Prolongation structure for nonlinear integrable couplings of a KdV soliton hierarchy*

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In this paper, a new nonlinear integrable coupling system of the soliton hierarchy is presented. From the Lax pairs, the coupled KdV equations are constructed successfully. Based on the prolongation method of Wahlquist and Estabrook, we study the prolongation structure of the nonlinear integrable couplings of the KdV equation.

Keywords: nonlinear integrable coupling system, prolongation structure, KdV soliton hierarchy

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1. Introduction

Wahlquist and Estabrook (WE) proposed a prolongation method for nonlinear evolution equations and applied it to the Kortewey de Vries (KdV) equation in their famous articles,[1,2] in which the concept of pseudoscience was introduced and the Bäcklund transformation of the KdV equation was obtained. They also showed that the prolongation structure can be determined by a set of Lie algebra relations.

The prolongation structure has been investigated from a geometric point of view. It has been shown that it is related to a variety of solving techniques for nonlinear differential equations, such as the Lax pairs, the conservation law and the Hamilton structure.[3−7] Wu, Guo and Wang considered the prolongation structures of nonlinear systems in higher dimensions.[8] In Ref. [9], the $Sl(2,R)$ principal prolongation structures of soliton equations and their conservation laws were proposed.

Many coupled equations appear in practical problems of various fields, such as biochemistry, physics and mechanics. For example, in order to systematically describe a kind of biochemistry model, Prigogine and Lefever proposed a coupled mathematical model in Ref. [10], which describes the biology-chemistry system; the well-known shallow water wave mathematic model and the coupled KdV model were given in Refs. [11]–[13]. In Ref. [14], Ma first proposed the perturbation method for establishing integrable couplings. Zhang presented the enlarged Lie algebra method to obtain the integrable couplings in Ref. [15]. Variational identities on semi-direct sums of the Lie algebras provide direct approaches to the Hamiltonian structures of the integrable equations in Refs. [15] and [16].

However, some integrable coupling systems are linear systems in the above papers. The method to generate nonlinear integrable couplings has been a puzzle to us. In this paper, a straightforward method to obtain the nonlinear integrable coupling system is proposed. Furthermore, a direct application to the KdV spectral problem leads to a novel soliton hierarchy of the integrable coupling system. Based on the prolongation structure method of WE, we study the prolongation structure of a nonlinear integrable coupling of the KdV equation. In addition, the Lax pairs of the nonlinear integrable couplings of the KdV hierarchy are constructed successfully.

2. Constructing new nonlinear integrable couplings for the soliton equation

Let $\boldsymbol{I}_N$ denote the unit matrix of order N, $N \in \mathrm{Z}$. For two matrices $\boldsymbol{A} = (a_{ij})_{pq}$, $\boldsymbol{B} = (b_{kl})_{rs}$, the Kronecker product $\boldsymbol{A} \otimes \boldsymbol{B}$ is defined by

$$\boldsymbol{A} \otimes \boldsymbol{B} = (a_{ij}\boldsymbol{B})_{(pr)\times(qs)}, \tag{1}$$

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or equivalently

$$(\boldsymbol{A}\otimes\boldsymbol{B})_{ij,kl}=a_{ik}b_{jl}. \tag{2}$$

The formula

$$\begin{aligned}&((\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D}))_{ij,kl}\\ =\;&(\boldsymbol{AC}\otimes\boldsymbol{BD})_{ij,kl}\\ =\;&\Big(\sum_p a_{ip}c_{pk}\Big)\Big(\sum_q b_{jq}d_{ql}\Big)\end{aligned} \tag{3}$$

shows that matrices $\boldsymbol{AC}$ and $\boldsymbol{BD}$ make sense. The relation will be used to construct a new integrable coupling system of the soliton equation hierarchy.

Assume that a continuous integrable model has two continuous integrable models and two continuous zero curvature representations

$$\begin{aligned}&\boldsymbol{U}_{1t}-\boldsymbol{V}_{1x}+[\boldsymbol{U}_1,\boldsymbol{V}_1]=0,\\ &\boldsymbol{U}_{2t}-\boldsymbol{V}_{2x}+[\boldsymbol{U}_2,V_2]=0,\end{aligned} \tag{4}$$

where $\boldsymbol{U}_1,\boldsymbol{V}_1$ and $\boldsymbol{U}_2,\boldsymbol{V}_2$ are $M\times M$ and $N\times N$ matrices, respectively. Ma defined new $\boldsymbol{U}_3$ and new $\boldsymbol{V}_3$ as

$$\begin{aligned}&\boldsymbol{U}_3=\boldsymbol{U}_1\otimes\boldsymbol{I}_N+\boldsymbol{I}_M\otimes\boldsymbol{U}_2,\\ &\boldsymbol{V}_3=\boldsymbol{V}_1\otimes\boldsymbol{I}_N+\boldsymbol{I}_M\otimes\boldsymbol{V}_2,\end{aligned} \tag{5}$$

then the same integrable model has another continuous zero curvature representation

$$\boldsymbol{U}_{3t}-\boldsymbol{V}_{3x}+[\boldsymbol{U}_3,\boldsymbol{V}_3]=0. \tag{6}$$

