# Four-mode coherent-entangled state and its application*

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A new kind of four-mode continuous variable coherent-entangled state is proposed in the Fock space by using the technique of integration within an ordered product, which exhibits both the properties of a coherent state and an entangled state, and spans a complete and orthonormal representation. The conjugate state of the four-mode continuous variable coherent-entangled state is derived by using the Fourier transformation. Moreover, a simple experimental protocol of generating a four-mode continuous variable coherent-entangled state is proposed by using beam splitters. As applications of this four-mode continuous variable coherent-entangled state, a four-mode entangled state and a four-mode squeezing-Fresnel operator are constructed.

**Keywords:** coherent-entangled state, beam splitter, squeezing-Fresnel operator, technique of integration within an ordered product

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## 1. Introduction

Various coherent states have been widely used in quantum mechanics and quantum optics since Glauber,[1−4] Klauder and Skagerstam[5] introduced the coherent state of the harmonic oscillator. The idea of quantum entangled states originated from Einstein, Podolsky, and Reson[6], and has stimulated physicists to construct various entangled state representations.[7−13] The two-mode coherent-entangled states $|\alpha, x\rangle$ and $|p, \beta\rangle$ are proposed in Refs. [14]–[16], which are common eigenvectors of $(\hat{X}_1 + \hat{X}_2)/2$, $(a_1 - a_2)$ and $(\hat{P}_1 - \hat{P}_2)/2$, $(a_1 + a_2)$, respectively, with $\hat{X}_j = (a_j + a_j^+)/\sqrt{2}$, $\hat{P}_j = (a_j - a_j^+)/\mathrm{i}\sqrt{2}$, $j = 1, 2$. Those states exhibit both the properties of a coherent state and an entangled state, and span the complete and orthonormal quantum mechanical representations in the Fock space. In Ref. [17], Fan and Wang presented a continuous variable three-mode coherent-entangled state, which is the common eigenstate of operators $\hat{X}_1+\hat{X}_2+\hat{X}_3$, $a_1-a_2$, and $a_2 - a_3$, and exhibits the properties of both the coherent and the entangled states. We have proposed a three-mode coherent-entangled state of continuous variables, $|x_2, x_3, \alpha\rangle$,[18] which is the common eigenvector of $\hat{X}_1 - \hat{X}_2$, $\hat{X}_1 - \hat{X}_3$, and $a_1 + a_2 + a_3$, and exhibits the properties of both the coherent and the entangled states, where $\hat{X}_i = (a_i + a_i^+)/\sqrt{2}$, $i = 1, 2, 3$.

In this work, we mainly construct a new kind of four-mode continuous variable coherent-entangled state (CV-CES) $|x_2, x_3, x_3, \alpha\rangle$ in the Fock space by using the technique of integration within an ordered product (IWOP),[7,19] and study the properties and the applications of this new four-mode coherent-entangled state.

In Section 2, we present the explicit form of the CV-CES in the four-mode Fock space and discuss its coherent and entangled properties. In Section 3, we propose a protocol for generating $|x_2, x_3, x_3, \alpha\rangle$ by using a single-mode vacuum state and three ideal single-mode squeezed states overlapping on beam splitters. In Section 4, we briefly discuss some applications of $|x_2, x_3, x_3, \alpha\rangle$ in quantum mechanics and quantum optics theories, such as generating a four-mode Einstein–

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Podolsky–Rosen (EPR) entangled state and deriving a new squeezing-Fresnel operator. Then we give a short summary in the end.

## 2. Four-mode CV-CES $|x_2, x_3, x_4, \alpha\rangle$

### 2.1. Common eigenstate of operators $\hat{X}_1 - \hat{X}_2, \hat{X}_1 - \hat{X}_3, \hat{X}_1 - \hat{X}_4$, and $a_1 + a_2 + a_3 + a_4$

After some effort, we acquire

$$\begin{aligned}
&|x_2, x_3, x_4, \alpha\rangle \\
&= \exp\Big\{ -\frac{3}{8}x_2^2 - \frac{3}{8}x_3^2 - \frac{3}{8}x_4^2 + \frac{1}{4}x_2x_3 \\
&+\frac{1}{4}x_2x_4 + \frac{1}{4}x_3x_4 \\
&-\frac{1}{8}|\alpha|^2 + \frac{\sqrt{2}}{4}(a_1^+ - 3a_2^+ + a_3^+ + a_4^+)x_2 \\
&+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ - 3a_3^+ + a_4^+)x_3 \\
&+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ + a_3^+ - 3a_4^+)x_4 \\
&+\frac{1}{4}(a_1^+ + a_2^+ + a_3^+ + a_4^+)\alpha - \frac{1}{8}[(a_1^+ - a_2^+)^2 \\
&+(a_1^+ - a_3^+)^2 + (a_1^+ - a_4^+)^2 + (a_2^+ - a_3^+)^2 \\
&+(a_2^+ - a_4^+)^2 + (a_3^+ - a_4^+)^2]\Big\}|0000\rangle. \qquad (1)
\end{aligned}$$

According to formulas

$$:\frac{\partial}{\partial a_i^+} f(a_i, a_i^\dagger) := [a_i, : f(a_i, a_i^+) :]$$

and $a_i|0000\rangle = 0$, $(i = 1, 2, 3, 4)$, from Eq. (1), we obtain

$$\begin{aligned}
&a_1|x_2, x_3, x_4, \alpha\rangle \\
&= \Big[\frac{\sqrt{2}}{4}(x_2 + x_3 + x_4) + \frac{\alpha}{4} \\
&-\frac{1}{4}(3a_1^+ - a_2^+ - a_3^+ - a_4^+)\Big]|x_2, x_3, x_4, \alpha\rangle, \\
&a_2|x_2, x_3, x_4, \alpha\rangle \\
&= \Big[\frac{\sqrt{2}}{4}(-3x_2 + x_3 + x_4) + \frac{\alpha}{4} \\
&-\frac{1}{4}(-a_1^+ + 3a_2^+ - a_3^+ - a_4^+)\Big]|x_2, x_3, x_4, \alpha\rangle, \\
&a_3|x_2, x_3, x_4, \alpha\rangle \\
&= \Big[\frac{\sqrt{2}}{4}(x_2 - 3x_3 + x_4) + \frac{\alpha}{4} \\
&-\frac{1}{4}(-a_1^+ - a_2^+ + 3a_3^+ - a_4^+)\Big]|x_2, x_3, x_4, \alpha\rangle, \\
&a_4|x_2, x_3, x_4, \alpha\rangle \\
&= \Big[\frac{\sqrt{2}}{4}(x_2 + x_3 - 3x_4) + \frac{\alpha}{4} \\
&-\frac{1}{4}(-a_1^+ - a_2^+ - a_3^+ + 3a_4^+)\Big]|x_2, x_3, x_4, \alpha\rangle. \qquad (2)
\end{aligned}$$

