New theorem relating two-mode entangled tomography to two-mode Fresnel operator*

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Based on the Fan–Hu's formalism, i.e., the tomogram of two-mode quantum states can be considered as the module square of the states' wave function in the intermediate representation, which is just the eigenvector of the Fresnel quadrature phase, we derive a new theorem for calculating the quantum tomogram of two-mode density operators, i.e., the tomogram of a two-mode density operator is equal to the marginal integration of the classical Weyl correspondence function of $F_2^{\dagger}\rho F_2$, where F_2 is the two-mode Fresnel operator. An application of the theorem in evaluating the tomogram of an optical chaotic field is also presented.

Keywords: two-mode quantum tomogram, two-mode Fresnel operator, Weyl correspondence

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1. Introduction

In quantum optics, the theory of tomography is a hot topic.[1−11] In Ref. [11], we found a new theorem in calculating the tomogram of density operators for quantum states, which not only is very convenient for the mixed state case, but also reveals the relationship between the tomogram of density operator ρ and the Fresnel transformed ρ's classical Weyl correspondence function. The theorem may be applied to calculating the tomogram of more complicated quantum states in a neat and concise way. The above theorem is for the single-mode case. The question naturally arises: can we extend the thorem to the two-mode case? Fortunately, a new formalism of two-mode quantum tomogram has been recently established.[12] It has been shown that for the tomographic approach, there exists a quantum mechanical representation $|\eta\rangle_{s,r}$ (named the entangled intermediate representation); and the Radon transform of the two-mode Wigner operator is just the pure-state density matrices $|\eta\rangle_{s,r\ s,r}\langle\eta|$. As a result, the tomogram of two-mode quantum states can be considered as the module square of the states' wave functions in the entangled intermediate representation. In a very recent paper,[13] Fan and Hu further found that $|\eta\rangle_{s,r} = F_2|\eta\rangle$, where F_2 is the two-mode Fresnel operator proposed as the quantum image of the classical Fresnel transformation,[14,15] thus the probability distribution for the Fresnel quadrature phase is the core of tomography. In this way, the Radon transform of the Wigner functions can be formulated much more simply and elegantly, which would enable us to evaluate the tomogram of quantum two-mode mixed states in a more direct way. We also find a new theorem, i.e., the two-mode quantum tomogram of a density operator ρ is equal to the marginal integration of the classical Weyl correspondence function of $F_2^{\dagger}\rho F_2$.

The rest of this paper is arranged as follows. In Section 2, we briefly review the relationship between the two-mode Fresnel transformation and the entangled quantum tomography. In Section 3, we derive the new theorem. As an application of the theorem, in Section 4, we calculate the tomogram of a two-mode optical chaotic field. Throughout the paper, we fully employ the technique of integration within the Weyl ordered product (IWOP) of operators and the Weyl ordering invariance under similar transformations.

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2. Relationship between two-mode Fresnel transformation and entangled quantum tomography

In Ref. [13], Fan and Hu have proved that the two-mode Wigner operator's Radon transform is

$$\pi\iint \mathrm{d}^2\gamma\mathrm{d}^2\sigma\Delta\left(\sigma,\gamma\right)\delta\left(\eta_2-D\sigma_2+B\gamma_1\right)\times\delta\left(\eta_1-D\sigma_1-B\gamma_2\right)=\left|\eta\right\rangle_{s,r\ s,r}\left\langle\eta\right|, \quad (1)$$

or

$$\pi\iint \mathrm{d}^2\gamma\mathrm{d}^2\sigma\Delta\left(\sigma,\gamma\right)\delta\left(\xi_1-A\sigma_1-C\gamma_2\right)\times\delta\left(\xi_2-A\sigma_2+C\gamma_1\right)=\left|\xi\right\rangle_{s,r\ s,r}\left\langle\xi\right|, \quad (2)$$

