatic search when the number of marked items in the unsorted database is larger than one. Later, they found that the above two adiabatic search algorithms had the same time complexity when there is only one marked item in the database. In the present paper, following the idea of Roland and Cerf [Roland J and Cerf N J 2002 *Phys. Rev.* A **65** 042308], if within the small symmetric evolution interval defined by Zhang et al., a local adiabatic evolution is performed instead of the original "global" one, this "new" algorithm exhibits slightly better performance, although they are progressively equivalent with M increasing. In addition, the proof of the optimality for this partial evolution based local adiabatic search when $M = 1$ is also presented. Two other special cases of the adiabatic algorithm obtained by appropriately tuning the evolution interval of partial adiabatic evolution based quantum search, which are found to have the same phenomenon above, are also discussed.

Keywords: partial adiabatic evolution, local adiabatic evolution, quantum search

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In around 2000, a quantum algorithm based on adiabatic evolution was proposed by Farhi *et al.*[1] trying to solve NP-complete problems. Their work is thought to be a novel approach to quantum computing. It is believed that adiabatic quantum computation is advantageous over standard quantum computation in that it is more robust against environmental noise,[2] decoherence and certain control errors,[3] though it is proven to be polynomially equivalent to gate model quantum computation.[4,5] In Ref. [6], a quantum search scheme by adiabatic evolution has been implemented with atoms in a cavity driven by laser.

Several quantum algorithms were implemented in this new paradigm, such as the Deutsch–Jozsa problem[7,8] and Grover's database search.[9] However, the latter was shown to have no advantage compared with the best classical search algorithm in Ref. [10]. Roland and Cerf[11] introduced a local adiabatic quantum search version recovering the quadratic speedup of Grover's algorithm and proved its optimality. Recently, according to the work of Ref. [12], Zhang and Lu[13] provided a partial adiabatic evolution based quantum search algorithm with which the local adiabatic search is further improved when there exists more than one item satisfying the search criterion. Nearly at the same time, they found that these two adiabatic algorithms had the same time complexity in Ref. [14].

In this paper, inspired by the early work of Roland and Cerf,[11] we try to find a kind of adiabatic search with which the results in Refs. [13] and [14] may be further improved. As a result, the time consumption for this "new" adiabatic algorithm is strictly lower, though progressively they demonstrate the same performance. Furthermore, we give the proof of optimality for this adiabatic search algorithm in the case where there is only one marked item in the database from the point of view of the time complexity.

Now we recall the basic ingredients of adiabatic quantum computation for the convenience of later presentation. The adiabatic theorem[15] indicates that if we consider a quantum system whose Hamiltonian "slowly" changes from initial Hamiltonian $H_{\rm i}$ to final Hamiltonian $H_{\rm f}$, then if we start with the system in the ground state which corresponds to the lowest

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eigenvalue of $H_{\rm i}$, let the state at any instantaneous time evolve according to the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi(t)\rangle = \widetilde{H}(t)|\varphi(t)\rangle, \tag{1}$$

where $|\varphi(t)\rangle (0 \leq t \leq T)$ is the state of the quantum system, and

$$\widetilde{H}(t) = H(s) = (1-s(t))H_{\rm i} + s(t)H_{\rm f}, \tag{2}$$

with $s = s(t)$ being a monotonic function with $s(0) = 0$ and $s(T) = 1$. Then at the end of the evolution the system will be "predominantly" in the ground state of $H_{\rm f}$, to which we encode the answer of the problem of interest. The total running time T of an adiabatic evolution can be estimated to be[15]

$$T \geq \frac{D_{\max}}{\varepsilon g_{\min}^2}, \tag{3}$$

where $0 < \varepsilon \ll 1$, and

$$\begin{aligned} D_{\max} &= \max_{s\in[0,1]}\left|\langle E(1,s)|\frac{\mathrm{d}H(s)}{\mathrm{d}s}|E(0,s)\rangle\right|, \\ g_{\min} &= \min_{s\in[0,1]}[E(1,s)-E(0,s)], \end{aligned} \tag{4}$$

with $E(k,s)$ and $|E(k,s)\rangle$ being the k-level eigenvalue and eigenstate of $H(s)(k = 0, 1)$. In the rest of the present context, we will use the following notations:

$$H_{\rm i} = I - |\Psi\rangle\langle\Psi|, \tag{5}$$

where

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle, \tag{6}$$

the equal superposition of all the elements that exist in the database;

$$H_{\rm f} = I - |\mu\rangle\langle\mu|, \tag{7}$$

where

$$|\mu\rangle = \frac{1}{\sqrt{M}}\sum_{j\in S}|j\rangle, \tag{8}$$

where S is the marked set, $|S| = M$.