Substituting $\hat{X}_i = (a_i + a_i^+)/\sqrt{2}$, $(i = 1, 2, 3, 4)$ into Eq. (2), we have

$$\begin{aligned}
&(a_1 + a_2 + a_3 + a_4)|x_2, x_3, x_4, \alpha\rangle = \alpha|x_2, x_3, x_4, \alpha\rangle, \\
&(\hat{X}_1 - \hat{X}_2)|x_2, x_3, x_4, \alpha\rangle = x_2|x_2, x_3, x_4, \alpha\rangle, \\
&(\hat{X}_1 - \hat{X}_3)|x_2, x_3, x_4, \alpha\rangle = x_3|x_2, x_3, x_4, \alpha\rangle, \\
&(\hat{X}_1 - \hat{X}_4)|x_2, x_3, x_4, \alpha\rangle = x_4|x_2, x_3, x_4, \alpha\rangle. \qquad (3)
\end{aligned}$$

Thus, we have proved that $|x_2, x_3, x_4, \alpha\rangle$ is the common eigenstate of the commutative operators of $\hat{X}_1 - \hat{X}_2$, $\hat{X}_1 - \hat{X}_3$, $\hat{X}_1 - \hat{X}_4$, and $a_1 + a_2 + a_3 + a_4$.

### 2.2. Completeness and orthogonality of $|x_2, x_3, x_4, \alpha\rangle$

Using the normal product form of the four-mode vacuum projector

$$\begin{aligned}
&|0000\rangle\langle 0000| \\
&= :\exp[-(a_1^+a_1 + a_2^+a_2 + a_3^+a_3 + a_4^+a_4)]:, \qquad (4)
\end{aligned}$$

from Eq. (1), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4}{2\pi\sqrt{\pi}} \int \frac{\mathrm{d}^2\alpha}{4\pi}|x_2, x_3, x_4, \alpha\rangle\langle x_2, x_3, x_4, \alpha| \\
&= :\int_{-\infty}^{\infty} \frac{\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4}{2\pi\sqrt{\pi}} \int \frac{\mathrm{d}^2\alpha}{4\pi} \exp\Big\{ -\frac{1}{4} \\
&\times [x_2 - (\hat{X}_1 - \hat{X}_2)]^2 \\
&-\frac{1}{4}[x_3 - (\hat{X}_1 - \hat{X}_3)]^2 - \frac{1}{4}[x_4 - (\hat{X}_1 - \hat{X}_4)]^2 \\
&-\frac{1}{4}[(x_3 - x_2) - (\hat{X}_2 - \hat{X}_3)]^2 \\
&-\frac{1}{4}[(x_4 - x_2) - (\hat{X}_2 - \hat{X}_4)]^2 \\
&-\frac{1}{4}[(x_4 - x_3) - (\hat{X}_3 - \hat{X}_4)]^2 \\
&-\frac{1}{4}[\alpha - (a_1 + a_2 + a_3 + a_4)] \\
&\times [\alpha^* - (a_1^+ + a_2^+ + a_3^+ + a_4^+)]\Big\} := 1. \qquad (5)
\end{aligned}$$

Thus, $|x_2, x_3, x_4, \alpha\rangle$ satisfies the completeness relation obviously.

Using Eq. (3), we obtain

$$\begin{aligned}
&\langle x_2', x_3', x_4', \alpha'|\hat{X}_1 - \hat{X}_i|x_2, x_3, x_4, \alpha\rangle \\
&= x_i\langle x_2', x_3', x_4', \alpha'|x_2, x_3, x_4, \alpha\rangle \\
&= x_i'\langle x_2', x_3', x_4', \alpha'|x_2, x_3, x_4, \alpha\rangle, \quad i = 2, 3, 4, \qquad (6)
\end{aligned}$$

which leads to

$$\langle x_2', x_3', x_4', \alpha'|x_2, x_3, x_4, \alpha\rangle(x_i - x_i') = 0. \quad (7)$$

From Eqs. (1) and (3), we have

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}|x_2, x_3, x_4, \alpha\rangle \\
&= \left[-\frac{\alpha^*}{8} + \frac{1}{4}(a_1^+ + a_2^+ + a_3^+ + a_4^+)\right]|x_2, x_3, x_4, \alpha\rangle, \\
&\frac{\partial}{\partial\alpha'^*}\langle x_2', x_3', x_4', \alpha'| \\
&= \langle x_2', x_3', x_4', \alpha'|\left[-\frac{\alpha'}{8} + \frac{1}{4}(a_1 + a_2 + a_3 + a_4)\right], \quad (8)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\langle x_2', x_3', x_4', \alpha'|x_2, x_3, x_4, \alpha\rangle \\
&= \left(-\frac{\alpha^*}{8} + \frac{\alpha'^*}{4}\right)\langle x_2', x_3', x_4', \alpha'|x_2, x_3, x_4, \alpha\rangle, \quad (9) \\
&\frac{\partial}{\partial\alpha'^*}\langle x_2', x_3', x_4', \alpha'|x_2, x_3, x_4, \alpha\rangle \\
&= \left(-\frac{\alpha'}{8} + \frac{\alpha}{4}\right)\langle x_2', x_3', x_4', \alpha'|x_2, x_3, x_4, \alpha\rangle. \quad (10)
\end{aligned}$$

From Eqs. (7)–(10) we finally derive

$$\begin{aligned}
&\langle x_2', x_3', x_4', \alpha'|x_2, x_3, x_4, \alpha\rangle \\
&= 2\pi^{3/2}\exp\left(-\frac{|\alpha|^2}{8} - \frac{|\alpha'|^2}{8} + \frac{1}{4}\alpha'^*\alpha\right) \\
&\quad\times\delta(x_2 - x_2')\delta(x_3 - x_3')\delta(x_4 - x_4'). \quad (11)
\end{aligned}$$

Thus, $|x_2, x_3, x_4, \alpha\rangle$ is only partially orthogonal. It exhibits the coherent state's non-orthogonal property with respect to parameter $\alpha$ and the orthogonal property with respect to parameters $x_2, x_3, x_4$.