where σ and γ are complex variables, $\sigma=\sigma_1+\mathrm{i}\sigma_2$, $\gamma=\gamma_1+\mathrm{i}\gamma_2$, and $\Delta\left(\sigma,\gamma\right)$ is the two-mode Wigner operator for the entangled two-body system, its normal ordering form[16] is

$$\begin{aligned}&\Delta\left(\sigma,\gamma\right)\\&=\frac{1}{\pi^2}:\exp\left[-\left(\sigma_1-\frac{Q_1-Q_2}{\sqrt{2}}\right)^2-\left(\sigma_2-\frac{P_1+P_2}{\sqrt{2}}\right)^2\right.\\&\left.-\left(\gamma_1-\frac{Q_1+Q_2}{\sqrt{2}}\right)^2-\left(\gamma_2-\frac{P_1-P_2}{\sqrt{2}}\right)^2\right]:,\end{aligned} \quad (3)$$

where $Q_i=(a_i+a_i^\dagger)/\sqrt{2}$ and $P_i=(a_i-a_i^\dagger)/\sqrt{2}\mathrm{i}$ are the quadratures for the two-body system ($i=1,2$), and s and r are two complex parameters which are related to $(A;B;C;D)$ through

$$\begin{aligned}A&=\frac{1}{2}\left(s^*-r^*+s-r\right),\\B&=\frac{1}{2\mathrm{i}}\left(s^*-s+r^*-r\right),\\C&=\frac{1}{2\mathrm{i}}\left(s-r-s^*+r^*\right),\\D&=\frac{1}{2}\left(s^*+s+r^*+r\right),\end{aligned} \quad (4)$$

with the constraint $|s|^2-|r|^2=1$ or $AD-BC=1$. The Radon projection in Eq. (1) is in the $(B;D)$ direction, while it is in the $(A;C)$ direction in Eq. (2). The state vector $\left|\eta\right\rangle_{s,r}$ is named as the entangled intermediate representation, and $\left|\xi\right\rangle_{s,r}$ is just its conjugate. We have

$$\int \mathrm{d}^2\eta\left|\eta\right\rangle_{s,r\ s,r}\left\langle\eta\right|=1,\quad \int \mathrm{d}^2\xi\left|\xi\right\rangle_{s,r\ s,r}\left\langle\xi\right|=1. \quad (5)$$

It has been revealed in Ref. [13] that

$$\left|\eta\right\rangle_{s,r}=F_2\left(r,s\right)\left|\eta\right\rangle,\quad \left|\xi\right\rangle_{s,r}=F_2\left(r,s\right)\left|\xi\right\rangle, \quad (6)$$

where

$$\left|\eta\right\rangle=\exp\left(-\frac{\left|\eta\right|^2}{2}+\eta a_1^\dagger-\eta^* a_2^\dagger+a_1^\dagger a_2^\dagger\right)\left|00\right\rangle \quad (7)$$

is in the entangled representation and is the common eigenvector of P_1+P_2 and Q_1-Q_2 for the two-body system, a_i ($i=1,2$) are the two field modes for the two-body system

$$\left|\xi\right\rangle=\exp\left(-\frac{1}{2}\left|\xi\right|^2+\xi a_1^\dagger+\xi^* a_2^\dagger-a_1^\dagger a_2^\dagger\right)\left|00\right\rangle \quad (8)$$

is in the frequency domain and is the common eigenvector of P_1-P_2 and Q_1+Q_2 for the two-body system. The $F_2\left(r,s\right)$ is the so-called two-mode Fresnel operator[14,15]

$$\begin{aligned}F_2\left(r,s\right)=&\exp\left(\frac{r}{s^*}a_1^\dagger a_2^\dagger\right)\exp\left(-\frac{r^*}{s^*}a_1a_2\right)\\&\times\exp\left[\left(a_1^\dagger a_1+a_2^\dagger a_2+1\right)\ln\left(s^*\right)^{-1}\right],\end{aligned} \quad (9)$$

