# Asymptotic iteration approach to supersymmetric bistable potentials

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We examine quasi exactly solvable bistable potentials and their supersymmetric partners within the framework of the asymptotic iteration method (AIM). It is shown that the AIM produces excellent approximate spectra and that sometimes it is found to be more useful to use the partner potential for computation. We also discuss the direct application of the AIM to the Fokker–Planck equation.

**Keywords:** bistable potentials, asymptotic iteration

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## 1. Introduction

The importance of bistable potentials hardly needs to be emphasized, and they have been extensively used in various branches of physics. In particular, they play an important role in the Fokker–Planck equation.[1−6] There have been many approaches to obtain the eigenvalues of such potentials,[7,8] but it is not always easy to determine them (especially the low lying ones). For a class of bistable potentials, this problem can be circumvented by using supersymmetry.[2,3,9]

A simple technique called the asymptotic iteration method (AIM) has been introduced[10] and applied for obtaining eigenvalues (and eigenfunctions) of a class of differential equations.[11−14] In particular, it has been applied successfully to obtain eigenvalues of the Schrödinger, the Klein–Gordon, and the Dirac equations with various potentials.[15−21] In most cases of solvable potentials, the AIM has been found to produce the exact spectra,[13,22,23] while for non exactly solvable potentials it yields reasonably good approximate solutions.[24−30] Although the AIM has been used to determine eigenvalues of bistable potentials,[24,25,30] it has not been employed to study supersymmetric potentials. Also there are other aspects, for example, the splitting between the first excited state and the ground state of a bistable potential, which need to be examined within the context of the AIM. Here our objective is to apply the AIM to supersymmetric systems (as well as the quasi exactly solvable ones) and critically examine different aspects of bistable potentials. In particular, we will apply the AIM to quasi exactly solvable (QES) potentials and their supersymmetric (SUSY) partners to examine the relative advantages of using the AIM to partners. This is of importance since quite often one partner is found to have double minima while the other partner has a single minimum. As mentioned earlier, bistable potentials play an important role in the Fokker–Planck equation. One way to solve the (time independent) Fokker–Planck equation is to transform it to a Schrödinger form. However, in view of the form of the Fokker–Planck equation, we shall apply the AIM directly to this equation. The organization of this paper is as follows. In Section 2 we briefly describe the AIM. In Section 3 we apply the AIM to QES potentials and their supersymmetric partners. In Section 4 we discuss a direct application of the AIM to the Fokker–Planck equation. Finally, Section 5 is devoted to a conclusion.

## 2. AIM

Although a complete description of the AIM can be found elsewhere,[10] for the sake of completeness, we present a brief outline here. Given that $\lambda_0(x)$ and

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$s_0(x)$ are sufficiently many times continuously differentiable, the second-order differential equation

$$y''(x) = \lambda_0(x)y'(x) + s_0(x)y(x) \tag{1}$$

has a general solution

$$\begin{aligned} & y(x) \\ &= \exp\left(-\int^x \alpha \mathrm{d}t\right)\left[C_2 + C_1\right. \\ &\left.\times \int^x \exp\left(\int^t (\lambda_0(\tau) + 2\alpha(\tau))\mathrm{d}\tau\right)\mathrm{d}t\right]. \end{aligned} \tag{2}$$

And for some $n > 0$,

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} \equiv \alpha, \tag{3}$$

where

$$\begin{aligned} \lambda_n &= \lambda'_{n-1} + s_{n-1} + \lambda_0\lambda_{n-1}, \\ s_n &= s'_{n-1} + s_0\lambda_{n-1}. \end{aligned} \tag{4}$$