## 2.3. Entanglement of $|x_2, x_3, x_4, \alpha\rangle$

By making the following Fourier transform

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_2}{\sqrt{8\pi}}\exp\left(-\frac{1}{4}\,\mathrm{i}\,u\alpha_2\right)|x_2, x_3, x_4, \alpha = \alpha_1 + \mathrm{i}\,\alpha_2\rangle \\
&= \exp\left\{-\frac{3}{8}x_2^2 - \frac{3}{8}x_3^2 - \frac{3}{8}x_4^2\right. \\
&\quad+\frac{1}{4}x_2x_3 + \frac{1}{4}x_2x_4 + \frac{1}{4}x_3x_4 - \frac{1}{8}\alpha_1^2 \\
&\quad+\frac{\sqrt{2}}{4}(a_1^+ - 3a_2^+ + a_3^+ + a_4^+)x_2 \\
&\quad+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ - 3a_3^+ + a_4^+)x_3 \\
&\quad+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ + a_3^+ - 3a_4^+)x_4 \\
&\quad+\frac{1}{4}(a_1^+ + a_2^+ + a_3^+ + a_4^+)\alpha_1 \\
&\quad-\frac{1}{8}[(a_1^+ - a_2^+)^2 + (a_1^+ - a_3^+)^2 + (a_1^+ - a_4^+)^2 \\
&\quad\left.+(a_2^+ - a_3^+)^2 + (a_2^+ - a_4^+)^2 + (a_3^+ - a_4^+)^2]\right\} \\
&\quad\times\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_2}{\sqrt{8\pi}}\exp\left[-\frac{1}{8}\alpha_2^2\right. \\
&\quad\left.+\frac{\mathrm{i}}{4}(a_1^+ + a_2^+ + a_3^+ + a_4^+ - u)\alpha_2\right]|0000\rangle, \quad (12)
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_2}{\sqrt{8\pi}}\exp\left(-\frac{1}{4}\,\mathrm{i}\,u\alpha_2\right)|x_2, x_3, x_4, \alpha\rangle \\
&= \pi\exp\left[-\frac{1}{16}(\alpha_1 - u)^2\right]|q_1\rangle\otimes|q_2\rangle\otimes|q_3\rangle\otimes|q_4\rangle. \quad (13)
\end{aligned}$$

where

$$\begin{aligned}
q_1 &= \frac{1}{4}\left[(x_2 + x_3 + x_4) + \frac{\alpha_1 + u}{\sqrt{2}}\right], \\
q_2 &= \frac{1}{4}\left[(-3x_2 + x_3 + x_4) + \frac{\alpha_1 + u}{\sqrt{2}}\right], \\
q_3 &= \frac{1}{4}\left[(x_2 - 3x_3 + x_4) + \frac{\alpha_1 + u}{\sqrt{2}}\right], \\
q_4 &= \frac{1}{4}\left[(x_2 + x_3 - 3x_4) + \frac{\alpha_1 + u}{\sqrt{2}}\right], \quad (14) \\
|q_i\rangle &= \pi^{-1/4}\exp\left\{-\frac{q_i^2}{2} + \sqrt{2}q_ia_i^+ - \frac{a_i^{+2}}{2}\right\}|0\rangle, \\
&\quad i = 1, 2, 3, 4.
\end{aligned}$$

Furthermore, by taking the inverse Fourier transform of Eq. (13), we can obtain

$$\begin{aligned}
&|x_2, x_3, x_4, \alpha\rangle \\
&= \pi\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{\sqrt{8\pi}}\exp\left[-\frac{1}{16}(\alpha_1 - u)^2 + \frac{1}{4}\,\mathrm{i}\,u\alpha_2\right] \\
&\quad\times|q_1\rangle\otimes|q_2\rangle\otimes|q_3\rangle\otimes|q_4\rangle, \quad (15)
\end{aligned}$$

which is the Schmidt decomposition of $|x_2, x_3, x_4, \alpha\rangle$ and confirms that $|x_2, x_3, x_4, \alpha\rangle$ is an entangled state.

## 2.4. Conjugate state of $|x_2, x_3, x_4, \alpha\rangle$

By making the following transform of $|x_2, x_3, x_4, \alpha\rangle$:

$$\begin{aligned}
|p_2, p_3, p_4, \beta\rangle &= \int\frac{\mathrm{d}^2\alpha}{8\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4}{4\pi\sqrt{2\pi}}|x_2, x_3, x_4, \alpha\rangle\exp\left(\frac{\beta\alpha * -\alpha\beta^*}{8}\right) \\
&\quad\times\exp\left[\frac{\mathrm{i}\,x_2(3p_2 - p_3 - p_4) + \mathrm{i}\,x_3(-p_2 + 3p_3 - p_4) + \mathrm{i}\,x_4(-p_2 - p_3 + 3p_4)}{4}\right], \quad (16)
\end{aligned}$$

and substituting Eq. (1) to Eq. (16), we have

$$\begin{aligned}
&|p_2,p_3,p_4,\beta\rangle\\
&=\exp\Big\{-\frac{3}{8}p_2^2-\frac{3}{8}p_3^2-\frac{3}{8}p_4^2+\frac{1}{4}p_2p_3\\
&+\frac{1}{4}p_2p_4+\frac{1}{4}p_3p_4-\frac{1}{8}|\beta|^2\\
&+\frac{\mathrm{i}\sqrt{2}}{4}(a_1^+-3a_2^++a_3^++a_4^+)p_2\\
&+\frac{\mathrm{i}\sqrt{2}}{4}(a_1^++a_2^+-3a_3^++a_4^+)p_3\\
&+\frac{\mathrm{i}\sqrt{2}}{4}(a_1^++a_2^++a_3^+-3a_4^+)p_4\\
&+\frac{1}{4}(a_1^++a_2^++a_3^++a_4^+)\beta\\
&+\frac{1}{8}[(a_1^+-a_2^+)^2+(a_1^+-a_3^+)^2\\
&+(a_1^+-a_4^+)^2+(a_2^+-a_3^+)^2\\
&+(a_2^+-a_4^+)^2+(a_3^+-a_4^+)^2]\Big\}|0000\rangle. \qquad (17)
\end{aligned}$$