which generates the two-dimensional kernel of the classical optical Fresnel diffraction

$$\begin{aligned}&\kappa_2^{(r,s)}\left(\eta',\eta\right)\\&=\frac{1}{2\mathrm{i}B\pi}\exp\left[\frac{\mathrm{i}}{2B}\left(A\left|\eta\right|^2-\left(\eta\eta'^*+\eta^*\eta'\right)\right)+D\left|\eta'\right|^2\right]\\&=\kappa_2^M\left(\eta',\eta\right),\end{aligned} \quad (10)$$

where superscript M only means that the parameters of κ_2^M are $[A,B;C,D]$, and subscript 2 implies the two dimensional kernel. The two-mode entangled Wigner operator[16,17] in the η representation is

$$\begin{aligned}&\Delta\left(\sigma,\gamma\right)\\&=\int\frac{\mathrm{d}^2\eta}{\pi^3}\left|\sigma-\eta\right\rangle\left\langle\sigma+\eta\right|\exp\left(\eta\gamma^*-\eta^*\gamma\right),\end{aligned} \quad (11)$$

in the ξ representation

$$\begin{aligned}&\Delta\left(\sigma,\gamma\right)\\&=\int\frac{\mathrm{d}^2\xi}{\pi^3}\left|-\xi+\gamma\right\rangle\left\langle\xi+\gamma\right|\exp\left(\xi^*\sigma-\xi\sigma^*\right).\end{aligned} \quad (12)$$

3. A new theorem relating quantum two-mode tomogram to two-mode Fresnel operator

Multiplying both sides of Eq. (1) by a density matrix and then performing the trace, noting the Wigner function $W(\sigma,\gamma)=\mathrm{Tr}(\rho\Delta(\sigma,\gamma))$, we have

$$\mathrm{Tr}\left(\rho\left|\eta\right\rangle_{s,r\ s,r}\left\langle\eta\right|\right)$$

$$
\begin{aligned}
&= \pi \iint \mathrm{d}^2\gamma \mathrm{d}^2\sigma \mathrm{Tr}\left(\rho \Delta\left(\sigma,\gamma\right)\right) \\
&\quad \times \delta\left(\eta_2 - D\sigma_2 + B\gamma_1\right)\delta\left(\eta_1 - D\sigma_1 - B\gamma_2\right) \\
&= {}_{s,r}\left\langle \eta\right| \rho \left|\eta\right\rangle_{s,r} \\
&= \left\langle \eta\right| F_2^{\dagger}\left(r,s\right)\rho F_2\left(r,s\right)\left|\eta\right\rangle \\
&= \pi \iint \mathrm{d}^2\gamma \mathrm{d}^2\sigma W\left(\sigma,\gamma\right)\delta\left(\eta_2 - D\sigma_2 + B\gamma_1\right) \\
&\quad \times \delta\left(\eta_1 - D\sigma_1 - B\gamma_2\right). \qquad (13)
\end{aligned}
$$

The right hand side of Eq. (13) is commonly defined as the tomogram of two-mode quantum states in the $(B;D)$ direction. So the calculation of the tomogram in the $(B;D)$ direction is ascribed to calculating

$$
\left\langle \eta\right| F_2^{\dagger}\left(r,s\right)\rho F_2\left(r,s\right)\left|\eta\right\rangle, \qquad (14)
$$

which is a concise and neat formula. Similarly, the tomogram in the $(A;C)$ direction is ascribed to

$$
\left\langle \xi\right| F_2^{\dagger}\left(r,s\right)\rho F_2\left(r,s\right)\left|\xi\right\rangle. \qquad (15)
$$