The termination condition of the method, as given in Eq. (3), can also be written as

$$\delta(x) = \lambda_n s_{n-1} - s_n\lambda_{n-1} = 0. \tag{5}$$

This $\delta(x)$ function has a crucial role for boundary-value problems: the coefficients $s_n$ and $\lambda_n$ now depend on the (unknown) eigenvalue $E$, which is determined by the vanishing condition (5). If, for a suitable choice of $E$, this equation is satisfied at every $x$ point, then the problem is considered to be exactly solvable. If the problem is not exactly solvable, this $\delta(x)$ function depends on both $x$ and $E$. In this case we solve Eq. (5) at a suitable $x_0$ point to obtain the eigenvalues. The method is developed with the idea of using minimal algebraic computations to solve second-order differential equations. Indeed, with the availability of mathematical software for performing symbolic calculations, the computation required by the asymptotic iteration presents few difficulties, even for the higher iteration steps. In the following sections, we shall consider the applications of the AIM to several potentials as well as to the Fokker–Planck equation.

## 3. QES potentials and their SUSY partners

Among the various quasi exactly solvable potentials, here we shall consider the double well sextic oscillator potentials and their supersymmetric partners. The general form of this family of potentials is given by[31,32]

$$V(x) = x^6 - (8j+3)x^2, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \tag{6}$$

Potentials (6) are considered quasi exactly solvable, because $(2j+1)$ energy levels can be determined analytically while the other levels remain unknown. Clearly the simplest member of this family has $j = 0$, and the corresponding superpotential is given by $W(x) = x^3$. In this case, the supersymmetric partner potentials are given by

$$V_\pm = W^2(x) \pm W'(x) = x^6 \pm 3x^2. \tag{7}$$

The potentials given by Eq. (7) are isospectral except for the zero energy ground state of $V_-(x)$, which is $\psi_{-0}(x) \approx \exp[-\int W(x)] = \exp[-x^4/4]$. The $V_-(x)$ represents the simplest quasi exactly solvable potential with exactly one known eigenvalue. All the other eigenvalues are unknown. By supersymmetry, $V_+(x)$ possesses exactly the same spectrum except for the ground state, i.e, $E_0^- = 0$, $E_{n+1}^- = E_n^+$. It was noted by Bernstein *et al.*[2] that $V_-(x)$ is a double well potential, the low lying eigenvalues are difficult to determine, and the partner potential $V_+(x)$, which has a single minimum, may be used to determine the eigenvalues of $V_-(x)$. Here we shall apply the AIM to both the original and the partner potentials to examine not only the accuracy of the method but also its sensitivity to the double well structures.

To apply the AIM to the potentials in Eq. (7), we may start with any wave function satisfying the boundary conditions. Here we consider a generic wave function of the form

$$\psi(x) = \mathrm{e}^{-x^4/4} g(\alpha, x), \quad g(\alpha, x) = \mathrm{e}^{-\alpha x^2/2} f(x). \tag{8}$$

Substituting Eq. (8) into the eigenvalue equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x), \tag{9}$$

we obtain

$$\begin{aligned} & f''(x) \\ &= 2(x^3 + x\alpha)f'(x) \\ &\quad - [E + (\alpha^2 + 8j)x^2 + 2\alpha x^4 - \alpha]f(x). \end{aligned} \tag{10}$$

The above equation can be identified with Eq. (1), and thus we have

$$\lambda_0 = 2(x^3 + \alpha x),$$

$$s_0 = -[E + (\alpha^2 + 8j)x^2 + 2\alpha x^4 - \alpha]. \quad (11)$$

To obtain the eigenvalues of the potentials in Eq. (7), we set $j = 0$. The results are given in Table 1. In our calculations, we have used $x_0 = 0$ and $\alpha = 4$.

