Asymptotic solving method for a sea–air oscillator model of atmospheric physics*

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In this paper, a class of coupled system for the El Niño/La Niña southern oscillation (ENSO) atmospheric physics oscillation model is considered. We propose an ENSO atmospheric physics model using a method from the asymptotic theory. It is indicated from the results that the asymptotic method can be used for analyzing the sea surface temperature anomaly and the thermocline depth anomaly of the atmosphere–ocean oscillation for the ENSO model in the equatorial Pacific.

Keywords: nonlinear, asymptotic theory, El Niña southern oscillation model

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1. Introduction

The El Niño/La Niña southern oscillation (ENSO) is an interannual phenomenon involved in tropical Pacific ocean–atmosphere interactions. The ENSO is a very attractive phenomenon and has been studied by many researchers.[1−5] The oscillatory nature of the ENSO requires both positive and negative ocean–atmosphere feedback. We consider an initial positive sea surface temperature (SST) anomaly in the equatorial eastern Pacific. This anomaly reduces the zonal SST gradient and hence the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn drive the ocean circulation changes, which reinforce the SST anomaly. This positive ocean–atmosphere feedback or coupled ocean–atmosphere instability leads to the equatorial Pacific' never-ending warm state. Lin *et al* have considered a class of problems in atmospheric physics, ocean weather, dynamic systems, and so on.[6−7] Recently, many scholars have investigated the nonlinear problem. Approximation methods have been developed and refined, including the method of averaging, the boundary layer method, the matched asymptotic expansion method and the multiple scale method.[8−12] Using the asymptotic method, Mo *et al.* have considered a class of reaction diffusion problems, the activator inhibitor systems, the ecological environment, the shock wave, the solitary wave, the laser pulse, the ocean science, atmospheric physics, etc.[13−19] In this paper, we consider a class of ENSO models by using a simple and special method for the asymptotic theory.

The conceptual oscillator model should consider the variations of both the eastern and the western Pacific anomaly patterns. The oscillator model can be reduced to a simple oscillator model by further simplification and using additional assumptions. The western Pacific oscillator proposed by Wang *et al.*[2,3] emphasizes the role of the western Pacific anomaly patterns in the ENSO. In particular, the off-equatorial daytime SST initiates equatorial wind anomalies in the western Pacific. Those wind anomalies force ocean responses that proceed eastward to affect anomalies in the equatorial eastern Pacific. If all delay parameters are zero, we know that the slow ocean dynamical adjustment does not necessarily need the explicit role

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of wave propagation. We now consider a generalized sea–air oscillator of the ENSO model:[4,5]

$$\frac{\mathrm{d}T}{\mathrm{d}t} = a\tau_1 + b\tau_2 - \delta_1 T^3, \tag{1}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -c\tau_1 - R_0 h - \delta_2 h^3, \tag{2}$$

$$\mu_1 \frac{\mathrm{d}\tau_1}{\mathrm{d}t} = dT - R_1\tau_1, \tag{3}$$

$$\mu_2 \frac{\mathrm{d}\tau_2}{\mathrm{d}t} = eh - R_2\tau_2, \tag{4}$$

where T is the SST anomaly in the equatorial eastern Pacific, h is the thermocline depth anomaly, τ_i, $(i = 1, 2,)$ are the equatorial zonal wind stress anomalies in the central and the western Pacific, respectively, the model parameters a, b, c, d, e, R_0, R_1, R_2 are positive constants with $aR_0R_2 \geq bce$, and $\delta = \delta_i$, $\mu = \mu_i$ $(i = 1, 2,)$ are small positive damping coefficients. We can obtain the asymptotic solution for the coupled system (1)–(4) of the ENSO model.

2. Asymptotic expansions of solution for ENSO

Set the solution $(T(t,\delta,\mu),\ h(t,\delta,\mu),\ \tau_1(t,\delta),\ \tau_2(t,\delta))$ of the nonlinear coupled system (1)–(4) to

$$T(t,\delta,\mu) = \sum_{i,j=0}^{\infty} T_{ij}(t)\delta^i\mu^j,$$

$$h(t,\delta,\mu) = \sum_{i,j=0}^{\infty} h_{ij}(t)\delta^i\mu^j, \tag{5}$$

$$\tau_l(t,\delta,\mu) = \sum_{i,j=0}^{\infty} \tau_{lij}(t)\delta^i\mu^j, \quad l = 1, 2. \tag{6}$$