However, Eq. (6) is not suitable for the definition of the integrable couplings. We will give a new pair of $\boldsymbol{U}_3$ and $\boldsymbol{V}_3$ by using the Kronecker product, and they can produce the integrable couplings.

The new Lax pair, $\boldsymbol{U}_3$ and $\boldsymbol{V}_3$, is presented as

$$\begin{aligned}&\boldsymbol{U}_3=\boldsymbol{I}_N\otimes\boldsymbol{U}_1+\boldsymbol{T}_M\otimes\boldsymbol{U}_2,\\ &\boldsymbol{V}_3=\boldsymbol{I}_N\otimes\boldsymbol{V}_1+\boldsymbol{T}_M\otimes\boldsymbol{V}_2,\end{aligned} \tag{7}$$

where $\boldsymbol{I}_N$ and $\boldsymbol{T}_M$ are matrices of the same rank. By using Eq. (7), we can obtain

$$\begin{aligned}[\boldsymbol{U}_3,\boldsymbol{V}_3]=\;&\boldsymbol{I}_N^2\otimes\boldsymbol{U}_1\boldsymbol{V}_1-\boldsymbol{I}_N^2\otimes\boldsymbol{V}_1\boldsymbol{U}_1\\ &+\boldsymbol{I}_N\boldsymbol{T}_M\otimes\boldsymbol{U}_1\boldsymbol{V}_2-\boldsymbol{I}_N\boldsymbol{T}_M\otimes\boldsymbol{V}_1\boldsymbol{U}_2\\ &+\boldsymbol{T}_M\boldsymbol{I}_N\otimes\boldsymbol{U}_2\boldsymbol{V}_1-\boldsymbol{T}_M\boldsymbol{I}_N\otimes\boldsymbol{V}_2\boldsymbol{U}_1\\ &+\boldsymbol{T}_M^2\otimes\boldsymbol{U}_2\boldsymbol{V}_2-\boldsymbol{T}_M^2\otimes\boldsymbol{V}_2\boldsymbol{U}_2.\end{aligned} \tag{8}$$

With $\boldsymbol{I}_N^2=\boldsymbol{I}_N$, $\boldsymbol{I}_N\boldsymbol{T}_M=\boldsymbol{T}_M\boldsymbol{I}_N=\boldsymbol{T}_M$, $\boldsymbol{T}_M^2=\boldsymbol{T}_M$, we have

$$\begin{aligned}[\boldsymbol{U}_3,\boldsymbol{V}_3]=\;&\boldsymbol{I}_N\otimes[\boldsymbol{U}_1,\boldsymbol{V}_1]+\boldsymbol{T}_M\otimes[\boldsymbol{U}_1,\boldsymbol{V}_2]\\ &+\boldsymbol{T}_M\otimes[\boldsymbol{U}_2,\boldsymbol{V}_1]+\boldsymbol{T}_M\otimes[\boldsymbol{U}_2,\boldsymbol{V}_2].\end{aligned} \tag{9}$$

Taking

$$\begin{aligned}&\boldsymbol{U}_{3t}=\boldsymbol{I}_N\otimes\boldsymbol{U}_{1t}+\boldsymbol{T}_M\otimes\boldsymbol{U}_{2t},\\ &\boldsymbol{V}_{3x}=\boldsymbol{I}_N\otimes\boldsymbol{V}_{1x}+\boldsymbol{T}_M\otimes\boldsymbol{V}_{2x},\end{aligned}$$

and Eq. (9) into the zero curvature equation

$$\boldsymbol{U}_{3t}-\boldsymbol{V}_{3x}+[\boldsymbol{U}_3,\boldsymbol{V}_3]=0, \tag{10}$$

we obtain the new zero curvature equations

$$\boldsymbol{U}_{1t}-\boldsymbol{V}_{1x}+[\boldsymbol{U}_1,\boldsymbol{V}_1]=0, \tag{11a}$$

$$\boldsymbol{U}_{2t}-\boldsymbol{V}_{2x}+[\boldsymbol{U}_1,\boldsymbol{V}_2]+[\boldsymbol{U}_2,\boldsymbol{V}_1]+[\boldsymbol{U}_2,\boldsymbol{V}_2]=0. \tag{11b}$$

So, from new spectral (8), we obtain a new form (Eq. (11)) of the zero curvature equations. Equation (11b) exactly presents a kind of integrable coupling system of Eq. (11a). It is normally a nonlinear integrable coupling, because matrix $\boldsymbol{V}_{2,x}-[\boldsymbol{U}_2,\boldsymbol{V}_2]$ often produces nonlinear terms.