The Weyl correspondence rule[18] is

$$
H\left(a_1^{\dagger},a_2^{\dagger};a_1,a_2\right) = \iint \mathrm{d}^2\sigma \mathrm{d}^2\gamma h\left[\sigma,\gamma\right]\Delta\left(\sigma,\gamma\right), \qquad (16)
$$

and the Weyl ordering form of $\Delta\left(\sigma,\gamma\right)$[19] is

$$
\begin{aligned}
&\Delta\left(\sigma,\gamma\right) \\
&= \vdots \delta^{(2)}\left(\sigma - a_1 + a_2^{\dagger}\right)\delta^{(2)}\left(\gamma - a_2^{\dagger} - a_1\right)\vdots \\
&= \vdots \delta\left(\sigma - a_1 + a_2^{\dagger}\right)\delta\left(\sigma^* - a_1^{\dagger} + a_2\right) \\
&\quad \times \delta\left(\gamma - a_2^{\dagger} - a_1\right)\delta\left(\gamma - a_2 - a_1^{\dagger}\right)\vdots, \qquad (17)
\end{aligned}
$$

where symbol $\vdots\ \vdots$ denotes the Weyl ordering. For the coordinate and the moment operators of the two-body system, $Q_i = (a_i + a_i^{\dagger})/\sqrt{2}$, $P_i = (a_i - a_i^{\dagger})/(\sqrt{2}\mathrm{i})$ $(i = 1,2)$, the Weyl correspondence rule can be rewritten as

$$
\begin{aligned}
&H\left[\left(Q_1 - Q_2, P_1 + P_2\right),\left(P_1 - P_2, Q_1 + Q_2\right)\right] \\
&= \iint \mathrm{d}^2\sigma \mathrm{d}^2\gamma h\left[\sqrt{2}\sigma,\sqrt{2}\gamma\right]\Delta\left(\sigma,\gamma\right), \qquad (18)
\end{aligned}
$$

and the classical correspondence of the Weyl ordered operator

$$
\vdots h\left[\left(Q_1 - Q_2, P_1 + P_2\right),\left(P_1 - P_2, Q_1 + Q_2\right)\right]\vdots \qquad (19)
$$

is obtained by the replacement of $(Q_1 - Q_2, P_1 + P_2) \to (\sqrt{2}\sigma_1, \sqrt{2}\sigma_2)$, $(P_1 - P_2, Q_1 + Q_2) \to (\sqrt{2}\gamma_1, \sqrt{2}\gamma_2)$ in h, i.e.,

$$
\begin{aligned}
&\vdots h\left[\left(Q_1 - Q_2, P_1 + P_2\right),\left(P_1 - P_2, Q_1 + Q_2\right)\right]\vdots \\
&= \iint \mathrm{d}^2\sigma \mathrm{d}^2\gamma h\left[\sqrt{2}\sigma,\sqrt{2}\gamma\right]\Delta\left(\sigma,\gamma\right). \qquad (20)
\end{aligned}
$$

Let the classical Weyl correspondence of $F_2^{\dagger}\left(r,s\right)\rho F_2\left(r,s\right)$ be $h\left[\sqrt{2}\sigma,\sqrt{2}\gamma\right]$

$$
\begin{aligned}
&F_2^{\dagger}\left(r,s\right)\rho F_2\left(r,s\right) \\
&= \iint \mathrm{d}^2\sigma \mathrm{d}^2\gamma h\left[\sqrt{2}\sigma,\sqrt{2}\gamma\right]\Delta\left(\sigma,\gamma\right), \qquad (21)
\end{aligned}
$$

then using Eqs. (14) and (21) and the orthogonal relation of the $|\eta\rangle$ representation

$$
\left\langle \eta'\right|\left.\eta\right\rangle = \pi\delta^{(2)}\left(\eta' - \eta\right), \qquad (22)
$$