Substituting Eqs. (5) and (6) into Eqs. (1)–(4) and letting $\delta = \mu = 0$, we obtain the reduced system of the original coupled system (1)–(4)

$$\frac{\mathrm{d}T_{00}}{\mathrm{d}t} = a\tau_{100} + b\tau_{200},$$

$$\frac{\mathrm{d}h_{00}}{\mathrm{d}t} = -c\tau_{100} - R_0 h_{00}, \tag{7}$$

$$\mathrm{d}T_{00} - R_1\tau_{100} = 0, \quad eh_{00} - R_2\tau_{200} = 0. \tag{8}$$

From Eq. (8), we have

$$\tau_{100} = \frac{d}{R_1}T_{00}, \quad \tau_{200} = \frac{e}{R_2}h_{00}. \tag{9}$$

Substituting Eq. (9) into Eq. (7), we obtain

$$\frac{\mathrm{d}T_{00}}{\mathrm{d}t} = \frac{ad}{R_1}T_{00} + \frac{be}{R_2}h_{00},$$

$$\frac{\mathrm{d}h_{00}}{\mathrm{d}t} = -\frac{cd}{R_1}T_{00} - R_0 h_{00}. \tag{10}$$

Thus Eq. (10) yields a set of solutions for the linear problem given by Eqs. (7) and (8)

$$T_{00}(t) = \frac{be}{R_2}C_{00}\exp(\lambda t),$$

$$h_{00}(t) = \left(\lambda - \frac{ad}{R_1}\right)C_{00}\exp(\lambda t), \tag{11}$$

$$\tau_{100}(t) = \frac{bde}{R_1R_2}C_{00}\exp(\lambda t),$$

$$\tau_{200}(t) = \frac{e}{R_2}\left(\lambda - \frac{ad}{R_1}\right)C_{00}\exp(\lambda t), \tag{12}$$

where C_{00} is an arbitrary constant. From $aR_0R_2 \geq bce$, we can have a negative characteristic value for Eq. (10)

$$\lambda = -\frac{R_0R_1 - ad}{2R_1} - \frac{1}{2}\left[\frac{(R_0R_1 - ad)^2}{R_1^2} + \frac{4d}{R_1R_2}(aR_0R_2 - bce)\right]^{\frac{1}{2}} < 0.$$

Thus we know that the atmosphere is in a steady state.

Substituting Eqs. (5) and (6) into Eqs. (1)–(4), for the coefficients of δ^0, μ^1, we have

$$\frac{\mathrm{d}T_{01}}{\mathrm{d}t} = a\tau_{101} + b\tau_{201},$$

$$\frac{\mathrm{d}h_{01}}{\mathrm{d}t} = -c\tau_{101} - R_0 h_{01}, \tag{13}$$

$$dT_{01} - R_1\tau_{101} = \frac{\mathrm{d}\tau_{100}}{\mathrm{d}t},$$

$$eh_{01} - R_2\tau_{201} = \frac{\mathrm{d}\tau_{200}}{\mathrm{d}t}. \tag{14}$$

From Eqs. (13) and (14), we have a set of solutions

$$T_{01}(t) = \left(\frac{be}{R_2}C_{01} + A_{01}t\right)\exp(\lambda t), \tag{15}$$

$$h_{01}(t) = \left[\left(\lambda - \frac{ad}{R_1}\right)C_{01} + B_{01}t\right]\exp(\lambda t), \tag{16}$$

$$\tau_{101}(t) = \frac{d}{R_1}\left(\frac{be}{R_2}C_{01} + A_{01}t\right)\exp(\lambda t), \tag{17}$$

$$\tau_{201}(t) = \frac{e}{R_2}\left[\left(\lambda - \frac{ad}{R_1}\right)C_{01} + B_{01}t\right]\exp(\lambda t)\Big], \tag{18}$$

where C_{01} is an arbitrary constant and

$$A_{01} = \frac{-[(\lambda + R_0)R_2D_{01} - beE_{01}]}{[bcde + (\lambda + R_0)(\lambda - ad)R_2]R_1R_2^2},$$

$$B_{01} = \frac{-[cdD_{01} + (\lambda R_1 - ad)E_{01}]}{[bcde + (\lambda + R_0)(\lambda - ad)R_2]R_2},$$
$$D_{01} = \frac{be\lambda C_{00}[adR_1 + (\lambda R_1 - ad)R_2]}{R_1^3 R_2},$$
$$E_{01} = \frac{ce\lambda C_{00}[\lambda R_1 - ad]}{R_1 R_2^2}.$$