Next let us shed light on the above general idea of constructing coupling systems by uisng a particular class of the Kronecker product. We consider the following kind of Kronecker product.

Example The case of 2×2 Kronecker product: let $\boldsymbol{U}$ and $\boldsymbol{V}$ have the forms

$$\boldsymbol{U}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\boldsymbol{U}_1+\begin{pmatrix}0&1\\0&1\end{pmatrix}\otimes\boldsymbol{U}_4, \tag{12}$$

$$\boldsymbol{V}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\boldsymbol{V}_1+\begin{pmatrix}0&1\\0&1\end{pmatrix}\otimes\boldsymbol{V}_4, \tag{13}$$

then we include a new pair of $\boldsymbol{U}$ and $\boldsymbol{V}$,

$$\begin{aligned}&\boldsymbol{U}=\begin{pmatrix}\boldsymbol{U}_1&\boldsymbol{U}_4\\0&\boldsymbol{U}_1+\boldsymbol{U}_4\end{pmatrix},\\ &\boldsymbol{V}=\begin{pmatrix}\boldsymbol{V}_1&\boldsymbol{V}_4\\0&\boldsymbol{V}_1+\boldsymbol{V}_4\end{pmatrix}.\end{aligned} \tag{14}$$

Therefore, the corresponding enlarged zero curvature equation $\boldsymbol{U}_t-\boldsymbol{V}_x+[\boldsymbol{U},\boldsymbol{V}]=0$ is equivalent to

$$\begin{cases}\boldsymbol{U}_{1t}-\boldsymbol{V}_{1x}+[\boldsymbol{U}_1,\boldsymbol{V}_1]=0,\\ \boldsymbol{U}_{4t}-\boldsymbol{V}_{4x}+[\boldsymbol{U}_1,\boldsymbol{V}_4]+[\boldsymbol{U}_4,\boldsymbol{V}_1]+[\boldsymbol{U}_4,\boldsymbol{V}_4]=0.\end{cases} \tag{15}$$

3. Nonlinear integrable couplings of the KdV hierarchy

By using the theory of continuous integrable couplings,[17−19] some integrable coupling systems of the soliton hierarchies have been obtained, such as the AKNS hierarchy, the KN hierarchy and the TD hierarchy.[20−23] In this paper, in order to obtain a nonlinear integrable coupling system, we illustrate a new approach by using the Kronecker product to obtain the nonlinear soliton equation hierarchy.

The new Lax pair forms, $\bar{\boldsymbol{U}}(\bar{\boldsymbol{u}},\lambda)$ and $\bar{\boldsymbol{V}}(\bar{\boldsymbol{u}},\lambda)$, are presented as

$$\begin{aligned}\bar{\boldsymbol{U}} &= \boldsymbol{I}_2\otimes\boldsymbol{U}_0+\boldsymbol{T}_2\otimes\boldsymbol{U}_1,\\ \bar{\boldsymbol{V}} &= \boldsymbol{I}_2\otimes\boldsymbol{V}_0+\boldsymbol{T}_2\otimes\boldsymbol{V}_1,\end{aligned}\tag{16}$$

where $\boldsymbol{U}_0,\boldsymbol{V}_0,\boldsymbol{U}_1$, and $\boldsymbol{V}_1$ are 2×2 matrices, and

$$\boldsymbol{I}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \boldsymbol{T}_2=\begin{pmatrix}0&1\\0&1\end{pmatrix}.\tag{17}$$

From the zero curvature equation, we have

$$\begin{cases}\boldsymbol{U}_{0t}-\boldsymbol{V}_{0x}+[\boldsymbol{U}_0,\boldsymbol{V}_0]=0,\\ \boldsymbol{U}_{1t}-\boldsymbol{V}_{1x}+[\boldsymbol{U}_0,\boldsymbol{V}_1]+[\boldsymbol{U}_1,\boldsymbol{V}_0]+[\boldsymbol{U}_1,\boldsymbol{V}_1]=0.\end{cases}\tag{18}$$

To illustrate our method, we consider the following isospectral problem

$$\begin{aligned}\boldsymbol{U}_0&=\begin{pmatrix}0&1\\ \lambda-u&0\end{pmatrix},\quad \boldsymbol{U}_1=\begin{pmatrix}0&0\\-v&0\end{pmatrix},\\ \boldsymbol{V}_0&=\begin{pmatrix}a&b\\c&-a\end{pmatrix},\quad \boldsymbol{V}_1=\begin{pmatrix}e&f\\g&-e\end{pmatrix},\end{aligned}\tag{19}$$

where u and v are potentials.