we have

$$
\begin{aligned}
\Xi &= \left\langle \eta\right| F_2^{\dagger}\left(r,s\right)\rho F_2\left(r,s\right)\left|\eta\right\rangle \\
&= \left\langle \eta\right|\left.\iint \mathrm{d}^2\eta' \mathrm{d}^2\xi h\left[\sqrt{2}\eta',\sqrt{2}\xi\right]\Delta\left(\eta',\xi\right)\right|\eta\right\rangle \\
&= \left\langle \eta\right|\left.\iint \mathrm{d}^2\eta' \mathrm{d}^2\xi h\left[\sqrt{2}\eta',\sqrt{2}\xi\right]\int \frac{\mathrm{d}^2\sigma}{\pi^3}\right|\eta' - \sigma\right\rangle \\
&\quad \times \left\langle \eta' + \sigma\right|\left.\exp\left(\sigma\xi^* - \sigma^*\xi\right)\right|\eta\right\rangle \\
&= \iint \mathrm{d}^2\eta' \mathrm{d}^2\xi h\left[\sqrt{2}\eta',\sqrt{2}\xi\right] \\
&\quad \times \int \frac{\mathrm{d}^2\sigma}{\pi^3}\exp\left(\sigma\xi^* - \sigma^*\xi\right)\left\langle \eta\right|\left.\eta' - \sigma\right\rangle\left\langle \eta' + \sigma\right|\left.\eta\right\rangle \\
&= \iint \mathrm{d}^2\eta' \mathrm{d}^2\xi h\left[\sqrt{2}\eta',\sqrt{2}\xi\right] \\
&\quad \times \int \frac{\mathrm{d}^2\sigma}{\pi^3}\exp\left(\sigma\xi^* - \sigma^*\xi\right) \\
&\quad \times \pi\delta^{(2)}\left(\eta - \eta' + \sigma\right)\pi\delta^{(2)}\left(\eta - \eta' - \sigma\right) \\
&= \frac{1}{\pi}\iint \mathrm{d}^2\eta' \mathrm{d}^2\xi h\left[\sqrt{2}\eta',\sqrt{2}\xi\right] \\
&\quad \times \exp[\left(\eta - \eta'\right)\xi^* - \left(\eta - \eta'\right)^*\xi]\delta^{(2)}\left(2\eta - 2\eta'\right) \\
&= \frac{1}{2\pi}\int \mathrm{d}^2\xi h\left[\sqrt{2}\eta,\sqrt{2}\xi\right]. \qquad (23)
\end{aligned}
$$

Thus we obtain a new theorem as follows. The tomogram of a two-mode density operator describing a two-body system is equal to the marginal integration of the classical Weyl correspondence $h\left[\sqrt{2}\eta,\sqrt{2}\xi\right]$ of $F_2^{\dagger}\left(r,s\right)\rho F_2\left(r,s\right)$, i.e.,

$$
\mathrm{Tr}\left(\rho\left|\eta\right\rangle_{s,r\ \ s,r}\left\langle \eta\right|\right) = \frac{1}{2\pi}\int \mathrm{d}^2\gamma h\left[\sqrt{2}\eta,\sqrt{2}\gamma\right], \qquad (24)
$$

or

$$
\mathrm{Tr}\left(\rho\left|\xi\right\rangle_{s,r\ \ s,r}\left\langle \xi\right|\right) = \frac{1}{2\pi}\int \mathrm{d}^2\sigma h\left[\sqrt{2}\sigma,\sqrt{2}\xi\right]. \qquad (25)
$$

When the two-mode operator reduces to the one-mode operator, we can see that the two-mode theorem reduces to the one-mode case[11] as expected.

4. Tomogram of two-mode optical chaotic field

As an application of the above theorem, we now consider the density operator of a two-mode chaotic field, which is expressed as

$$\rho_{\rm c} = \left(1 - {\rm e}^{\theta}\right)^2 {\rm e}^{\theta\left(a^{\dagger}a + b^{\dagger}b\right)}, \tag{26}$$

