Elegant Ince–Gaussian beams in a quadratic-index medium*

Bai Zhi-Yong(白志勇), Deng Dong-Mei(邓冬梅), and Guo Qi(郭 旗)†

Key Laboratory of Photonic Information Technology of Guangdong Higher Education Institutes, South China Normal University, Guangzhou 510631, *China*

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Elegant Ince–Gaussian beams, which are the exact solutions of the paraxial wave equation in a quadratic-index medium, are derived in elliptical coordinates. These kinds of beams are the alternative form of standard Ince–Gaussian beams and they display better symmetry between the Ince-polynomials and the Gaussian function in mathematics. The transverse intensity distribution and the phase of the elegant Ince–Gaussian beams are discussed.

Keywords: elegant Ince–Gaussian beams, complex variables, quadratic-index media, elliptic coordinates

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1. Introduction

The problem of the propagation of optical beams in free space and other media has been investigated.[1−18] In free space, the well-known Hermite–Gaussian (HG), Laguerre–Gaussian (LG)[3] and Ince–Gaussian (IG) beams,[4] which are the exact solutions of the free-space paraxial wave equation (PWE) in Cartesian, circular cylindrical and elliptic coordinates, respectively, have already been obtained. Each kind of beam is a complete family of exact and orthogonal solutions of the PWE, and any paraxial field can be described by a superposition of those beams with appropriate weighting. In 1973, Siegman[5] reported the elegant HG beams, which were the alternative forms of standard HG solutions and had better symmetry in mathematical expression. These solutions form a complete biorthogonal set of the solutions of the PWE and can also describe any paraxial field through a proper superposition. This motivated researchers to study the elegant form solutions. Then the elegant LG,[6] IG[7] and Hermite–Laguerre–Gaussian beams,[8] which are also the exact solutions of the PWE in free space, were reported one after another. These elegant form solutions are closely related to the multipole field,[9] which can describe the propagation of optical beams through a complex parabolic medium and arise in the higher-order connection terms of perturbation expansions of the solutions to the wave equation whose leading term is the fundamental Gaussian beam function.[6,7,10]

The quadratic medium is one of the ideal media for the long-distance transmission of optical beams. Some authors have studied the paraxial optical field in a quadratic-medium and obtained standard HG,[11] LG[12] and IG[13] beams. However, the elegant form solutions have not been reported. In this paper, we are concerned with the transmission of optical beams through a quadratic-index medium and we derive the elegant form solutions of the PWE in a quadratic medium in elliptic coordinates. We find some differences between the standard and elegant IG beams.

2. The elegant Ince–Gaussian solutions

In an inhomogeneous medium, we consider the distribution of the refractive index as $n(r) = n_0(1 - a^2r^2/2)$, where n_0 is the refractive index along the spatial axis of the medium, r is the transverse distance from the spatial axis and a is a measure of the

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†Corresponding author. E-mail: guoq@scnu.edu.cn

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parabolic dependence (rate of decrease) of the index $n(r)$ with increasing distance from the axis.[14] We assume that the light beams input the medium along the spatial axis and write the paraxial electric field as $E(\boldsymbol{r},z,t)=\psi(\boldsymbol{r},z)\exp(\mathrm{i}kz-\mathrm{i}\omega t)$. Using the approximation $n^2(r)\sim n_0^2(1-a^2r^2)$ and the expression of the PWE of $E(\boldsymbol{r},z,t)$, one can obtain the paraxial wave equation[13] of the slowly varying envelope $\psi(\boldsymbol{r},z)$ in the quadratic-index medium

$$2\mathrm{i}k\frac{\partial\psi(\boldsymbol{r},z)}{\partial z}+\Delta_\perp\psi(\boldsymbol{r},z)-k^2a^2r^2\psi(\boldsymbol{r},z)=0, \quad (1)$$

where $\boldsymbol{r}$ is the two-dimensional (2D) transverse coordinate vector, z is the longitudinal (propagation direction) coordinate, $k=n_0\omega/c$ is the wave number along the axis and $\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the 2D transverse Laplacian operator.