Set

$$\begin{aligned}a&=\sum_{k=0}^{\infty}a_k\lambda^k,\quad b=\sum_{k=0}^{\infty}b_k\lambda^k,\quad c=\sum_{k=0}^{\infty}c_k\lambda^k,\\ e&=\sum_{k=0}^{\infty}e_k\lambda^k,\quad f=\sum_{k=0}^{\infty}f_k\lambda^k,\quad g=\sum_{k=0}^{\infty}g_k\lambda^k,\end{aligned}\tag{20}$$

the corresponding enlarged stationary zero curvature equation is obtained as

$$\bar{\boldsymbol{V}}_x=[\bar{\boldsymbol{U}},\bar{\boldsymbol{V}}],\tag{21}$$

i.e.,

$$\begin{cases}\boldsymbol{V}_{0x}=[\boldsymbol{U}_0,\boldsymbol{V}_0],\\ \boldsymbol{V}_{1x}=[\boldsymbol{U}_0,\boldsymbol{V}_1]+[\boldsymbol{U}_1,\boldsymbol{V}_0]+[\boldsymbol{U}_1,\boldsymbol{V}_1].\end{cases}\tag{22}$$

Obviously, we have

$$[\boldsymbol{U}_0,\boldsymbol{V}_0]=\begin{pmatrix}c_n-b_{n+1}+ub_n & -2a_n\\ 2a_{n+1}-ua_n & -c_n+b_{n+1}-ub_n\end{pmatrix},\tag{23a}$$

$$[\boldsymbol{U}_0,\boldsymbol{V}_1]+[\boldsymbol{U}_1,\boldsymbol{V}_0]+[\boldsymbol{U}_1,\boldsymbol{V}_1]=\begin{pmatrix}g_n-f_{n+1}+uf_n+vb_n+vf_n & -2e_n\\ 2e_{n+1}-2ue_n-2va_n-2ve_n & -g_n+f_{n+1}-uf_n-vb_n-vf_n\end{pmatrix}.\tag{23b}$$

Thus, from Eq. (23) we obtain

$$\begin{cases}a_{nx}=c_n-b_{n+1}+ub_n,\\ b_{nx}=-2a_n,\\ c_{nx}=2a_{n+1}-2ua_n,\\ e_{nx}=g_n-f_{n+1}+uf_n+vb_n+vf_n,\\ f_{nx}=-2e_n,\\ g_{nx}=2e_{n+1}-2ue_n-2va_n-2ve_n.\end{cases}\tag{24}$$

If $a_0=b_0=0$, $e_0=f_0=g_0=0$, $c_0=1$, we can see that all sets of a_n, b_n, c_n, e_n, f_n, g_n are uniquely determined. In particular, the first few sets are

$$\begin{cases}a_1=0,\ b_1=1,\ c_1=-\dfrac{1}{2}u,\ f_1=0,\\ b_2=\dfrac{1}{2}u, b_3=\dfrac{1}{8}u_{xx}+\dfrac{3}{8}u^2,\\ b_4=\dfrac{1}{32}u_{xxxx}+\dfrac{5}{32}u_x^2+\dfrac{5}{16}uu_{xx}+\dfrac{5}{16}u^3,\\ f_2=\dfrac{1}{2}v_x, f_3=\dfrac{1}{8}v_{xx}+\dfrac{3}{4}uv++\dfrac{1}{2}v^2,\\ f_4=\dfrac{1}{32}v_{xxxx}+\dfrac{5}{16}u_xv_x+\dfrac{5}{16}u_{xx}v\\ \qquad+\dfrac{5}{16}uv_{xx}+\dfrac{15}{16}u^2v.\end{cases}\tag{25}$$

Equation (22) can be written as

$$\begin{cases}-(\lambda^n\boldsymbol{V})_{+0x}+[\boldsymbol{U}_0,\boldsymbol{V}_0]_+=(\lambda^n\boldsymbol{V})_{-0x}-[\boldsymbol{U}_0,\boldsymbol{V}_0]_-,\\ -(\lambda^n\boldsymbol{V})_{+1x}+[\boldsymbol{U}_0,\boldsymbol{V}_1]_++[\boldsymbol{U}_1,\boldsymbol{V}_0]_++[\boldsymbol{U}_1,\boldsymbol{V}_1]_+=(\lambda^n\boldsymbol{V})_{-1x}-[\boldsymbol{U}_0,\boldsymbol{V}_1]_--[\boldsymbol{U}_1,\boldsymbol{V}_0]_--[\boldsymbol{U}_1,\boldsymbol{V}_1]_-.\end{cases}\tag{26}$$