where $\theta = -\omega\hbar/kT$, k is the Boltzmann constant, ω is the frequency, and T is the temperature of the two-mode chaotic field. The $\rho_{\rm c}$ is qualified to be a density operator, and ${\rm Tr}\rho_{\rm c} = 1$. In order to evaluate the classical Weyl correspondence of $F_2^{\dagger}(r,s)\,\rho_{\rm c}F_2(r,s)$, we first present the Weyl ordered form[20] of $\left(1 - {\rm e}^{\theta}\right)^2 {\rm e}^{\theta\left(a^{\dagger}a + b^{\dagger}b\right)}$

$$\begin{aligned}
&\left(1 - {\rm e}^{\theta}\right)^2 {\rm e}^{\theta\left(a^{\dagger}a + b^{\dagger}b\right)} \\
&= \left(2\frac{1 - {\rm e}^{\theta}}{1 + {\rm e}^{\theta}}\right)^2 \\
&\quad \times \vdots \exp\left[-\frac{1 - {\rm e}^{\theta}}{1 + {\rm e}^{\theta}}\left(P_1^2 + P_2^2 + Q_1^2 + Q_2^2\right)\right]\vdots .
\end{aligned} \tag{27}$$

Then using operators' Weyl ordering invariance under similar transformations[21] and the property under the two-mode Fresnel transformation

$$F_2^{\dagger}a_1F_2 = ra_2^{\dagger} + sa_1, \quad F_2^{\dagger}a_2F_2 = ra_1^{\dagger} + sa_2, \tag{28}$$

noticing $Q_i = (a_i + a_i^{\dagger})/\sqrt{2}$, $P_i = (a_i - a_i^{\dagger})/{\rm i}\sqrt{2}$ $(i = 1, 2)$, and letting $\beta \equiv (1 - {\rm e}^{\theta})/(1 + {\rm e}^{\theta})$, we can convert $F_2^{\dagger}(r,s)\,\rho_{\rm c}F_2(r,s)$ into its Weyl ordering form

$$\begin{aligned}
&F_2^{\dagger}(r,s)\,\rho_{\rm c}F_2(r,s) \\
&= F_2^{\dagger}(r,s)\left(2\frac{1-e^{\theta}}{1+e^{\theta}}\right)^2 \\
&\quad \times \vdots \exp\left[-\frac{1-e^{\theta}}{1+e^{\theta}}\left(P_1^2 + P_2^2 + Q_1^2 + Q_2^2\right)\right]\vdots F_2(r,s) \\
&= 4\beta^2 \vdots \exp\left\{-\beta\left[\left(ss^* + rr^*\right)\left(Q_1^2 + Q_2^2 + P_1^2 + P_2^2\right)\right.\right. \\
&\quad + 4\Re\left(sr^*\right)\left(Q_1Q_2 - P_1P_2\right) \\
&\quad \left.\left. - 4\Im\left(sr^*\right)\left(Q_1P_2 + Q_2P_1\right)\right]\right\}\vdots .
\end{aligned} \tag{29}$$

So according to Eq. (20), the classical Weyl correspondence function $h_{\rm c}(\sqrt{2}\eta;\sqrt{2}\gamma)$ of $F_2^{\dagger}(r,s)\,\rho_{\rm c}F_2(r,s)$ is

$$\begin{aligned}
&4\beta^2 \exp\left(-\beta\left\{\left(ss^* + rr^*\right)\left[\left(\frac{\eta_1 + \gamma_1}{\sqrt{2}}\right)^2 + \left(\frac{\eta_1 - \gamma_1}{\sqrt{2}}\right)^2 + \left(\frac{\eta_2 + \gamma_2}{\sqrt{2}}\right)^2 + \left(\frac{\eta_2 - \gamma_2}{\sqrt{2}}\right)^2\right]\right.\right. \\
&\quad \left.\left. + 4\Re\left(sr^*\right)\left(\frac{\eta_1 + \gamma_1}{\sqrt{2}}\frac{\gamma_1 - \eta_1}{\sqrt{2}} - \frac{\eta_2 + \gamma_2}{\sqrt{2}}\frac{\eta_2 - \gamma_2}{\sqrt{2}}\right) - 4\Im\left(sr^*\right)\left(\frac{\eta_1 + \gamma_1}{\sqrt{2}}\frac{\eta_2 - \gamma_2}{\sqrt{2}} + \frac{\gamma_1 - \eta_1}{\sqrt{2}}\frac{\eta_2 + \gamma_2}{\sqrt{2}}\right)\right\}\right) \\
&= 4\beta^2 \exp\left\{-\beta\left[\left(ss^* + rr^*\right)\left(|\eta|^2 + |\gamma|^2\right) + 4\Re\left(sr^*\right)\left(\frac{|\gamma|^2}{2} - \frac{|\eta|^2}{2}\right) - 4\Im\left(sr^*\right)\left(\gamma_1\eta_2 - \eta_1\gamma_2\right)\right]\right\}.
\end{aligned} \tag{30}$$

