# Dynamic bifurcation of a modified Kuramoto–Sivashinsky equation with higher-order nonlinearity

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Under the periodic boundary condition, dynamic bifurcation and stability in the modified Kuramoto–Sivashinsky equation with a higher-order nonlinearity $\mu(u_x)^p u_{xx}$ are investigated by using the centre manifold reduction procedure. The result shows that as the control parameter crosses a critical value, the system undergoes a bifurcation from the trivial solution to produce a cycle consisting of locally asymptotically stable equilibrium points. Furthermore, for cases in which the distances to the bifurcation points are small enough, one-order approximations to the bifurcation solutions are obtained.

**Keywords:** Kuramoto–Sivashinsky equation, centre manifold reduction, dynamic bifurcation

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## 1. Introduction

The Kuramoto–Sivashinsky (KS) equation arises in many different physical fields, such as frame front pagation,[1] reaction–diffusion systems[2]and growth interfaces.[3] A rich variety of dynamic behaviours, such as global attractors, inertial manifolds,[4] travelling wave solutions,[5] bifurcations[6,7] and chaos,[8] have been found in the systems. More recently, the generalized or modified KS equations in physics, especially for systems under external forces[9] and nonlinear perturbations,[10] have also received great attention.

In this paper, we consider the following modified KS equation (see Ref. [10]):

$$u_t = u_{xxxx} + \lambda u_{xx} + uu_x + \mu(u_x)^p u_{xx}, \tag{1}$$

where $u(x,t)$ is a real scalar function, $p \geq 1$ is an integer, $x \in (0, 2\pi)$, $t \geq 0$, $\lambda$ and $\mu$ are two parameters. When $\mu = 0$, Eq. (1) reduces to the standard KS equation. Equation (1) is supplemented with the periodic boundary condition,

$$u(x + 2\pi) = u(x). \tag{2}$$

Furthermore, we assume in our case that the average flow vanishes,

$$\int_0^{2\pi} u \, \mathrm{d}x = 0. \tag{3}$$

When the initial-value problem is considered, we supplement these equations with

$$u(x, 0) = u_0(x). \tag{4}$$

Politi and Misbah[10] have considered a more general form of the modified KS equation by adding a term like $\partial_x G(u_x)$ (this includes the term $\partial_x (u_x)^{p+1}$ as a particular case) to it and found general criteria to understand its dynamics by analyzing the steady state solutions only. In Ref. [11], the approximate symmetry reductions and the infinity series symmetry reduction travelling wave solution symmetry perturbation of Eq. (1) have been constructed by applying an approximate symmetry reduction approach and using the Lie SYMGRP package. In our previous work,[12−14] by using the multiple scale method, we have described the approximate bifurcation solutions of a modified KS equation with a weak dissipation term $\varepsilon u_x u_{xx}$ near the trivial solution. Additionally, the solitary wave solutions for some Burgers type equations have been investigated.[15−19] Equation (1) studied in this paper is related to the phase transition of a non-equilibrium system incidentally from one steady state to another.

The aim of this paper is to study the effect of the higher-order nonlinearity on the dynamic bifurcation near the zero solution. Hence, we need to describe the explicit structure of the bifurcated attractor near the change of the stability. Our technical tool is the

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one-order approximation formula of the centre manifold function developed by Ma *et al.*[6] Using it, we can find a reduced system and perform the bifurcation analysis on the two-dimensional centre manifold near $\lambda = 1$. In face, the reduced system on the centre manifold locally governs the dynamic behaviour near the zero solution and can provide complete information about the dynamic transition of Eq. (1). Finally, some new steady state solutions, which locally attract all of the phase space, are obtained.

The paper is organized as follows. In Section 2, we establish appropriate infinite-dimensional function spaces and reduce Eq. (1) to its local centre manifold. In Section 3, by stating and proving our main theorems, we give the bifurcation analysis and some approximations to the asymptotically stable bifurcation solutions of Eq. (1) in the cases of $p = 1$, $p = 2$ and $p \geq 3$. Section 4 concludes this paper.