We find that the terms on the left-hand side of Eq. (26) are of nonnegative degree, while the terms on the right-hand side of (26) are of nonpositive degree. Therefore, it gives rise to

$$-\boldsymbol{V}_{0x}^{(n)}+[\boldsymbol{U}_0,\boldsymbol{V}_0^{(n)}]=\begin{pmatrix} b_{n+1} & 0\\ -2a_{n+1} & -b_{n+1}\end{pmatrix}, \quad (27a)$$

$$-\boldsymbol{V}_{1x}^{(n)}+[\boldsymbol{U}_0,\boldsymbol{V}_1^{(n)}]+[\boldsymbol{U}_1,\boldsymbol{V}_0^{(n)}]+[\boldsymbol{U}_1,\boldsymbol{V}_1^{(n)}] = \begin{pmatrix} f_{n+1} & 0\\ -2e_{n+1} & -f_{n+1}\end{pmatrix}. \quad (27b)$$

Taking $\boldsymbol{V}_0^{(n)}=\boldsymbol{V}_{0+}^{(n)}+\begin{pmatrix} 0 & 0\\ -b_{n+1} & 0\end{pmatrix}$, $\boldsymbol{V}_1^{(n)}=\boldsymbol{V}_{1+}^{(n)}+\begin{pmatrix} 0 & 0\\ -f_{n+1} & 0\end{pmatrix}$, from the stationary zero curvature equation (Eq. (22)), we have

$$-\boldsymbol{V}_{0x}^{(n)}+[\boldsymbol{U}_0,\boldsymbol{V}_0^{(n)}]=\begin{pmatrix} 0 & 0\\ 2b_{m+1,x} & 0\end{pmatrix}, \quad (28a)$$

$$-\boldsymbol{V}_{1x}^{(n)}+[\boldsymbol{U}_0,\boldsymbol{V}_1^{(n)}]+[\boldsymbol{U}_1,\boldsymbol{V}_0^{(n)}]+[\boldsymbol{U}_1,\boldsymbol{V}_1^{(n)}] = \begin{pmatrix} 0 & 0\\ 2f_{m+1,x} & 0\end{pmatrix}. \quad (28b)$$

Thus the zero curvature equation (22) determines the following system:

$$\bar{\boldsymbol{u}}_t=\begin{pmatrix} u\\ v\end{pmatrix}_t=2\boldsymbol{J}\begin{pmatrix} b_{n+1}\\ f_{n+1}\end{pmatrix} =2\begin{pmatrix} \partial & 0\\ 0 & \partial\end{pmatrix}\begin{pmatrix} e_{n+1}\\ f_{n+1}\end{pmatrix}. \quad (29)$$

From Eq. (24), we obtain a recurrence operator,

$$\begin{pmatrix} b_{n+1}\\ f_{n+1}\end{pmatrix}=\boldsymbol{L}\begin{pmatrix} b_n\\ f_n\end{pmatrix}, \quad (30)$$

with

$$\boldsymbol{L}=\begin{pmatrix} \frac{1}{4}\partial^2+u-\frac{1}{2}\partial^{-1}u_x & 0\\ \frac{1}{2}v+\frac{1}{2}\partial^{-1}v\partial & \frac{1}{4}\partial^2+u+v-\frac{1}{2}\partial^{-1}u_x\end{pmatrix}.$$

The equation hierarchy (29) is derived from the zero curvature Eq. (18), therefore it is integrable. Comparing $\boldsymbol{J}$ and $\boldsymbol{L}$ in Eqs. (29) and (30), we can see that Eq. (29) is a nonlinear integrable coupling system of the KdV equation hierarchy.

When $n=2$, we obtain

$$\begin{cases} u_t=\frac{3}{2}uu_x+\frac{1}{4}u_{xxx},\\ v_t=\frac{3}{2}(uv)_x+\frac{1}{4}v_{xxx}+2vv_x.\end{cases} \quad (31)$$

We obtain the nonlinear integrable couplings of the KdV equation hierarchy by using the Kronecker product. So the Kronecker product is an efficient and straightforward method to construct the integrable coupling system of the soliton hierarchy.