In Ref. [12], the author considered a particular class of quantum adiabatic evolution where the final Hamiltonian was a one-dimensional projector Hamiltonian on the corresponding ground state. A faster evolution was obtained by performing a partial adiabatic evolution within a narrow interval around the transition point where the energy gap was minimum. Following this idea, Zhang and Lu proposed a partial adiabatic evolution based quantum search algorithm.[13] It was found that the algorithm improved the time complexity of local adiabatic search, which was from $O(\sqrt{N/M})$, to $O(\sqrt{N}/M)$ when $M > 1$. They further found that the two adiabatic search schemes had the same time complexity when $M = 1$ in Ref. [14]. In fact, from the execution procedure of partial adiabatic evolution, we could find it actually evolves "globally" within the small interval specified. Thus, naturally, following the idea of the local adiabatic in Ref. [11], if this global evolution is replaced by a local one within the same interval, what result will we have? We now answer this question in detail below. For convenience, firstly we describe how this new algorithm works and name it micro-local adiabatic search.

This micro-local adiabatic search consists of the following four steps:

1) the initial state is prepared to be the stable ground state $|\Psi\rangle$ of $H_{\rm i}$;

2) at $t = 0$, the quantum Hamiltonian is suddenly changed to $H(s^-)$, without disturbing the initial state;[15]

3) the Hamiltonian evolves from $H(s^-)$ to $H(s^+)$ locally in infinitesimal time intervals $\mathrm{d}t$ by dividing T';

4) measure the state of the system.

Repeat these four steps until a marked state may be found.

Now, we calculate the one-round time consumption of this adiabatic evolution search. Following the same steps in Ref. [11], the time duration for the current algorithm can be estimated to be

$$T' = \frac{1}{\varepsilon}\int_{s^-}^{s^+}\frac{1}{g^2(s)}\,\mathrm{d}s, \tag{9}$$

where

$$s^- = \frac{1}{2} - \frac{1}{2\sqrt{N}}, \quad s^+ = \frac{1}{2} + \frac{1}{2\sqrt{N}}, \tag{10}$$

according to Ref. [13], and

$$g(s) = \sqrt{1 - 4s(1-s)\frac{N-M}{N}}. \tag{11}$$

After some explicit integral calculations, we can have

$$T' = \frac{N}{\varepsilon\sqrt{M(N-M)}}\arctan\sqrt{\frac{N-M}{MN}}.$$

And the one-round success probability

$$P = |\langle\Psi|E(0,s^-)\rangle|^2 \times |\langle\mu|E(0,s^+)\rangle|^2 \tag{12}$$

can be lower bounded by a constant.[13] So the overall running time now is

$$T_{\rm ml} = \frac{T'}{P} = \frac{N}{\varepsilon P\sqrt{M(N-M)}} \arctan\sqrt{\frac{N-M}{MN}}. \quad (13)$$

We point out here that the reason why we keep the "arctan" part in $T_{\rm ml}$ is that it is very small with M increasing, although function $\arctan x$ is a bounded quantity. However, it is well known that if the variable x tends to positive infinity, we can then replace $\arctan x$ with $\frac{\pi}{2}$, which is the same method used by the authors of Ref. [11].