## 2. Mathematical setting and centre manifold reduction

For the mathematical setting of the problem given in Eqs. (1)–(4), we establish

$$H_1 = \{u \in H^4(0, 2\pi) \,|u \text{ satisfies Eqs. } (2)-(4)\}$$

and

$$H = \{u \in L^2(0, 2\pi) \,|u \text{ satisfies Eqs. } (2)-(4)\}\ ,$$

where $H^4(0, 2\pi)$ is the usual real-valued Sobolev space. We can see that $H_1$ is dense in $H$ and the injection of $H_1$ in $H$ is compact. We define the linear operators

$$\mathrm{L}_\lambda u = -u_{xxxx} - \lambda u_{xx} : H_1 \to H$$

and

$$G(u) = -uu_x - \mu(u_x)^p u_{xx} : H_1 \to H.$$

The eigenvalues and the eigenvectors of $\mathrm{L}_\lambda$ are respectively given by

$$\beta_{2n-1}(\lambda) = \beta_{2n}(\lambda) = n^2(\lambda - n^2)$$

and

$$e_{2n-1}(x) = \sin nx, \quad e_{2n}(x) = \cos nx,$$

where $n = 1, 2, \ldots$ It is easy to see that the eigenvector sequence $\{e_n\}_{n=1}^{\infty}$ is an orthogonal basis of $H$. Moreover, the eigenvalues have the following properties near $\lambda = 1$,

$$\beta_{1,2}(\lambda) \quad \begin{cases} < 0, & \text{if } \lambda < 1, \\ = 0, & \text{if } \lambda = 1, \\ > 0, & \text{if } \lambda > 1, \end{cases}$$
$$\beta_j(1) < 0, \quad j \geq 3.$$

The $G(u) = \mathrm{O}(\|u\|_H)$. In addition, spaces $H$ and $H_1$ can be decomposed into the following form,

$$H_1 = E_1 \oplus E_2, \quad H = E_1 \oplus \bar{E}_2,$$

where

$$\begin{aligned} E_1 \ &= \ \{u_1 \sin x + u_2 \cos x \,|u_1, u_2 \in \mathbf{R}, \}, \\ E_2 \ &= \ \Big\{ \sum_{k=2}^{\infty} (u_{2k-1} \sin kx + u_{2k} \cos kx) \\ &\qquad \Big| \sum_{k=2}^{\infty} k^8 (u_{2k-1}^2 + u_{2k}^2) < \infty, \\ &\qquad u_{2k-1} \in \mathbf{R},\ u_{2k} \in \mathbf{R} \Big\}, \\ \bar{E}_2 \ &= \ \Big\{ \sum_{k=2}^{\infty} (u_{2k-1} \sin kx + u_{2k} \cos kx) \\ &\qquad \Big| \sum_{k=2}^{\infty} (u_{2k-1}^2 + u_{2k}^2) < \infty, \\ &\qquad u_{2k-1} \in \mathbf{R},\ u_{2k} \in \mathbf{R} \Big\}. \end{aligned}$$

Therefore, by using the centre manifold theorem,[6] we reduce Eq. (1) to its local centre manifold near $(u, \lambda) = (0, 1)$, and get a system of equations

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = (\lambda - 1)u_1 + \frac{1}{\pi} \int_0^{2\pi} G(u) \sin x \,\mathrm{d}x, \quad (5)$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = (\lambda - 1)u_2 + \frac{1}{\pi} \int_0^{2\pi} G(u) \cos x \,\mathrm{d}x, \quad (6)$$

where $u = u_1 \sin x + u_2 \cos x + \Phi(u_1, u_2, \lambda)$, $\Phi(u_1, u_2, \lambda)$ is a centre manifold function depending continuously on $\lambda$ in $E_2$.

## 3. Main results

In this section, the dynamic bifurcation and the stability of the reduced equations on the two-dimensional centre manifold are presented near $\lambda = 1$. In order to show the impact of the perturbation term $\mu(u_x)^p u_{xx}$ on the non-equilibrium phase transitions, we need to track the behaviour of the bifurcation solutions with respect to the perturbation parameter $\mu$.