4. Prolongation structure of the integrable couplings of the KdV equation

In Refs. [3]–[5], the prolongation structures of the coupled KdV equations were considered. In this section, we will investigate the prolongation structures and the integrable couplings of the KdV equation. For Eq. (31), we define a new set of independent variables

$$p=u_x,\quad q=u_{xx}=p_x,\quad r=v_x,\quad s=v_{xx}=r_x. \quad (32)$$

Equation (31) can be represented by the following set of two-forms:

$$\begin{cases} m_1=\mathrm{d}u\wedge\mathrm{d}t-p\mathrm{d}x\wedge\mathrm{d}t,\\ m_2=\mathrm{d}p\wedge\mathrm{d}t-q\mathrm{d}x\wedge\mathrm{d}t,\\ m_3=\mathrm{d}v\wedge\mathrm{d}t-r\mathrm{d}x\wedge\mathrm{d}t,\\ m_4=\mathrm{d}r\wedge\mathrm{d}t-s\mathrm{d}x\wedge\mathrm{d}t,\\ m_5=\mathrm{d}x\wedge\mathrm{d}u-\frac{1}{4}\mathrm{d}q\wedge\mathrm{d}t-\frac{3}{2}\mathrm{d}x\wedge\mathrm{d}t,\\ m_6=\mathrm{d}x\wedge\mathrm{d}v-\frac{1}{4}\mathrm{d}s\wedge\mathrm{d}t-\frac{3}{2}ur\mathrm{d}x\wedge\mathrm{d}t\\ \quad\quad -\frac{3}{2}vp\mathrm{d}x\wedge\mathrm{d}t-2vr\mathrm{d}x\wedge\mathrm{d}t,\end{cases} \quad (33)$$

which is closed under the exterior differential

$$\mathrm{d}m_i=\sum_{j=1}^{6}g_i^j\wedge m_j, \quad (34)$$

where g_i^j $(i,j=1,2,\ldots,6)$ are the differential 1-forms.

According to the prolongation method, we add new 1-forms

$$\omega^i=\mathrm{d}y^i-F^i(u,v,p,q,r,s,y^i)\mathrm{d}x -G^i(u,v,p,q,r,s,y^i)\mathrm{d}t, \quad (35)$$

where $\boldsymbol{y}$ is a pseudopotential. In the case of the integrable couplings of the KdV equation, $\boldsymbol{F}$ and $\boldsymbol{G}$ are restricted to implicit functions of independent variables x and t. However, here we require that $\boldsymbol{F}$ and $\boldsymbol{G}$ are explicit functions of t. Then, the exterior derivatives of $\boldsymbol{\omega}$ are required to lie in the augmented ideal of forms $\boldsymbol{I}'=\{\boldsymbol{I},\boldsymbol{\omega}\}$,

$$\mathrm{d}\omega^i=\sum_{k=1}^{6}f_k^i m_k+\eta^i\wedge\omega^i,\quad i=1,2,\ldots \quad (36)$$

From Eq. (36), we obtain the following system of the overdetermined differential equation:

$$\boldsymbol{F}_p = \boldsymbol{F}_q = \boldsymbol{F}_r = \boldsymbol{F}_s = 0, \tag{37a}$$

$$-\frac{1}{4}\boldsymbol{F}_u + \boldsymbol{G}_q = 0, \quad -\frac{1}{4}\boldsymbol{F}_v + \boldsymbol{G}_s = 0, \tag{37b}$$

$$\boldsymbol{G}_r s + \boldsymbol{G}_v r + \boldsymbol{G}_p q + \boldsymbol{G}_u p - 6up\boldsymbol{G}_q - 6ur\boldsymbol{G}_s - 6vp\boldsymbol{G}_s - 8vr\boldsymbol{G}_s = 0, \tag{37c}$$

where $\boldsymbol{F}\boldsymbol{G}_y - \boldsymbol{G}\boldsymbol{F}_y$ can be denoted by $[\boldsymbol{F}, \boldsymbol{G}]$, then equation (37) gives rise to a Lie algebra for determination. With the knowledge of the Lie algebra, we find that $\boldsymbol{F}$ and $\boldsymbol{G}$ can be expressed as

$$\boldsymbol{F} = u\boldsymbol{X}_1 + v\boldsymbol{X}_2 + \boldsymbol{X}_3, \tag{38a}$$

$$\boldsymbol{G} = \frac{1}{4}\boldsymbol{X}_1 q + \frac{1}{4}\boldsymbol{X}_2 s + \boldsymbol{H}(u, v, p, r), \tag{38b}$$

where $\boldsymbol{X}_i$ $(i = 1, 2, 3)$ are $n \times n$ matrixes to be determined.