**Table 1.** Eigenvalues for potential (6) with $j = 0$. Number of iterations for each partner is 50.

| Potentials | $E_{\rm AIM}$ | $E^{33}_{\rm exact}$ |
|---|---|---|
| $x^6 - 3x^2$ | 0 | 0 |
| | 1.93548 | 1.9354 |
| | 6.2985 | 6.2985 |
| | 11.681 | 11.6809 |
| | 18.0426 | 18.0426 |
| | 25.2546 | 25.2546 |
| $x^6 + 3x^2$ | – | – |
| | 1.9354 | – |
| | 6.2985 | – |
| | 11.6809 | – |
| | 18.0426 | – |
| | 25.2546 | – |

It is seen from Table 1 that the eigenvalues obtained by the AIM completely agree with the exact numerical results. Also the values are better than those obtained by the semiclassical approximations. There is also a complete matching of the values between the partners. Interestingly, the number of the iterations required for either partner is 50. We thus conclude that as far as the AIM is concerned, the double well structure of the original potential $V_-(x)$ does not pose any particular problem.

We now consider other potentials given by Eq. (6). It can be shown that for $j > 0$, one partner of the SUSY system is given by Eq. (6), while the other partner is of a non-polynomial character. For example, for $j = 1/2$, the superpotential is given by $W = x^3 - 2\sqrt{2}x/(1 + \sqrt{2}x^2)$. The corresponding partner potentials are given by[32]

$$V_-(x) = x^6 - 7x^2, \quad (12)$$

$$V_+(x) = x^6 - x^2 + \frac{4\sqrt{2}}{1 + \sqrt{2}x^2} - \frac{8\sqrt{2}}{(1 + \sqrt{2}x^2)^2}. \quad (13)$$

The potential $V_-(x)$ has $(2 \times (1/2) + 1) = 2$ exactly known eigenstates, namely, the ground and the second excited states

$$E_{0,2} = \mp\sqrt{2},$$
$$\psi_{0,2}(x) = N_{0,2}(1 \pm \sqrt{2}x^2)\ \mathrm{e}^{-x^4/4}. \quad (14)$$

While the partner potential given by Eq. (13) shares all the energies except the ground state. Similarly, if we consider the $j = 1$ case, we would have $V_-(x) = x^6 - 11x^2$, while the partner potential would be of a non-polynomial nature. We would like to point out that partner potentials of the Eq. (13) type are not particularly suitable for the AIM treatment (or for that matter by any other method) as a consequence of SUSY, and we may use $V_-(x)$ to determine their eigenvalues. We have computed the eigenvalues of the potentials given by Eq. (6) with $j = 1/2, 1, 3/2$ using the generic wave function of the Eq. (8) type with $x_0 = 0$ and $\alpha = 4$. The results are given in Table 2.

**Table 2.** Eigenvalues of potential (6) with $j = 1/2, 1, 3/2$. Numbers of iterations for the potentials are 50, 60, and 62, respectively.

| Potentials | $E_{\rm AIM}$ | $E^{33}_{\rm exact}$ |
|---|---|---|
| $x^6 - 7x^2$ | −2.82843 | −2.82843 |
| | −2.26508 | – |
| | 2.82843 | 2.82843 |
| | 7.19806 | 7.19806 |
| | 13.0093 | 13.0093 |
| | 19.7215 | 19.7215 |
| | 27.24 | 27.24 |
| $x^6 - 11x^2$ | −8 | −8 |
| | −7.91735 | – |
| | 0 | 0 |
| | 2.52036 | 2.52036 |
| | 8 | 8 |
| | 14.113 | 14.113 |
| | 21.1575 | 21.1575 |
| | 28.9748 | 28.9748 |
| $x^6 - 15x^2$ | −15.0775 | – |
| | −15.0686 | – |
| | 0 | – |
| | −3.55932 | – |
| | 2.73313 | – |
| | 3.55932 | – |
| | 8.52705 | – |
| | 15.0775 | – |
| | 22.4227 | – |

From Table 2, we find that the AIM results are in excellent agreement with the exact numerical results. In fact, they are better compared to the Wentzel–Kramers–Brilloum (WKB) or the supersymmetric WKB values.[33] All the potentials in Table 2 are double well potentials with two symmetric minima, and the numbers of the iterations required in these cases are 50, 60, and 62. As in the $j = 0$ case, the double well nature of these potentials is not really a serious problem as far as the determination of the eigenvalues is concerned. In this context, the next

question is about the number of the iterations and the depth of the potential, which is defined as the distance between the location of the minima and the $x$ axis. From Table 3, it can be seen that as the depth of the well increases, the number of the iterations required also increases. However, although the energy splitting between the first two levels decreases as the depth increases, nevertheless it is still appreciable for potential $V(x) = x^6 - 15x^2$. Finally, we note that all the potentials in Table 2 have non-polynomial partner potentials like Eq. (13). Because of supersymmetry, the energy levels of these non-polynomial potentials are the same as those in Table 2.