Secondly, we compare $T_{\rm ml}$ with the result obtained by Zhang and Lu. For simplicity, we denote the time consumption in Ref. [13] as $T_{\rm p}$, and $T_{\rm p} = \frac{\sqrt{N}}{\varepsilon PM}$. The following result is easy to obtain:

$$\frac{T_{\rm ml}}{T_{\rm p}} = \frac{\arctan\sqrt{\frac{N-M}{NM}}}{\sqrt{\frac{N-M}{NM}}}. \quad (14)$$

Under the assumption of $N \gg M \geq 1$, the above equation can be rewritten as

$$\frac{T_{\rm ml}}{T_{\rm p}} = \frac{\arctan\sqrt{\frac{1}{M}}}{\sqrt{\frac{1}{M}}}. \quad (15)$$

In fact, $\arctan x < x$, for $\forall x \in (0,1]$, we know that $T_{\rm ml} < T_{\rm p}$ strictly holds, although with M increasing, the difference between them progressively disappears. In particular, when $M = 1$, $T_{\rm ml} = \frac{\pi}{4} \times T_{\rm p}$, which is also the maximum "speedup" provided by the present algorithm over the algorithm given in Ref. [14]. In fact, only when M tends to infinity, will our algorithm have the same performance as the one presented in Ref. [13].

We now turn to another viewpoint of the micro-local adiabatic quantum search above: its time complexity. This "new" adiabatic algorithm in fact is optimal for $M = 1$, and can be proved as follows. The idea is in close analogy to that in Ref. [11], with some modifications. Let $|\varphi_\mu, t\rangle$ be the state of the quantum register during the computation when the solution state is $|\mu\rangle$. After one-round running time $T' = t^+ - t^-$, where t^+ and t^- are two time points corresponding to the start point s^- and end point s^+ of the narrow evolution interval, respectively, the states corresponding to different states $|\mu\rangle$ and $|\mu'\rangle$ should be sufficiently different,

$$1 - |\langle\varphi_\mu, t^+|\varphi_{\mu'}, t^+\rangle|^2 \geq \epsilon, \ \forall \mu \neq \mu'. \quad (16)$$

It should be pointed out that the reason why we could safely make this assumption is that the one-round execution of the algorithm almost determines the whole time complexity, a fact which is due to the nature of the current algorithm. In fact, the original partial evolution based adiabatic search also enjoys this property. $H(s)$ can be rewritten as

$$H(s) = H_1(s) + H_2(s), \quad (17)$$

where

$$H_1(s) = I - (1-s)(|\Psi\rangle\langle\Psi|), \quad (18)$$

and

$$H_2(s) = -s|\mu\rangle\langle\mu|. \quad (19)$$

$|\varphi_\mu, t\rangle, |\varphi_{\mu'}, t\rangle$ are the solutions of the Schrödinger equations

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi_\mu, t\rangle = (H_1 + H_2)|\varphi_\mu, t\rangle, \quad (20)$$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi_{\mu'}, t\rangle = (H_1 + H_2')|\varphi_{\mu'}, t\rangle, \quad (21)$$

with the initial condition

$$|\varphi_\mu, t^-\rangle = |\varphi_{\mu'}, t^-\rangle = |\Psi\rangle, \quad (22)$$

also determined by the nature of algorithm in step 2). From Eqs. (20) and (21), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[1 - |\langle\varphi_\mu, t|\varphi_{\mu'}, t\rangle|^2] \\ &= 2\,\mathrm{Im}[\langle\varphi_\mu, t|H_2 - H_2'|\varphi_{\mu'}, t\rangle\langle\varphi_{\mu'}, t|\varphi_\mu, t\rangle] \\ &\leq 2[\,|\langle\varphi_\mu, t|H_2|\varphi_{\mu'}, t\rangle| + |\langle\varphi_\mu, t|H_2'|\varphi_{\mu'}, t\rangle|\,]. \end{aligned} \quad (23)$$

Summing over μ and μ', together with Cauchy–Schwartz inequality, and with $|\varphi_\mu, t\rangle$, $|\varphi_{\mu'}, t\rangle$ being normalized, we will have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\mu,\mu'}[1 - |\langle\varphi_\mu, t|\varphi_{\mu'}, t\rangle|^2] \\ &\leq 4N\sum_\mu ||H_2|\varphi_\mu, t\rangle||. \end{aligned} \quad (24)$$