Now we respectively state and prove our main results in the cases of $p = 1$, $p = 2$, and $p \geq 3$.

**Theorem 1** Assume that $p = 1$, $-\frac{1}{2} < \mu < 1$. As $\lambda$ crosses 1, Eq. (1) undergoes a bifurcation from the trivial solution to produce a cycle consisting of equilibrium points, which are locally asymptotically stable in $H$. Furthermore, let $\lambda = 1 + \varepsilon R$, $0 < \varepsilon \ll 1$, $R = \mathrm{O}(1) > 0$, the steady state bifurcation solutions can be expressed as

$$u = \sqrt{\frac{48\,\varepsilon R}{(1+2\mu)(1-\mu)}}(\sin\theta \sin x + \cos\theta \cos x) + \mathrm{O}(\varepsilon^{1/2}),$$

where $-\infty < \theta < +\infty$.

**Proof** When $p = 1$, $G(u) = G_2(u) = -uu_x - \mu u_x u_{xx}$ is the two-multilinear operator. Then according to the one-order approximation formula of the centre manifold function,[6] near $\lambda = 1$, we have

$$\Phi(u_1, u_2, \lambda) = \sum_{n=3}^{\infty} \varphi_n(u_1, u_2, \lambda) e_n + \mathrm{O}(|(u_1, u_2)|^3, |(u_1, u_2)|^2 |\beta_1(\lambda)|),$$

where

$$\varphi_n(u_1, u_2, \lambda) = -\frac{1}{\beta_n(\lambda)\pi} \int_0^{2\pi} G_2(u_1 \sin x + u_2 \cos x) e_n \,\mathrm{d}x.$$

Careful calculations give

$$\begin{aligned} \varphi_3 &= -\frac{1}{\beta_3(\lambda)} \int_0^{2\pi} G_2(u_1 \sin x + u_2 \cos x) \sin 2x \,\mathrm{d}x \\ &= \frac{1-\mu}{\beta_3(\lambda)}(u_1^2 - u_2^2), \\ \varphi_4 &= -\frac{1}{\beta_4(\lambda)} \int_0^{2\pi} G_2(u_1 \sin x + u_2 \cos x) \cos 2x \,\mathrm{d}x \\ &= \frac{1-\mu}{\beta_4(\lambda)} u_1 u_2, \end{aligned}$$

and $\varphi_j = 0$, $j = 5, 6, \ldots$ Thus the approximate centre manifold function becomes

$$\Phi = \varphi_3 \sin 2x + \varphi_4 \cos 2x + \mathrm{O}(|(u_1, u_2)|^3, |(u_1, u_2)|^2 |\beta_1(\lambda)|). \quad (7)$$

Substituting Eq. (7) into Eqs. (5) and (6) yields the following reduced system on the two-dimensional centre manifold in $H$:

$$\begin{aligned} \frac{\mathrm{d}u_1}{\mathrm{d}t} &= (\lambda - 1)u_1 + \frac{1}{\pi} \int_0^{2\pi} G(u_1 \sin x + u_2 \cos x + \Phi) \sin x \,\mathrm{d}x \\ &= (\lambda - 1)u_1 + \frac{(1+2\mu)(1-\mu)}{4\beta_3(\lambda)} u_1(u_1^2 + u_2^2) + \mathrm{O}(|(u_1, u_2)|^4, |(u_1, u_2)|^3 |\beta_1(\lambda)|), \quad (8) \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} &= (\lambda - 1)u_2 + \frac{1}{\pi} \int_0^{2\pi} G(u_1 \sin x + u_2 \cos x + \Phi) \cos x \,\mathrm{d}x \\ &= (\lambda - 1)u_2 + \frac{(1+2\mu)(1-\mu)}{4\beta_3(\lambda)} u_2(u_1^2 + u_2^2) + \mathrm{O}(|(u_1, u_2)|^4, |(u_1, u_2)|^3 |\beta_1(\lambda)|). \quad (9) \end{aligned}$$