Substituting Eq. (30) into Eq. (24), we derive the tomogram of the two-mode chaotic field

$$\begin{aligned}
&{\rm Tr}\left(\rho_{\rm c}\left|\eta\right\rangle_{s,r\ s,r}\left\langle\eta\right|\right) \\
&= \frac{1}{2\pi}\int {\rm d}^2\gamma h\left[\sqrt{2}\eta, \sqrt{2}\gamma\right] \\
&= 4\beta^2\frac{1}{2\pi}\int {\rm d}^2\gamma \exp\left\{-\beta\left[\left(ss^* + rr^*\right)\left(|\eta|^2 + |\gamma|^2\right) + 4\Re\left(sr^*\right)\left(\frac{|\gamma|^2}{2} - \frac{|\eta|^2}{2}\right) - 4\Im\left(sr^*\right)\left(\gamma_1\eta_2 - \eta_1\gamma_2\right)\right]\right\} \\
&= 4\beta^2\frac{1}{2\pi}\exp\left\{-\beta\left[\left(ss^* + rr^*\right)|\eta|^2 - 2\Re\left(sr^*\right)|\eta|^2\right]\right\} \\
&\quad \times \int {\rm d}\gamma_1 \exp\left[-\beta\tau\gamma_1^2 + 4\beta\Im\left(sr^*\right)\eta_2\gamma_1\right]\int {\rm d}\gamma_2 \exp\left[-\beta\tau\gamma_2^2 - 4\beta\Im\left(sr^*\right)\eta_1\gamma_2\right] \\
&= \frac{\beta}{\tau}\exp\left\{-\beta\left[\left(ss^* + rr^*\right)|\eta|^2 - 2\Re\left(sr^*\right)|\eta|^2\right] + \frac{4\beta}{\tau}\Im^2\left(sr^*\right)|\eta|^2\right\},
\end{aligned} \tag{31}$$

where

$$\left(ss^* + rr^*\right) + 2\Re\left(sr^*\right) = \tau. \tag{32}$$

By noticing Eq. (4),

$$\begin{aligned}
\Re\left(r^*s\right) &= \frac{1}{4}\left(-A^2 - C^2 + B^2 + D^2\right), \\
\Im\left(r^*s\right) &= -\frac{1}{2}\left(BA + CD\right),
\end{aligned}$$

$$(ss^* + rr^*) = \frac{1}{2}\left(A^2 + C^2 + B^2 + D^2\right), \tag{33}$$

and $AD - BC = 1$, we can rewrite Eq. (31) as

$$\begin{aligned}\mathrm{Tr}\left(\rho_{\mathrm{c}} |\eta\rangle_{s,r\ s,r}\langle\eta|\right) &= \frac{1}{2\pi}\int \mathrm{d}^2\gamma h_{\mathrm{c}}(\sqrt{2}\eta, \sqrt{2}\gamma) \\ &= \frac{\beta}{(B^2+D^2)}\exp\left[-\beta\frac{\left(A^2+C^2\right)\left(B^2+D^2\right)-(BA+CD)^2}{(B^2+D^2)}|\eta|^2\right] \\ &= \frac{\beta}{(B^2+D^2)}\exp\left[-\frac{\beta}{(B^2+D^2)}|\eta|^2\right], \end{aligned} \tag{34}$$

which is the the tomogram of ρ_{c} in the $(B; D)$ direction. It is a normal distribution and

$$\int \frac{\mathrm{d}^2\eta}{\pi}\frac{\beta}{(B^2+D^2)}\exp\left[-\frac{\beta}{(B^2+D^2)}|\eta|^2\right] = 1. \tag{35}$$