Substituting the above equations into Eq. (37c) and collecting the coefficients of q, s, we have

$$\boldsymbol{H}_r - \frac{1}{4}[\boldsymbol{X}_3, \boldsymbol{X}_2] - \frac{1}{4}u[\boldsymbol{X}_1, \boldsymbol{X}_2] = 0, \tag{39a}$$

$$\boldsymbol{H}_p - \frac{1}{4}v[\boldsymbol{X}_1, \boldsymbol{X}_2] - \frac{1}{4}[\boldsymbol{X}_3, \boldsymbol{X}_1] = 0, \tag{39b}$$

$$\boldsymbol{H}_v r - \frac{3}{2}\boldsymbol{X}_2 ur - \frac{3}{2}\boldsymbol{X}_1 up - \frac{3}{2}\boldsymbol{X}_2 vp - v[\boldsymbol{X}_2, \boldsymbol{H}] + \boldsymbol{H}_u p - [\boldsymbol{X}_3, h] - u[\boldsymbol{X}_1, h] = 0. \tag{39c}$$

Solving the above set of nonlinear partial differential equations of $\boldsymbol{H}$ by the same procedure, we have

$$\begin{aligned}\boldsymbol{G} = {} & \left(\frac{3}{4}u^2 + \frac{1}{4}q\right)\boldsymbol{X}_1 + \left(\frac{3}{2}uv + \frac{1}{4}s\right)\boldsymbol{X}_2 \\ & + \frac{1}{4}p\boldsymbol{X}_3 + \frac{1}{4}r\boldsymbol{X}_4 - \left(\frac{1}{4}pv + \frac{1}{4}ru\right)\boldsymbol{X}_5 \\ & + \frac{1}{4}u\boldsymbol{X}_6 + \frac{1}{4}v\boldsymbol{X}_7 - \frac{1}{4}vu\boldsymbol{X}_8 \\ & + \frac{1}{8}u^2\boldsymbol{X}_9 + \frac{1}{4}uv\boldsymbol{X}_{10} + \frac{1}{8}v^2\boldsymbol{X}_{11},\end{aligned} \tag{40}$$

with

$$\begin{aligned}&[\boldsymbol{X}_1, \boldsymbol{X}_2] = \boldsymbol{X}_4, \quad [\boldsymbol{X}_3, \boldsymbol{X}_1] = \boldsymbol{X}_5, \\ &[\boldsymbol{X}_3, \boldsymbol{X}_2] = \boldsymbol{X}_6, [\boldsymbol{X}_3, \boldsymbol{X}_7] = 0, \\ &[\boldsymbol{X}_1, \boldsymbol{X}_8] = [\boldsymbol{X}_2, \boldsymbol{X}_9] = [\boldsymbol{X}_1, \boldsymbol{X}_4] = [\boldsymbol{X}_2, \boldsymbol{X}_4] = 0, \\ &[\boldsymbol{X}_1, \boldsymbol{X}_7] + [\boldsymbol{X}_3, \boldsymbol{X}_{11}] = 0, \\ &[\boldsymbol{X}_2, \boldsymbol{X}_7] + [\boldsymbol{X}_3, \boldsymbol{X}_{12}] = 0, \\ &[\boldsymbol{X}_3, \boldsymbol{X}_8] + [\boldsymbol{X}_1, \boldsymbol{X}_{11}] = 0, \\ &[\boldsymbol{X}_1, \boldsymbol{X}_9] + [\boldsymbol{X}_2, \boldsymbol{X}_{10}] = 0, \\ &[\boldsymbol{X}_3, \boldsymbol{X}_9] + [\boldsymbol{X}_2, \boldsymbol{X}_{12}] = 0, \\ &[\boldsymbol{X}_1, \boldsymbol{X}_{10}] + [\boldsymbol{X}_2, \boldsymbol{X}_8] = 0, \\ &[\boldsymbol{X}_3, \boldsymbol{X}_{10}] + [\boldsymbol{X}_1, \boldsymbol{X}_{12}] + [\boldsymbol{X}_2, \boldsymbol{X}_{11}] = 0,\end{aligned} \tag{41}$$