As $\lambda$ slightly crosses 1, it is easy to see that the coefficient matrix of the linear parts of the above coupled system have real eigenvalues only. We transform the above reduced system into

$$\begin{aligned} &\frac{\mathrm{d}(u_1^2 + u_2^2)}{\mathrm{d}t} \\ &= (\lambda - 1)(u_1^2 + u_2^2) + \frac{(1+2\mu)(1-\mu)}{4\beta_3(\lambda)}(u_1^2 + u_2^2)^2 + \mathrm{O}(|(u_1, u_2)|^5, |(u_1, u_2)|^4 |\beta_1(\lambda)|). \quad (10) \end{aligned}$$

Consequently, when $(1+2\mu)(1-\mu)/4\beta_3(\lambda) < 0$, from Eq. (10) we know that the reduced system of Eqs. (8) and (9) has a supercritical bifurcation from the trivial solution $(0, 0)$ to produce some asymptotically stable steady state solutions, which lie on the cycle

$$u_1^2 + u_2^2 = -\frac{4\beta_3(\lambda)}{(1+2\mu)(1-\mu)}(\lambda - 1) + \mathrm{O}(\lambda - 1).$$

Furthermore, let $\lambda - 1 = \varepsilon R$, $0 < \varepsilon \ll 1$, $R = \mathrm{O}(1) > 0$, since $u = u_1 \sin x + u_2 \cos x + \Phi$, the KS equation (1) has some bifurcation solutions that can be expressed as

$$u = \sqrt{\frac{48\,\varepsilon R}{(1+2\mu)(1-\mu)}}(\sin\theta \sin x + \cos\theta \cos x) + \mathrm{O}(\varepsilon),$$

which locally attract some bounded sets around them in $H$. Here $\varepsilon R$ denotes the distance to the bifurcation point, $-\infty < \theta < +\infty$. Those are the first-order approximations to the non-trivial steady state solutions near a bifurcation point as time goes to infinity. We can also see that the bifurcation solutions are contained within an attractor in Hilbert space $H$. Owing to the arbitrary value of $\theta$, the bifurcated equilibrium solution set is shown as a cycle on the two-dimensional centre manifold near the trivial solution. This completes the proof.

**Theorem 2** Assume that $p = 2$, $\mu < \frac{1}{12}$. As $\lambda$ crosses 1, Eq. (1) undergoes a bifurcation from the

trivial solution to produce a cycle consisting of the equilibrium points, which are locally asymptotically stable in $H$. Furthermore, let $\lambda = 1 + \varepsilon R$, $0 < \varepsilon \leq 1$, $R = \mathrm{O}(1) > 0$, the bifurcation solutions can be expressed as

$$u = \sqrt{\frac{48\,\varepsilon R}{1 - 12\mu}}(\sin\theta \sin x + \cos\theta \cos x) + \mathrm{O}(\varepsilon^{1/2}),$$

where $-\infty < \theta < +\infty$.

**Proof** When $p = 2$, $G(u) = -uu_x - \mu(u_x)^2 u_{xx}$. We denote the two-multilinear operator $-uu_x$ by $G_2(u)$. Similar to the previous proof, we can derive an approximation of the centre manifold function near $(u, \lambda) = (0, 1)$,

$$\begin{aligned}\Phi = {} & \frac{u_1^2 - u_2^2}{2\beta_3(\lambda)} \sin 2x + \frac{u_1 u_2}{\beta_4(\lambda)} \cos 2x \\ & + \mathrm{O}(|(u_1, u_2)|^3, |(u_1, u_2)|^2 |\beta_1(\lambda)|),\end{aligned}$$

and correspondingly a reduced system on the centre manifold in $H$,

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}t} = {} & (\lambda - 1)u_1 + \frac{1 + \mu\beta_3(\lambda)}{4\beta_3(\lambda)} u_1(u_1^2 + u_2^2) \\ & + \mathrm{O}(|(u_1, u_2)|^4, |(u_1, u_2)|^3 |\beta_1(\lambda)|), \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} = {} & (\lambda - 1)u_2 + \frac{1 + \mu\beta_3(\lambda)}{4\beta_3(\lambda)} u_2(u_1^2 + u_2^2) \\ & + \mathrm{O}(|(u_1, u_2)|^4, |(u_1, u_2)|^3 |\beta_1(\lambda)|).\end{aligned}$$