### 3.1. QES potentials with angular momentum barrier

We now consider the sextic oscillator potential with exactly one known eigenstate again, but with an angular momentum term. The superpotential for such a potential is given by

$$W(x) = x^3 - \frac{d}{x}. \tag{15}$$

In this case, the partner potentials are given by

$$\begin{aligned} V_-(x) &= x^6 - (2d+3)x^2 + \frac{d(d-1)}{x^2}, \\ V_+(x) &= x^6 - (2d-3)x^2 + \frac{d(d+1)}{x^2}. \end{aligned} \tag{16}$$

We note that for $d = 1$, the first of these potentials can be interpreted as a one-dimensional potential or a three-dimensional one with angular momentum $l = 0$, the second one can only be treated in three dimensions with angular momentum $l = 1$. Thus for the pair of potentials given in Eq. (16) to be interpreted as SUSY partners, it is necessary to consider these potentials on the half line. In this case the potentials do not have the double well structure. On the other hand, viewed as radial potentials, the energy eigenvalues obtained using Eq. (16) denote eigenvalues $E_{n,l+1}$ corresponding to the spherically symmetric potentials

$$\begin{aligned} V_-(r) &= r^6 - (2d+1)r^2, \\ V_+(r) &= r^6 - (2d-3)r^2. \end{aligned} \tag{17}$$

Now to obtain the eigenvalues of the Schrödinger equations with the potentials given in Eq. (16) using the AIM, we use the wave functions

$$\begin{aligned} &\psi_-(x) = x^d \, \mathrm{e}^{-x^4/4 - \alpha x^2/2} f_-(x) \quad \text{for} \quad V_-(x), \\ &\psi_+(x) = x^{d+1} \, \mathrm{e}^{-x^4/4 - \alpha x^2/2} f_+(x) \quad \text{for} \quad V_+(x). \end{aligned} \tag{18}$$

In Table 3, we present the computed spectrum obtained using $x_0 = 0$ and $\alpha = 4$. It is seen from Table 3 that the original potential ($V_-(x)$) requires more iterations than the partner potential ($V_+(x)$) in order to produce eigenvalues of the same accuracy. It should be noted that in spite of having no double well structure, the number of iterations is fewer for the partner. This is perhaps due to the fact that for the partner potential, the angular momentum value is increased by one unit. The situation is similar to the one encountered in the shifted $1/N$ expansion in supersymmetric quantum mechanics.[34] There is another interesting point to be noted. It can be seen that as $d$ increases, i.e, for larger values of angular momentum, the number of iterations also increases for both partners (although the iteration number for $V_-$ is larger than that for $V_+$).

**Table 3.** Eigenvalues of the pair of potentials in Eq. (16) with different values of $d$, where $N_-$ and $N_+$ denote the number of iterations corresponding to $V_-$ and $V_+$, respectively.