Substituting Eqs. (5) and (6) into Eqs. (1)–(4), for the coefficients of δ^1, μ^0, we have

$$\frac{\mathrm{d}T_{10}}{\mathrm{d}t} = a\tau_{110} + b\tau_{210} - T_{00}^3,$$
$$\frac{\mathrm{d}h_{10}}{\mathrm{d}t} = -c\tau_{110} - R_0 h_{10} - h_{00}^3, \quad (19)$$
$$dT_{10} - R_1\tau_{110} = 0, \quad eh_{10} - R_2\tau_{210} = 0. \quad (20)$$

From Eqs. (19) and (20), we have a set of solutions

$$T_{10}(t) = \frac{be}{R_2}C_{10}\exp(\lambda t) + A_{10}\exp(3\lambda t), \quad (21)$$
$$h_{10}(t) = \left(\lambda - \frac{ad}{R_1}\right)C_{10}\exp(\lambda t) + B_{10}\exp(3\lambda t), \quad (22)$$
$$\tau_{110}(t) = \frac{d}{R_1}\left[\frac{be}{R_2}C_{10}\exp(\lambda t) + A_{10}\exp(3\lambda t)\right], \quad (23)$$
$$\tau_{210}(t) = \frac{e}{R_2}\left[\left(\lambda - \frac{ad}{R_1}\right)C_{10}\exp(\lambda t) + B_{10}\exp(3\lambda t)\right], \quad (24)$$

where C_{10} is an arbitrary constant and

$$A_{10} = \frac{-be[(be)^2(3\lambda + R_0)R_1^3 + (\lambda R_0 - ad)^3 R_2^2]C_{00}^3}{[bcde + (3\lambda + R_0)(3\lambda R_1 - ad)R_2]R_1^2 R_2^2},$$
$$B_{10} = \frac{-[cd(be)^3 R_1^3 + (3\lambda R_1 - ad)(\lambda R_1 - ad)^3 R_2^3]C_{00}^3}{[bcde + (3\lambda + R_0)(3\lambda R_1 - ad)R_2]R_1^3 R_2^2}.$$

From Eqs. (11), (12), (15)–(18) and (21)–(24), we can find the asymptotic expansions of a solution set for coupled systems (1)–(4) as

$$T_{\mathrm{asy}}(t) = \frac{be}{R_2}C_{00}\exp(\lambda t) + \left(\frac{be}{R_2}C_{01} + A_{01}t\right)\exp(\lambda t)\delta + \left[\frac{be}{R_2}C_{10}\exp(\lambda t) + A_{10}\exp(3\lambda t)\right]\mu + \mathrm{O}(\gamma), \quad (25)$$
$$h_{\mathrm{asy}}(t) = \left(\lambda - \frac{ad}{R_1}\right)C_{00}\exp(\lambda t) + \frac{-[cdD_{01} + (\lambda R_1 - ad)E_{01}]}{[bcde + (\lambda + R_0)(\lambda - ad)R_2]R_2}\delta + \left[\left(\lambda - \frac{ad}{R_1}\right)C_{10}\exp(\lambda t) + B_{10}\exp(3\lambda t)\right]\mu + \mathrm{O}(\gamma), \quad (26)$$
$$\tau_{1\mathrm{asy}}(t) = \frac{bde}{R_1 R_2}C_{00}\exp(\lambda t) + \frac{d}{R_1}\left(\frac{be}{R_2}C_{01} + A_{01}t\right)\exp(\lambda t)\delta + \frac{d}{R_1}\left[\frac{be}{R_2}C_{10}\exp(\lambda t) + A_{10}\exp(3\lambda t)\right]\mu + \mathrm{O}(\gamma), \quad (27)$$
$$\tau_{2\mathrm{asy}}(t) = \frac{e}{R_2}\left(\lambda - \frac{ad}{R_1}\right)C_{00}\exp(\lambda t) + \frac{e}{R_2}\left[\left(\lambda - \frac{ad}{R_1}\right)C_{01} + B_{01}t\right]\exp(\lambda t)]\delta + \frac{e}{R_2}\left[\left(\lambda - \frac{ad}{R_1}\right)C_{10}\exp(\lambda t) + B_{10}\exp(3\lambda t)\right]\mu + \mathrm{O}(\gamma), \quad (28)$$

where $\gamma = \max\{\delta^2, \delta\mu, \mu^2\}$.