In fact, using

$$:\frac{\partial}{\partial a_i^+}f(a_i,a_i^+):=[a_i,:f(a_i,a_i^+):]$$

and $a_i|0000\rangle=0$, $(i=1,2,3,4)$, we can see that $|p_2,p_3,p_4,\beta\rangle$ is the common eigenstate of $\hat{P}_1-\hat{P}_2$, $\hat{P}_1-\hat{P}_3$, $\hat{P}_1-\hat{P}_4$, and $a_1+a_2+a_3+a_4$, i.e.,

$$\begin{aligned}
&(a_1+a_2+a_3+a_4)|p_2,p_3,p_4,\beta\rangle=\beta|p_2,p_3,p_4,\beta\rangle,\\
&(\hat{P}_1-\hat{P}_2)|p_2,p_3,p_4,\beta\rangle=p_2|p_2,p_3,p_4,\beta\rangle,\\
&(\hat{P}_1-\hat{P}_3)|p_2,p_3,p_4,\beta\rangle=p_3|p_2,p_3,p_4,\beta\rangle,\\
&(\hat{P}_1-\hat{P}_4)|p_2,p_3,p_4,\beta\rangle=p_4|p_2,p_3,p_4,\beta\rangle. \qquad (18)
\end{aligned}$$

Using the IWOP technique, we can also easily prove

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4}{2\pi\sqrt{\pi}}\int\frac{\mathrm{d}^2\beta}{4\pi}|p_2,p_3,p_4,\beta\rangle\langle p_2,p_3,p_4,\beta|\\
&=1. \qquad (19)
\end{aligned}$$

Comparing Eq. (18) with Eq.(3), we know that $|p_2,p_3,p_4,\beta\rangle$ is the conjugate state of $|x_2,x_3,x_4,\alpha\rangle$.

## 3. Generating of $|x_2,x_3,x_4,\alpha\rangle$ by using beam splitters

We shall now discuss an experimental protocol to generate $|x_2,x_3,x_4,\alpha\rangle$. The role of the beam splitter operator can be expressed as

$$\hat{B}_{1234}=\hat{B}_{34}(\theta_{34})\hat{B}_{23}(\theta_{23})\hat{B}_{12}(\theta_{12}), \qquad (20)$$

where

$$\begin{aligned}
\hat{B}_{12}(\theta_{12})&=\exp[-\theta_{12}(a_1^+a_2-a_1a_2^+)],\\
\hat{B}_{23}(\theta_{23})&=\exp[-\theta_{23}(a_2^+a_3-a_2a_3^+)],\\
\hat{B}_{34}(\theta_{34})&=\exp[-\theta_{34}(a_3^+a_4-a_3a_4^+)]. \qquad (21)
\end{aligned}$$

Let the ideal single-mode squeezed state[17,20–22] in modes 1, 3, 4 expressed by

$$\begin{aligned}
|x=0\rangle_1&=\exp\left(-\frac{1}{2}a_1^{+2}\right)|0\rangle_1,\\
|x=0\rangle_3&=\exp\left(-\frac{1}{2}a_3^{+2}\right)|0\rangle_3,\\
|x=0\rangle_4&=\exp\left(-\frac{1}{2}a_4^{+2}\right)|0\rangle_4,
\end{aligned}$$

and the vacuum state $|0\rangle_2$ in mode 2, we firstly enter the direct product $|x=0\rangle_1\otimes|0\rangle_2\otimes|x=0\rangle_3\otimes|x=0\rangle_4$ to beam splitter $\hat{B}_{12}$, which then go through the second beam splitter $\hat{B}_{23}$ and the third beam splitter $\hat{B}_{34}$ after passing through beam splitters $\hat{B}_{12}$ and $\hat{B}_{23}$.

Using

$$\begin{aligned}
\hat{B}_{12}a_1^+\hat{B}_{12}^{-1}&=a_1^+\cos\theta_{12}+a_2^+\sin\theta_{12},\\
\hat{B}_{23}a_2^+\hat{B}_{23}^{-1}&=a_2^+\cos\theta_{23}+a_3^+\sin\theta_{23},\\
\hat{B}_{34}a_3^+\hat{B}_{34}^{-1}&=a_3^+\cos\theta_{34}+a_4^+\sin\theta_{34},\\
\hat{B}_{23}a_3^+\hat{B}_{23}^{-1}&=a_3^+\cos\theta_{23}-a_2^+\sin\theta_{23},\\
\hat{B}_{34}a_4^+\hat{B}_{34}^{-1}&=a_4^+\cos\theta_{34}-a_3^+\sin\theta_{34}, \qquad (22)
\end{aligned}$$

and setting

$$\begin{aligned}
&\cos\theta_{12}=\sqrt{\frac{3}{4}},\quad \sin\theta_{12}=-\sqrt{\frac{1}{4}},\quad \cos\theta_{23}=\sqrt{\frac{1}{3}},\\
&\sin\theta_{23}=\sqrt{\frac{2}{3}},\quad \cos\theta_{34}=\sqrt{\frac{1}{2}},\quad \sin\theta_{34}=\sqrt{\frac{1}{2}},
\end{aligned}$$

we obtain the four-mode squeezed vacuum state

$$\begin{aligned}
&\hat{B}_{1234}\exp\left(-\frac{1}{2}a_1^{+2}-\frac{1}{2}a_3^{+2}-\frac{1}{2}a_4^{+2}\right)|0\rangle_1\otimes|0\rangle_2\otimes|0\rangle_4\\
&=\exp\Big\{-\frac{1}{8}[(a_1^+-a_2^+)^2+(a_1^+-a_3^+)^2\\
&+(a_1^+-a_4^+)^2+(a_2^+-a_3^+)^2\\
&+(a_2^+-a_4^+)^2+(a_3^+-a_4^+)^2]\Big\}|0000\rangle. \qquad (23)
\end{aligned}$$