As $\lambda$ slightly crosses 1, we have $(1 + \mu\beta_3(\lambda))/4\beta_3(\lambda) < 0$. This means that the above reduced system has a supercritical bifurcation from the trivial solution to some asymptotically stable steady state solutions, which lie on the cycle

$$u_1^2 + u_2^2 = -\frac{4\beta_3(\lambda)}{1 + \mu\beta_3(\lambda)}(\lambda - 1) + \mathrm{O}(\lambda - 1).$$

Similar to the proof in the previous theorem, we can get the approximations to the bifurcation solutions of Eq. (1) sufficiently near $\lambda = 1$.

**Theorem 3** Assume that $p \geq 3$. As $\lambda$ crosses 1, Eq. (1) undergoes a bifurcation from the trivial solution to produce a cycle consisting of the equilibrium points, which are locally asymptotically stable in $H$. Furthermore, let $\lambda = 1 + \varepsilon R$, $0 < \varepsilon \leq 1$, $R = \mathrm{O}(1) > 0$, the bifurcation solutions can be expressed as

$$u = \sqrt{48\,\varepsilon R}(\sin\theta \sin x + \cos\theta \cos x) + \mathrm{O}(\varepsilon^{1/2}),$$

where $-\infty < \theta < +\infty$.

**Proof** When $p \geq 3$, $G(u) = -uu_x - \mu(u_x)^p u_{xx}$, $G_2(u) = -uu_x$. Then we can get an approximation of the centre manifold function near $(u, \lambda) = (0, 1)$ as

$$\begin{aligned}\Phi = {} & \frac{u_1^2 - u_2^2}{2\beta_3(\lambda)} \sin 2x + \frac{u_1 u_2}{\beta_4(\lambda)} \cos 2x \\ & + \mathrm{O}(|(u_1, u_2)|^3, |(u_1, u_2)|^2 |\beta_1(\lambda)|),\end{aligned}$$

and correspondingly a reduced system on the centre manifold in space $H$,

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}t} = {} & (\lambda - 1)u_1 + \frac{1}{4\beta_3(\lambda)} u_1(u_1^2 + u_2^2) \\ & + \mathrm{O}(|(u_1, u_2)|^4, |(u_1, u_2)|^3 |\beta_1(\lambda)|), \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} = {} & (\lambda - 1)u_2 + \frac{1}{4\beta_3(\lambda)} u_2(u_1^2 + u_2^2) \\ & + \mathrm{O}(|(u_1, u_2)|^4, |(u_1, u_2)|^3 |\beta_1(\lambda)|).\end{aligned}$$

The rest of the proof is similar to that of theorem 1.

# 4. Conclusion

Given some ranges of perturbation parameter $\mu$, the approximations and the stability of the steady state bifurcation solutions to the modified KS system (1) near $(u, \lambda) = (0, 1)$ are discussed by using the centre manifold reduction method. It is easy to see that the branches have changed in the presence of the higher-order nonlinearity. When $p \geq 3$, from theorem 3 we know that the first term in $\varepsilon$ expansions for the branching solutions are unaffected by the higher-order nonlinear term. Our results are useful to the bifurcation control by varying the turning parameters $\lambda$ and $\mu$. In addition, based on the results of Ref. [6], the research tool used in this paper is also applicable to the bifurcation analysis of the modified KS equation in the two-dimensional spatial domain. In that case, more complex bifurcation behaviours can be obtained because of the multiple eigenvectors of the linearized problem. It should be noted that when the system is restricted to some special subspaces (for example, a subspace of the even functions[7]), since $u(x, y)$ is invariant under shifts in both $x$ and $y$, the bifurcation problem in the two-dimensional spatial domain can be simplified.

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