Similarly, we can derive the tomogram of ρ_{c} in the $(A; C)$ direction

$$\begin{aligned}&\mathrm{Tr}\left(\rho|\xi\rangle_{s,r\ s,r}\langle\xi|\right) \\ &= \frac{1}{2\pi}\int \mathrm{d}^2\sigma h\left[\sqrt{2}\sigma, \sqrt{2}\xi\right] \\ &= \frac{\beta}{(A^2+C^2)}\exp\left[-\frac{\beta}{(A^2+C^2)}|\xi|^2\right], \end{aligned} \tag{36}$$

which is also a normal distribution with

$$\int \frac{\mathrm{d}^2\xi}{\pi}\frac{\beta}{(A^2+C^2)}\exp\left[-\frac{\beta}{(A^2+C^2)}|\xi|^2\right] = 1. \tag{37}$$

By choosing $\beta = 1/2$ and $(A; B; C; D) = (1/2; 2; 1/2; 4)$, which satisfy relation $AD - BC = 1$, we draw curves representing the tomogram of ρ_{c} in the $(B; D)$ and the $(A; C)$ directions in Figs. 1 and 2, respectively. Not only the normal distributions, but also the difference between the two directions are shown. Note that the two curves are complemented to each other, the shape of the Fig. 1 curve is short and fat, while that of the Fig. 2 curve is tall and slim, since $AD - BC = 1$.

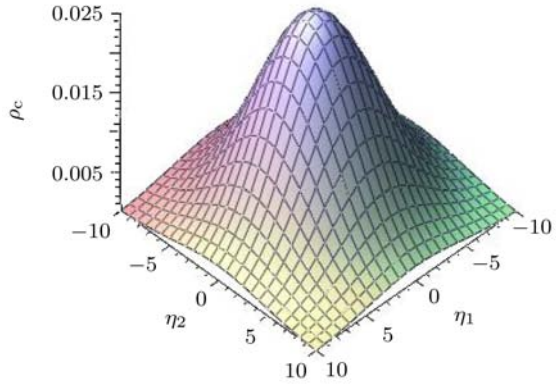


Fig. 1. (colour online) Tomogram of ρ_{c} in the (B, D) direction.

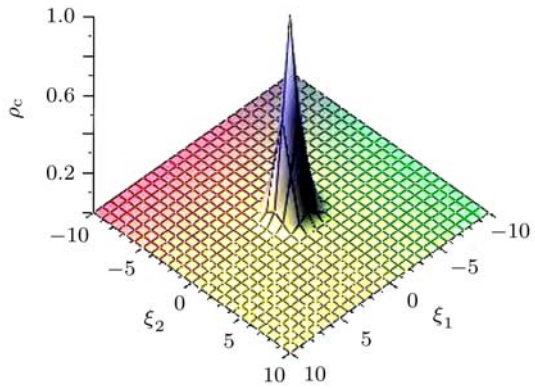


Fig. 2. (colour online) Tomogram of ρ_{c} in the (A, C) direction.

5. Conclusion

In summary, we have found a new theorem for calculating the tomogram for the density operators of a two-mode system, which is not only convenient for the two-mode mixed state cases, but also reveals the relationship between the tomogram of the two-mode density operator ρ and the Fresnel transformed ρ's classical Weyl correspondence function. The theorem may be applied to calculating the tomogram of more complicated quantum states in a neat and concise way.

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