It is clear that

$$\begin{aligned}
&|x_2,x_3,x_4,\alpha\rangle\\
&=\hat{D}(z_1)\hat{D}(z_2)\hat{D}(z_3)\hat{D}(z_4)\exp\Big\{-\frac{1}{8}[(a_1^+-a_2^+)^2\\
&+(a_1^+-a_3^+)^2+(a_1^+-a_4^+)^2+(a_2^+-a_3^+)^2\\
&+(a_2^+-a_4^+)^2+(a_3^+-a_4^+)^2]\Big\}|0000\rangle, \qquad (24)
\end{aligned}$$

where $\hat{D}_i(z_i) = \exp(z_i a_i^+ - z_i^* a_i)$, $i = 1, 2, 3, 4$ is the displacement operator, and

$$
\begin{aligned}
z_1 &= \frac{1}{4}\left[\frac{1}{\sqrt{2}}(x_2 + x_3 + x_4) + \alpha\right],\\
z_2 &= \frac{1}{4}\left[\frac{1}{\sqrt{2}}(-3x_2 + x_3 + x_4) + \alpha\right],\\
z_3 &= \frac{1}{4}\left[\frac{1}{\sqrt{2}}(x_2 - 3x_3 + x_4) + \alpha\right],\\
z_4 &= \frac{1}{4}\left[\frac{1}{\sqrt{2}}(x_2 + x_3 - 3x_4) + \alpha\right].
\end{aligned}
$$

According to $\hat{D}_i(z_i) a_i^+ \hat{D}^{-1}(z_i) = a_i^+ - z_i^*$, and

$$
\begin{aligned}
&\hat{D}(z_1)\hat{D}(z_2)\hat{D}(z_3)\hat{D}(z_4)|0000\rangle\\
&= \exp\left[\sum_{i=1}^{4}\left(-\frac{1}{2}|z_i|^2 + z_i a_i^+\right)\right]|0000\rangle, \qquad (25)
\end{aligned}
$$

we can have

$$
\begin{aligned}
&\hat{D}(z_1)\hat{D}(z_2)\hat{D}(z_3)\hat{D}(z_4)\exp\Big\{-\frac{1}{8}[(a_1^+ - a_2^+)^2\\
&+(a_1^+ - a_3^+)^2 + (a_1^+ - a_4^+)^2 + (a_2^+ - a_3^+)^2\\
&+(a_2^+ - a_4^+)^2 + (a_3^+ - a_4^+)^2]\Big\}|0000\rangle\\
&= \exp\Big\{-\frac{3}{8}x_2^2 - \frac{3}{8}x_3^2 - \frac{3}{8}x_4^2 + \frac{1}{4}x_2x_3\\
&+\frac{1}{4}x_2x_4 + \frac{1}{4}x_3x_4 - \frac{1}{8}|\alpha|^2\\
&+\frac{\sqrt{2}}{4}(a_1^+ - 3a_2^+ + a_3^+ + a_4^+)x_2\\
&+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ - 3a_3^+ + a_4^+)x_3\\
&+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ + a_3^+ - 3a_4^+)x_4\\
&+\frac{1}{4}(a_1^+ + a_2^+ + a_3^+ + a_4^+)\alpha\\
&-\frac{1}{8}[(a_1^+ - a_2^+)^2 + (a_1^+ - a_3^+)^2\\
&+(a_1^+ - a_4^+)^2 + (a_2^+ - a_3^+)^2\\
&+(a_2^+ - a_4^+)^2 + (a_3^+ - a_4^+)^2]\Big\}|0000\rangle\\
&= |x_2, x_3, x_4, \alpha\rangle. \qquad (26)
\end{aligned}
$$

Experimentally, we can achieve displacement $D(z_i)$, $i = 1$, 2, 3, 4 by reflecting the light field

$$
\begin{aligned}
&\exp\Big\{-\frac{1}{8}[(a_1^+ - a_2^+)^2 + (a_1^+ - a_3^+)^2 + (a_1^+ - a_4^+)^2\\
&+(a_2^+ - a_3^+)^2 + (a_2^+ - a_4^+)^2 + (a_3^+ - a_4^+)^2]\Big\}|0000\rangle
\end{aligned}
$$

from a reflecting mirror which is partially reflective and partially transmissive and adding through the mirror a field that has been phase and amplitude modulated according to values $x_2, x_3, x_4$ and $\alpha$ in $D(z_i)$, $i = 1, 2, 3, 4$. In Refs. [23] and [24], a possible schematic diagram of such an experimental implementation is shown.

# 4. Applications of $|x_2, x_3, x_4, \alpha\rangle$

## 4.1. Generating four-mode EPR entangled state by superposition of $|x_2, x_3, x_4, \alpha\rangle$

By making the transform

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_1}{\sqrt{8\pi}}\exp\left(-\frac{1}{4}\,\mathrm{i}\,u\alpha_1\right)|x_2, x_3, x_4, \alpha = \alpha_1 + \mathrm{i}\,\alpha_2\rangle\\
&= \exp\Big\{-\frac{3}{8}x_2^2 - \frac{3}{8}x_3^2 - \frac{3}{8}x_4^2\\
&+\frac{1}{4}x_2x_3 + \frac{1}{4}x_2x_4 + \frac{1}{4}x_3x_4 - \frac{\alpha_2^2}{8}\\
&+\frac{\sqrt{2}}{4}(a_1^+ - 3a_2^+ + a_3^+ + a_4^+)x_2\\
&+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ - 3a_3^+ + a_4^+)x_3\\
&+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ + a_3^+ - 3a_4^+)x_4\\
&+\frac{\mathrm{i}}{4}(a_1^+ + a_2^+ + a_3^+ + a_4^+)\alpha_2\\
&-\frac{1}{8}[(a_1^+ - a_2^+)^2 + (a_1^+ - a_3^+)^2 + (a_1^+ - a_4^+)^2\\
&+(a_2^+ - a_3^+)^2 + (a_2^+ - a_4^+)^2 + (a_3^+ - a_4^+)^2]\Big\}\\
&\times\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_1}{\sqrt{8\pi}}\exp\Big[-\frac{1}{8}\alpha_1^2 + \frac{1}{4}(a_1^+ + a_2^+\\
&+a_3^+ + a_4^+ - \mathrm{i}\,u)\alpha_1\Big]|0000\rangle, \qquad (27)
\end{aligned}
$$

and letting $p = \frac{\alpha_2 - u}{\sqrt{2}}$, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_1}{\sqrt{8\pi}}\exp\left(-\frac{1}{4}\,\mathrm{i}\,u\alpha_1\right)|x_2, x_3, x_4, \alpha = \alpha_1 + \mathrm{i}\,\alpha_2\rangle\\
&= \exp\left[-\frac{1}{16}(\alpha_2 + u)^2\right]|p, x_2, x_3, x_4\rangle, \qquad (28)
\end{aligned}
$$