We can obtain higher asymptotic expansions of the solution for the sea–air oscillator of the ENSO model (1)–(4) using the same method.

3. Accuracy of the approximate solution

In order to explain the accuracy of the expressions of the first approximate solution (25)–(28), we now select some peculiar parameters for the model:[3] $a = 1.5 \times 10^{-2}$ °C · m^2 · N^{-1} · a^{-1}, $b = 7.5 \times 10^2$ °C · m^2 · N^{-1} · a^{-1}, $c = 1.5 \times 10^3$ m^3 · N^{-1} · a^{-1}, $d = 3.6 \times 10^{-2}$ °C · m^2 · N^{-1} · a^{-1}, $e = 3.0 \times 10^{-1}$ N · m^{-3} · a^{-1}, $R_0 = 5.0$ a^{-1}, $R_1 = R_2 = 2.0$ a^{-1}, $\delta_1 = 0.1$ °C · m^2 · N^{-1} · a^{-1}, $\delta_2 = 0.1$ °C · m^2 · N^{-1} · a^{-1}, $\mu_i = 0.05$ °C · m^2 · N^{-1} · a^{-1}, $i = 1, 2$, $t \in [0, 4]$, $[T, h, \tau_1, \tau_2]_{t=0} = [1, 0, 0, 0]$. Then we obtain the curves of the SST anomaly T shown in

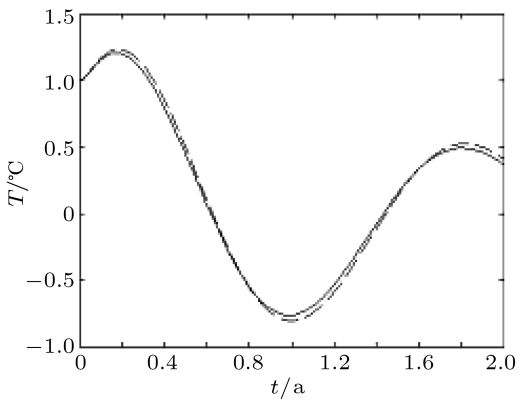


Fig. 1. SST anomaly T obtained in simulation (solid line) and asymptotic approximate (dashed line) solutions.

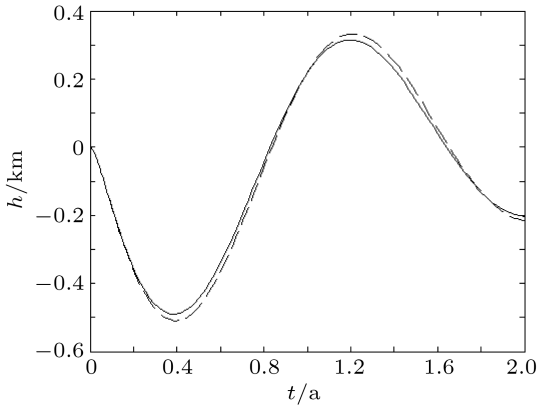


Fig. 2. Thermocline depth anomaly h obtained in simulation (solid line) and asymptotic approximate (dashed line) solutions.

Fig. 1 and those of the thermocline depth anomaly h shown in Fig. 2. A comparison between the simulation and the asymptotic approximate solutions is made. From Figs. 1 and 2, we can see that the first approximate solution using the asymptotic method has a good level of accuracy.

4. Summary and discussion

It is easy to see that the solution given by Eqs. (5) and (6) obtained by using the asymptotic method for the coupled system (1)–(4) at $t \to \infty$ tends to the singular point and can be used for analyzing the SST anomaly and the thermocline depth anomaly of the atmosphere–ocean oscillation for the ENSO model in the equatorial eastern Pacific.

The ENSO is a complicated natural phenomenon of the global climate in atmospheric physics. Hence we need to reduce it to basic models. We can solve them by using the approximate method. The asymptotic method is a good and succinct approximate analytic method, which differs from the general numerical method. The expansions of solution using the asymptotic method can be sequentially analyzed. Thus, from the approximate analytic solution given by Eqs. (5) and (6), we can further study the qualitative and quantitative behaviours of the SST anomaly in the equatorial eastern Pacific, the thermocline depth anomaly and the zonal wind stress anomaly.

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