Furthermore, from Cauchy's inequality

$$\begin{aligned} &\sum_\mu ||H_2|\varphi_\mu, t\rangle||^2 = s^2 \\ &\Rightarrow \sum_\mu ||H_2|\varphi_\mu, t\rangle|| \leq \sqrt{N}s, \end{aligned} \quad (25)$$

we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\mu,\mu'}[1 - |\langle\varphi_\mu, t|\varphi_{\mu'}, t\rangle|^2] \leq 4sN^{\frac{3}{2}} \quad (26)$$

holds. We integrate this inequality using initial condition (22), the following result is thus obvious:

$$\sum_{\mu,\mu'}[1-|\langle\varphi_\mu,t^+|\varphi_{\mu'},t^+\rangle|^2]\leq 4N^{\frac{3}{2}}\int_{t^-}^{t^+}s(t)\mathrm{d}t. \quad (27)$$

By inequality (16) and $0\leq s(t)\leq 1$, we find

$$t^+-t^-\geq\frac{\epsilon}{4}\sqrt{N}\quad(\epsilon>0) \quad (28)$$

under the assumption of $N\gg 1$. Thus, using some simple algebra, the way the same as that in Ref. [13], the one-round success probability can be bounded as

$$P>\frac{1}{10}. \quad (29)$$

We thus arrive at the total time complexity, which now is

$$T=\frac{t^+-t^-}{P}=O(\sqrt{N}). \quad (30)$$

Therefore, the proof is completed.

We now discuss the global adiabatic and local adiabatic algorithm which is considered as two special cases of partial evolution based quantum search. The evidence for this consideration can be found in Refs. [12] and [14]. For the former, the evolution interval should obviously be specified as $s^-=0$ and $s^+=1$. In addition, our micro-local search and partial adiabatic evolution now degenerate to the original local adiabatic in Ref. [11] and global adiabatic in Ref. [10], respectively. From Ref. [11], we know that

$$\frac{T_{\mathrm{ml}}}{T_{\mathrm{p}}}=\frac{\pi}{2}\sqrt{\frac{M}{N}} \quad (31)$$

holds in this situation, which is also the maximum speedup we can gain, as we will see. For the later adiabatic algorithm, s^- and s^+ are specified as $s^-=\frac{1}{2}-\frac{1}{2}\sqrt{M/N}, s^+=\frac{1}{2}+\frac{1}{2}\sqrt{M/N}$ according to Ref. [12]. After some simple calculations as we did before, we have

$$\frac{T_{\mathrm{ml}}}{T_{\mathrm{p}}}=\frac{\arctan(\sqrt{1-\frac{M}{N}})}{\sqrt{1-\frac{M}{N}}}\approx\frac{\pi}{4} \quad (32)$$

under the assumption of $N\gg M\geq 1$. Combining Eq. (15) with Eqs. (31) and (32), we can obtain the following conclusion coarsely: the larger the evolution interval, the more obvious speedup can be obtained.

In conclusion, based on the work in Ref. [13], combining with the early result of Ref. [11], we provide a kind of partial evolution based local adiabatic search: the micro-local adiabatic evolution algorithm. Even though this new algorithm progressively exhibits the same performance, we show that its time consumption is actually strictly less than the results obtained by Refs. [13] and [14]. Particularly, when $M=1$, it has an obvious advantage over the one given in Ref. [14]. In fact, the ingredient we consider for applying a local adiabatic instead of global evolution in the narrow interval is possibly useful for physical implementation interests, besides possible running time improvement considerations: as long as the Hamiltonian dynamics is present, a local adiabatic quantum search demonstrates some robustness to the decoherence in the instantaneous eigenbasis of the search Hamiltonian, which is discussed in detail by Åberg *et al.*[16] In addition, we give the proof of optimality for this micro-local evolution search in the case of $M=1$ from the point of view of the time complexity aspect. Finally, we generalize the relation, micro-local adiabatic versus partial adiabatic evolution, to two other special cases which have been mentioned explicitly or implicitly in relevant articles,[12,14] and drawn a corresponding tentative conclusion: micro-local adiabatic evolution speeds up partial adiabatic most severely in the whole evolution interval, i.e., $[0,1]$.

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