where

$$
\begin{aligned}
&|p, x_2, x_3, x_4\rangle\\
&= \exp\Big[-\frac{3}{8}x_2^2 - \frac{3}{8}x_3^2 - \frac{3}{8}x_4^2 + \frac{3}{8}x_2x_3\\
&+\frac{3}{8}x_2x_4 + \frac{3}{8}x_3x_4 - \frac{1}{8}p^2\\
&+\frac{\sqrt{2}}{4}(a_1^+ - 3a_2^+ + a_3^+ + a_4^+)x_2\\
&+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ - 3a_3^+ + a_4^+)x_3
\end{aligned}
$$

$$+\frac{\sqrt{2}}{4}(a_1^+ + a_2^+ + a_3^+ - 3a_4^+)x_4 + \frac{\mathrm{i}\sqrt{2}}{4}(a_1^+ + a_2^+ + a_3^+ + a_4^+)p - \frac{1}{4}(a_1^+ + a_2^+ + a_3^+ + a_4^+)^2 + (a_1^+a_2^+ + a_1^+a_3^+ + a_1^+a_4^+ + a_2^+a_3^+ + a_2^+a_4^+ + a_3^+a_4^+)\Big]|0000\rangle, \quad (29)$$

which is a common eigenstate of operators $\hat{X}_1 - \hat{X}_2$, $\hat{X}_1 - \hat{X}_3$, $\hat{X}_1 - \hat{X}_4$, and $\hat{P}_1 + \hat{P}_2 + \hat{P}_3 + \hat{P}_4$ and a standard four-mode entangled state,[24] i.e.,

$$\begin{aligned}(\hat{X}_1 - \hat{X}_2)|p, x_2, x_3, x_4\rangle &= x_2|p, x_2, x_3, x_4\rangle,\\ (\hat{X}_1 - \hat{X}_3)|p, x_2, x_3, x_4\rangle &= x_3|p, x_2, x_3, x_4\rangle,\\ (\hat{X}_1 - \hat{X}_4)|p, x_2, x_3, x_4\rangle &= x_4|p, x_2, x_3, x_4\rangle\\ (\hat{P}_1 + \hat{P}_2 + \hat{P}_3 + \hat{P}_4)|p, x_2, x_3, x_4\rangle &= p|p, x_2, x_3, x_4\rangle. \end{aligned} \quad (30)$$

### 4.2. Four-mode squeezing operators derived by $|x_2, x_3, x_4, \alpha\rangle$

Making classical transformation $x_i \to \mu x_i$ $(i = 2, 3, 4)$ in $|x_2, x_3, x_4, \alpha\rangle$, we now construct the following ket–bra integration of a new squeezing-Fresnel operator:

$$\hat{U}(\mu) = \mu^{3/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4}{2\pi\sqrt{\pi}}\int\frac{\mathrm{d}^2\alpha}{4\pi}|\mu x_2, \mu x_3, \mu x_4, \alpha\rangle\langle x_2, x_3, x_4, \alpha|. \quad (31)$$

According to Eq. (11), we can prove that $\hat{U}(\mu)$ is unitary, i.e., $\hat{U}^+(\mu)\hat{U}(\mu) = \hat{U}(\mu)\hat{U}^+(\mu) = 1$. Substituting Eq. (1) into Eq. (31) and letting $e^\lambda = \mu$ , with the IWOP technique, we obtain

$$\begin{aligned}\hat{U}(\mu) =\ & \operatorname{sech}\lambda\exp\left\{\frac{\tanh\lambda}{8}[(a_1^+ - a_2^+)^2 + (a_1^+ - a_3^+)^2 + (a_1^+ - a_4^+)^2 + (a_2^+ - a_3^+)^2\right.\\ &\left. + (a_2^+ - a_4^+)^2 + (a_3^+ - a_4^+)^2]\right\} : \exp[a_i^+(\boldsymbol{M} - \boldsymbol{I})_{ij}a_j] : \exp\left\{-\frac{\tanh\lambda}{8}[(a_1 - a_2)^2\right.\\ &\left. + (a_1 - a_3)^2 + (a_1 - a_4)^2 + (a_2 - a_3)^2 + (a_2 - a_4)^2 + (a_3 - a_4)^2]\right\}, \end{aligned} \quad (32)$$

where $\operatorname{sech}\lambda = \frac{2\mu}{\mu^2+1}$, $\tanh\lambda = \frac{\mu^2-1}{\mu^2+1}$, $\boldsymbol{I}$ is the unitary matrix, and

$$\boldsymbol{M} = \frac{1}{4}\begin{pmatrix} 3\operatorname{sech}\lambda + 1 & 1 - \operatorname{sech}\lambda & 1 - \operatorname{sech}\lambda & 1 - \operatorname{sech}\lambda \\ 1 - \operatorname{sech}\lambda & 3\operatorname{sech}\lambda + 1 & 1 - \operatorname{sech}\lambda & 1 - \operatorname{sech}\lambda \\ 1 - \operatorname{sech}\lambda & 1 - \operatorname{sech}\lambda & 3\operatorname{sech}\lambda + 1 & 1 - \operatorname{sech}\lambda \\ 1 - \operatorname{sech}\lambda & 1 - \operatorname{sech}\lambda & 1 - \operatorname{sech}\lambda & 3\operatorname{sech}\lambda + 1 \end{pmatrix}. \quad (33)$$

The inverse matrix of $\boldsymbol{M}$ is

$$\boldsymbol{M}^{-1} = \frac{1}{4}\begin{pmatrix} 3\cosh\lambda + 1 & 1 - \cosh\lambda & 1 - \cosh\lambda & 1 - \cosh\lambda \\ 1 - \cosh\lambda & 3\cosh\lambda + 1 & 1 - \cosh\lambda & 1 - \cosh\lambda \\ 1 - \cosh\lambda & 1 - \cosh\lambda & 3\cosh\lambda + 1 & 1 - \cosh\lambda \\ 1 - \cosh\lambda & 1 - \cosh\lambda & 1 - \cosh\lambda & 3\cosh\lambda + 1 \end{pmatrix}. \quad (34)$$