| $d$ | $N_-$ | $N_+$ | $n$ | $E^-_{\mathrm{AIM}}$ | $E^+_{\mathrm{AIM}}$ |
|---|---|---|---|---|---|
| 1 | 40 | 30 | 0 | 0 | – |
| | | | 1 | 9.45554 | 9.45554 |
| | | | 2 | 22.5039 | 22.5039 |
| | | | 3 | 38.7054 | 38.7054 |
| | | | 4 | 57.5386 | 57.5386 |
| 2 | 40 | 31 | 0 | 0 | – |
| | | | 1 | 12.0796 | 12.0796 |
| | | | 2 | 26.6481 | 26.6481 |
| | | | 3 | 43.9442 | 43.9442 |
| | | | 4 | 63.6794 | 63.6794 |
| 3 | 41 | 34 | 0 | 0 | – |
| | | | 1 | 14.3375 | 14.3375 |
| | | | 2 | 30.4825 | 30.4825 |
| | | | 3 | 48.9339 | 48.9339 |
| | | | 4 | 69.6100 | 69.6100 |
| 6 | 43 | 37 | 0 | 0 | – |
| | | | 1 | 19.7888 | 19.7888 |
| | | | 2 | 40.4403 | 40.4403 |
| | | | 3 | 62.487 | 62.487 |
| | | | 4 | 86.145 | 86.145 |

## 4. Fokker–Planck equation and AIM

The general form of the Fokker–Planck equation is given by[1,2]

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} + \beta U'(x)\right] P(x,t). \tag{19}$$

In the stationary case, the probability density can be written as $P(x,t) = \mathrm{e}^{-Et}\psi(x)$, and equation (19) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta U'(x)\frac{\mathrm{d}}{\mathrm{d}x} + \beta U''(x)\right]\psi(x) = -E\psi(x). \quad (20)$$

Now using transformation $\psi(x) = \mathrm{e}^{-\beta U/2}\phi(x)$, we can bring Eq. (20) into the Schrödinger form

$$\begin{aligned} &A^\dagger A\phi = \lambda\phi, \\ &A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x), \\ &W(x) = \frac{1}{2}\beta U'(x). \end{aligned} \quad (21)$$

It is now possible to solve Eq. (21) using various methods available for the Schrödinger equation. However, from Eq. (20), it is clear that the form of the Fokker–Planck equation is particularly suitable for application of the AIM. Here our objective would be to apply the AIM directly to Eq. (20) and examine the relative advantages of applying the AIM directly to Eq. (20) or its equivalent form in Eq. (21).

To apply the AIM, we first have to remove the asymptotic part from Eq. (20). To do this, we consider a wave function of the form $\psi(x) = \exp(-\beta U - \alpha x^2/2)f(x)$. After substituting this in Eq. (20), we obtain

$$\begin{aligned} f''(x) = &(2\alpha x + \beta U')f' \\ &- (E - \alpha + \alpha\beta x U' + \alpha^2 x^2)f. \end{aligned} \quad (22)$$

Now to make a comparison with the earlier results, we consider

$$\frac{1}{2}\beta U'(x) = x^3. \quad (23)$$

Then from Eq. (22), we obtain

$$\begin{aligned} f''(x) = &2(x^3 + \alpha x)f'(x) \\ &- (E - \alpha + 2\alpha x^4 + \alpha^2 x^2)f(x). \end{aligned} \quad (24)$$

We now apply the AIM to Eq. (24). In our calculations, we have taken $\alpha = 4$ and $x_0 = 0$. We show the computed eigenvalues in Table 4.

**Table 4.** Eigenvalues of the Fokker–Planck equation (24).

| $n$ | $E_{\mathrm{AIM}}$ |
|---|---|
| 0 | 0 |
| 1 | 1.93548 |
| 2 | 6.2985 |
| 3 | 11.681 |
| 4 | 18.0426 |

From Table 4, we find that the eigenvalues are exactly the same as those in the equivalent Schrödinger case. Thus the AIM can be directly applied to the Fokker–Planck equation.

## 5. Conclusion

We have applied the AIM to several QES potentials and their supersymmetric partners. In all the cases, the eigenvalues found are in excellent agreement with the existing results. Interestingly, it has been found that for most potentials, the AIM requires fewer iterations when applied to the partner potentials, which are often the potentials with a single minimum. Finally, we have also discussed the direct application of the AIM to the Fokker–Planck equation. We believe it would be interesting to apply the AIM to the Fokker–Planck equation when the drift coefficient has a non-polynomial nature.

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