and

$$\begin{aligned}&-\frac{1}{8}[\boldsymbol{X}_3, \boldsymbol{X}_9] - \frac{3}{4}\boldsymbol{X}_3 - \frac{1}{4}[\boldsymbol{X}_1, \boldsymbol{X}_6] = 0, \\ &-\frac{1}{4}[\boldsymbol{X}_2, \boldsymbol{X}_8] + \frac{1}{8}[\boldsymbol{X}_1, \boldsymbol{X}_{11}] + \frac{1}{4}[\boldsymbol{X}_2, \boldsymbol{X}_{10}] = 0, \\ &[\boldsymbol{X}_2, \boldsymbol{X}_1 2] = -\frac{1}{4}[\boldsymbol{X}_3, \boldsymbol{X}_{11}], \\ &[\boldsymbol{X}_3, \boldsymbol{X}_{12}] = 0, \quad [\boldsymbol{X}_3, \boldsymbol{X}_{11}] = 0, \\ &[\boldsymbol{X}_1, \boldsymbol{X}_{12}] = -\frac{1}{4}[\boldsymbol{X}_3, \boldsymbol{X}_6], \quad [\boldsymbol{X}_1, \boldsymbol{X}_9] = 0, \\ &-\frac{1}{2}[\boldsymbol{X}_2, \boldsymbol{X}_9] + [\boldsymbol{X}_1, \boldsymbol{X}_8] - [\boldsymbol{X}_1, \boldsymbol{X}_{10}] = 0, \\ &-\frac{1}{2}[\boldsymbol{X}_3, \boldsymbol{X}_{11}] = [\boldsymbol{X}_2, \boldsymbol{X}_7], [\boldsymbol{X}_3, \boldsymbol{X}_6] = 0, \\ &-\frac{1}{4}[\boldsymbol{X}_3, \boldsymbol{X}_8] + \frac{1}{4}[\boldsymbol{X}_1, \boldsymbol{X}_7] + \frac{1}{4}[\boldsymbol{X}_3, \boldsymbol{X}_{10}] + \frac{3}{2}\boldsymbol{X}_4 + \frac{1}{4}[\boldsymbol{X}_2, \boldsymbol{X}_6] = 0,\end{aligned} \tag{42}$$

where $[\boldsymbol{X}_i, \boldsymbol{X}_j] = \boldsymbol{X}_i\boldsymbol{X}_j - \boldsymbol{X}_j\boldsymbol{X}_i$, and $\boldsymbol{X}_i$ $(i = 1, 2, \ldots, 12)$ determine an incomplete Lie algebra $\boldsymbol{L}$, which is called the prolongation algebra.

It is worth noting that in the prolongation structure of the integrable couplings of the KdV equation, $\boldsymbol{X}_i$ depends only on prolongation variable x, while $\boldsymbol{X}_i$ in Eq. (40) depends on x and t.

To find the matrix representation of $\boldsymbol{X}_i$ $(i = 1, 2, \ldots, 12)$, we try to embed prolongation algebra $\boldsymbol{L}$ represented by Eqs. (41) and (42) in semi-simple algebra $sl(4, \mathrm{C})$. We examine the $sl(4, \mathrm{C})$ algebra and obtain the matrix representation of $\boldsymbol{X}_i$ $(i = 1, 2, 3, 4)$ as

$$\boldsymbol{X}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 0 \end{pmatrix}, \quad \boldsymbol{X}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{43}$$

$$\boldsymbol{X}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{X}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

and $\boldsymbol{X}_j$ $(j = 5, 6, \ldots, 12)$ can be determined from Eqs. (41) and (42).

Equation (31), which is the nonlinear integrable couplings of the KdV equation, gives rise to

$$\boldsymbol{y}_x = \boldsymbol{F}\boldsymbol{y}, \quad \boldsymbol{y}_t = \boldsymbol{G}\boldsymbol{y}, \tag{45}$$

where

$$\boldsymbol{y} = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}, \tag{46}$$

and $\boldsymbol{F}$, $\boldsymbol{G}$ are

$$\boldsymbol{F} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ u-\lambda & 0 & v & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & u+v-\lambda & 0 \end{pmatrix}, \tag{47}$$

$$\boldsymbol{G} = \begin{pmatrix} a+u_x & -\lambda-\frac{1}{2}u & \frac{1}{4}v_x & -\frac{1}{2}v \\ \frac{1}{4}u_x-\frac{1}{2}\lambda u-\frac{1}{2}u^2+\lambda^2 & a-\frac{1}{2}u_x & -\frac{1}{2}\lambda v-uv+\frac{1}{4}v_{xx} & -\frac{1}{4}v_x \\ 0 & 0 & a+\frac{1}{4}u_x+\frac{1}{4}v_x & -\lambda-\frac{1}{2}u-\frac{1}{2}v \\ 0 & 0 & \frac{1}{4}u_x-\frac{1}{2}\lambda u-\frac{1}{2}u^2+\lambda^2-\frac{1}{2}\lambda v-uv+\frac{1}{4}v_{xx} & a-\frac{1}{2}u_x-\frac{1}{4}v_x \end{pmatrix}, \tag{48}$$

with λ being a spectral parameter.

5. Conclusion

In summary, a novel nonlinear integrable coupling system of the KdV equation hierarchy is proposed. Based on the prolongation structure, we have investigated the prolongation structures of a nonlinear integrable coupling system of the KdV equation.

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