According to the formula: $\exp[a_i^+(\mathrm{e}^{\Omega} - I)_{ij}a_j] := \exp[a_i^+\Omega_{ij}a_j]$, $(i, j = 1, 2, 3, 4)$,[26] we have

$$\begin{aligned}\hat{U}(\mu) =\ & \operatorname{sech}\lambda\exp\left\{\frac{\tanh\lambda}{8}[(a_1^+ - a_2^+)^2 + (a_1^+ - a_3^+)^2 + (a_1^+ - a_4^+)^2 + (a_2^+ - a_3^+)^2\right.\\ &\left. + (a_2^+ - a_4^+)^2 + (a_3^+ - a_4^+)^2]\right\}\exp[a_i^+\Omega_{ij}a_j]\exp\left\{-\frac{\tanh\lambda}{8}[(a_1 - a_2)^2\right.\\ &\left. + (a_1 - a_3)^2 + (a_1 - a_4)^2 + (a_2 - a_3)^2 + (a_2 - a_4)^2 + (a_3 - a_4)^2]\right\}. \end{aligned} \quad (35)$$

where $\boldsymbol{\Omega} = \ln\boldsymbol{M}$. Based on operator formulas[26]

$$\exp[a_i^+\Omega_{ij}a_j]a_l\exp[-a_i^+\Omega_{ij}a_j] = (\mathrm{e}^{-\boldsymbol{\Omega}})_{li}a_i, \quad \exp[a_i^+\Omega_{ij}a_j]a_l^+\exp[-a_i^+\Omega_{ij}a_j] = a_i^+(\mathrm{e}^{\boldsymbol{\Omega}})_{li}, \quad (36)$$

and

$$\mathrm{e}^{\hat{A}}\hat{B}\,\mathrm{e}^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \cdots,$$

we obtain

$$\begin{aligned}
&\hat{U}(\mu)(\hat{X}_1 - \hat{X}_2)\hat{U}^{-1}(\mu) = (\hat{X}_1 - \hat{X}_2)/\mu,\\
&\hat{U}(\mu)(\hat{X}_1 - \hat{X}_3)\hat{U}^{-1}(\mu) = (\hat{X}_1 - \hat{X}_3)/\mu,\\
&\hat{U}(\mu)(\hat{X}_1 - \hat{X}_4)\hat{U}^{-1}(\mu) = (\hat{X}_1 - \hat{X}_4)/\mu,
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&\hat{U}(\mu)(\hat{P}_1 - \hat{P}_2)\hat{U}^{-1}(\mu) = \mu(\hat{P}_1 - \hat{P}_2),\\
&\hat{U}(\mu)(\hat{P}_1 - \hat{P}_3)\hat{U}^{-1}(\mu) = \mu(\hat{P}_1 - \hat{P}_3),\\
&\hat{U}(\mu)(\hat{P}_1 - \hat{P}_4)\hat{U}^{-1}(\mu) = \mu(\hat{P}_1 - \hat{P}_4).
\end{aligned} \tag{38}$$

In order to compare the squeezing effect caused by $\hat{U}(\mu)$ with the ordinary four-mode squeezing, we define three pairs quadrature components of the fields as

$$\begin{aligned}
&\hat{Q}_2 = \frac{\hat{X}_1 - \hat{X}_2}{\sqrt{2}}, \quad \hat{Q}'_2 = \frac{\hat{P}_1 - \hat{P}_2}{\sqrt{2}},\\
&\hat{Q}_3 = \frac{\hat{X}_1 - \hat{X}_3}{\sqrt{2}}, \quad \hat{Q}'_3 = \frac{\hat{P}_1 - \hat{P}_3}{\sqrt{2}},\\
&\hat{Q}_4 = \frac{\hat{X}_1 - \hat{X}_4}{\sqrt{2}}, \quad \hat{Q}'_4 = \frac{\hat{P}_1 - \hat{P}_4}{\sqrt{2}},
\end{aligned} \tag{39}$$

with $[\hat{Q}_j, \hat{Q}'_j] = \mathrm{i}$ $(j = 2,3,4)$. When the field is in the squeezed vacuum state

$$\begin{aligned}
&\hat{U}(\mu)|0000\rangle\\
&= \operatorname{sech}\lambda \exp\left\{\frac{\tanh\lambda}{8}[(a_1^+ - a_2^+)^2 + (a_1^+ - a_3^+)^2 + (a_1^+ - a_4^+)^2 + (a_2^+ - a_3^+)^2 + (a_2^+ - a_4^+)^2 + (a_3^+ - a_4^+)^2]\right\}|0000\rangle,
\end{aligned} \tag{40}$$

from Eqs. (37)–(40), we can calculate the uncertainty relation of the quadrature components

$$\begin{aligned}
&(\Delta\hat{Q}_2)^2 = (\Delta\hat{Q}_3)^2 = (\Delta\hat{Q}_4)^2 = \frac{\mathrm{e}^{2\lambda}}{2},\\
&(\Delta\hat{Q}'_2)^2 = (\Delta\hat{Q}'_3)^2 = (\Delta\hat{Q}'_4)^2 = \frac{1}{2\,\mathrm{e}^{2\lambda}},
\end{aligned} \tag{41}$$

which is just related to

$$\Delta\hat{Q}_2 \cdot \Delta\hat{Q}'_2 = \Delta\hat{Q}_3 \cdot \Delta\hat{Q}'_3 = \Delta\hat{Q}_4 \cdot \Delta\hat{Q}'_4 = \frac{1}{2}.$$

On the other hand, making classical transformation $\alpha \to s\alpha - r\alpha^*$ with $ss^* - rr^* = 1$, we construct the following ket–bra integration:

$$\begin{aligned}
&\hat{U}(s)\\
&= \sqrt{s}\int_{-\infty}^{\infty}\frac{\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4}{2\pi\sqrt{\pi}}\int\frac{\mathrm{d}^2\alpha}{4\pi}|x_2,x_3,x_4,s\alpha - r\alpha^*\rangle\\
&\quad\times\langle x_2,x_3,x_4,\alpha|.
\end{aligned} \tag{42}$$

We can prove that $\hat{U}(s)$ is unitary, i.e., $\hat{U}^+(s)\hat{U}(s) = \hat{U}(s)\hat{U}^+(s) = 1$. Performing the integration in Eq. (42) , we have

$$\begin{aligned}
\hat{U}(s) = {}& \frac{1}{\sqrt{s^*}}\exp\left[-\frac{r}{8s^*}(a_1^+ + a_2^+ + a_3^+ + a_4^+)^2\right]\\
&\times :\exp[a_i^+(\boldsymbol{M'} - \boldsymbol{I})_{ij}a_j]:\\
&\times\exp\left[\frac{r^*}{8s^*}(a_1 + a_2 + a_3 + a_4)^2\right],
\end{aligned} \tag{43}$$

where

$$\boldsymbol{M'} = \frac{1}{4}\begin{pmatrix} 3+\frac{1}{s^*} & 1-\frac{1}{s^*} & 1-\frac{1}{s^*} & 1-\frac{1}{s^*}\\ 1-\frac{1}{s^*} & 3+\frac{1}{s^*} & 1-\frac{1}{s^*} & 1-\frac{1}{s^*}\\ 1-\frac{1}{s^*} & 1-\frac{1}{s^*} & 3+\frac{1}{s^*} & 1-\frac{1}{s^*}\\ 1-\frac{1}{s^*} & 1-\frac{1}{s^*} & 1-\frac{1}{s^*} & 3+\frac{1}{s^*}\end{pmatrix}, \tag{44}$$

and the inverse matrix of $\boldsymbol{M'}$ is

$$M'^{-1} = \frac{1}{4}\begin{pmatrix} 3+s^* & 1-s^* & 1-s^* & 1-s^*\\ 1-s^* & 3+s^* & 1-s^* & 1-s^*\\ 1-s^* & 1-s^* & 3+s^* & 1-s^*\\ 1-s^* & 1-s^* & 1-s^* & 3+s^*\end{pmatrix}. \tag{45}$$

According to formula

$$:\exp[a_i^+(\,\mathrm{e}^{\boldsymbol{\Omega'}} - I)_{ij}a_j] := \exp[a_i^+\Omega'_{ij}a_j],$$

$(i,j = 1,2,3,4)$, we have

$$\begin{aligned}
\hat{U}(s) = {}& \frac{1}{\sqrt{s^*}}\exp\left[-\frac{r}{8s^*}(a_1^+ + a_2^+ + a_3^+ + a_4^+)^2\right]\\
&\times\exp[a_i^+\Omega'_{ij}a_j]\\
&\times\exp\left[\frac{r^*}{8s^*}(a_1 + a_2 + a_3 + a_4)^2\right],
\end{aligned} \tag{46}$$

where $\boldsymbol{\Omega'} = \ln\boldsymbol{M'}$. When the field is in the squeezed vacuum state

$$\begin{aligned}
&\hat{U}(s)|0000\rangle\\
&= \frac{1}{\sqrt{s^*}}\exp[-\frac{r}{8s^*}(a_1^+ + a_2^+ + a_3^+ + a_4^+)^2]|0000\rangle,
\end{aligned} \tag{47}$$

we define a pair of quadrature components of the fields as

$$\hat{Q}_1 = \frac{\hat{X}_1 + \hat{X}_2 + \hat{X}_3 + \hat{X}_4}{2},$$

$$\hat{Q}_1' = \frac{\hat{P}_1 + \hat{P}_2 + \hat{P}_3 + \hat{P}_4}{2}. \quad (48)$$

Using Eq. (36) and

$$\mathrm{e}^{\hat{A}}\hat{B}\,\mathrm{e}^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \cdots,$$

we obtain

$$(\Delta\hat{Q}_1)^2 = \frac{|s - r^*|^2}{2}, \quad (\Delta\hat{Q}_1')^2 = \frac{|s + r^*|^2}{2}, \quad (49)$$

and

$$\Delta\hat{Q}_1 \cdot \Delta\hat{Q}_1' = \frac{1}{2}\sqrt{(s^2 - r^{*2}) \cdot (s^{*2} - r^2)}, \quad (50)$$

which are related to $s$ and $r$. Equations (49) and (50) appear to be the same as the squeezing effect caused by the ordinary four-mode squeezing operator, but equations (49) and (50) are quite different from that of the ordinary one. To see this more clearly, we rewrite $s$ and $r$ with real parameters $A$, $B$, $C$, $D$

$$s = \frac{1}{2}[(A + D) - \mathrm{i}(B - C)],$$
$$r = \frac{1}{2}[(D - A) - \mathrm{i}(B + C)], \quad (51)$$

where $AD - BC = 1$. Then equations (49) and (50) can be written respectively as

$$(\Delta\hat{Q}_1)^2 = \frac{A^2 + B^2}{2}, \quad (\Delta\hat{Q}_1')^2 = \frac{C^2 + D^2}{2}, \quad (52)$$

and

$$\Delta\hat{Q}_1 \cdot \Delta\hat{Q}_1' = \frac{1}{2}\sqrt{(A^2 + B^2) \cdot (C^2 + D^2)}. \quad (53)$$

Only when $B = C = 0$, $A = \mathrm{e}^{\sigma}$, and $D = \mathrm{e}^{-\sigma}$, equations (52) and (53) can lead to the quantum fluctuation of

$$(\Delta\hat{Q}_1)^2 = \frac{\mathrm{e}^{2\sigma}}{2}, \quad (\Delta\hat{Q}_1')^2 = \frac{1}{2\,\mathrm{e}^{2\sigma}},$$
$$\Delta\hat{Q}_1 \cdot \Delta\hat{Q}_1' = \frac{1}{2}. \quad (54)$$

## 5. Summary

In conclusion, with the aid of the IWOP technique, we have proposed a new kind of four -mode CV-CES $|x_2, x_3, x_4, \alpha\rangle$. We prove the completeness relation and the partial orthogonal of $|x_2, x_3, x_4, \alpha\rangle$. We have shown that $|x_2, x_3, x_4, \alpha\rangle$ makes up a complete representation in the four-mode Fock space and exhibits both the properties of coherent and entangled states. The conjugate state of $|x_2, x_3, x_4, \alpha\rangle$ can be derived from $|x_2, x_3, x_4, \alpha\rangle$ by using the Fourier transformation. Moreover, a simple experimental protocol of generating a four-mode CV-CES is proposed by using beam splitters. The superposition of $|x_2, x_3, x_4, \alpha\rangle$ along $\alpha_1$ can generate a four-mode entangled state. By making a classical transformation in $|x_2, x_3, x_4, \alpha\rangle$, a four-mode squeezing-Fresnel operator is constructed, which can play the role of special squeezing.

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