The Gaussian-shaped solution[13,15–18] of Eq. (1) is

$$\phi_{\mathrm{G}}=\frac{w_0}{w(z)}\exp\left[\frac{\mathrm{i}k}{2q(z)}r^2+\mathrm{i}\theta\right], \quad (2)$$

where

$$w(z)=w_0\left[\cos^2(az)+\frac{1}{a^2z_{\mathrm{R}}^2}\sin^2(az)\right]^{1/2}$$

is beam width,

$$q(z)=\frac{1}{a}\frac{aq_0\cos(az)+\sin(az)}{\cos(az)-aq_0\sin(az)}$$

is a complex parameter, $q_0=q(0)=-\mathrm{i}z_{\mathrm{R}}=-\mathrm{i}kw_0^2/2$ is initial complex parameter, z_{R} is the Rayleigh range, w_0 is the initial beam width and

$$\theta=-\arctan\left[\frac{1}{az_{\mathrm{R}}}\tan(az)\right].$$

To obtain the elegant form solutions of Eq. (1) in elliptic coordinates, we constitute the solutions ψ_{eIG} as the multiplications of four functions

$$\psi_{\mathrm{eIG}}=A(z)F(\xi)N(\eta)\phi_{\mathrm{G}}, \quad (3)$$

and define the complex elliptical variables as $x=f(z)\cosh\xi\cos\eta$, $y=f(z)\sinh\xi\sin\eta$, where $A(z),F(\xi),N(\eta)$ are complex functions, (x,y,z) represent the real Cartesian coordinates, ξ ($\xi\geq0$) and η ($0\leq\eta\leq2\pi$) denote the radial and angular elliptic variables, respectively, and $f(z)=\sqrt{\epsilon}\chi(z)$ which has the dimension of length as a semifocal separation. In the expression of $f(z)$, $\epsilon=f_0^2/w_0^2$ is the elliptic parameter, f_0 is the semifocal separation at waist plane $z=0$ and

$$\chi(z)=\sqrt{w_0^2\cos(az)\left[\cos(az)+\frac{1}{aq_0}\sin(az)\right]}.$$

When $z=0$, variables (ξ,η) reduce to the real elliptical coordinates because $\chi(0)=w_0$ and $f(0)=f_0$.

Then, inserting Eq. (3) into Eq. (1) and using the method of separation of variables, we can obtain

$$\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2}-\epsilon\sinh2\xi\frac{\mathrm{d}F}{\mathrm{d}\xi}-(b-p\epsilon\cosh2\xi)F=0, \quad (4)$$

$$\frac{\mathrm{d}^2N}{\mathrm{d}\eta^2}+\epsilon\sinh2\eta\frac{\mathrm{d}N}{\mathrm{d}\eta}+(b-p\epsilon\cosh2\eta)N=0, \quad (5)$$

$$\frac{\mathrm{i}k\chi(z)}{A}\frac{\mathrm{d}A}{\mathrm{d}z}=p, \quad (6)$$

where p and b are separation constants. Equation (11) can be solved easily

$$\begin{aligned}A(z)&=\left(\frac{aq_0}{aq_0+\tan(az)}\right)^{p/2}\\&=\cos^{p/2}(az)\left(\frac{w_0}{w(z)}\right)^{p/2}\exp\left(\frac{\mathrm{i}p\theta}{2}\right).\end{aligned} \quad (7)$$

Equation (5) is the Ince equation.[4,7,13,19] We can obtain Eq. (4) from Eq. (5) by supposing $\eta=\mathrm{i}\xi$ and vice versa. The solutions of Eq. (5) are known as the even and odd Ince polynomials of order p and degree m, usually expressed as C_p^m and S_p^m, respectively, where $0\leq m\leq p$ for even function, $1\leq m\leq p$ for odd function, and the indices $(p,\ m)$ have the same parity, i.e.,$(-1)^{p-m}=1$. Equation (1) corresponds to the mathematical description of high-order elegant IG beams. To search for three-dimensional solutions, we combine the solutions of the same parity in ξ and η, which show continuity in the whole space. Then, the elegant IG solutions can be obtained

$$\begin{aligned}\psi_{p,m}^{\mathrm{e}}&=c_{pm}^{\mathrm{e}}\frac{\cos^{p/2}(az)}{w^{p/2+1}(z)}C_p^m(\mathrm{i}\xi,\epsilon)C_p^m(\eta,\epsilon)\\&\quad\times\exp\left[-\frac{r^2}{w^2(z)}+\frac{\mathrm{i}kr^2}{2R(z)}\right.\\&\quad\left.+\mathrm{i}\left(1+\frac{p}{2}\right)\theta(z)\right],\end{aligned} \quad (8)$$

$$\begin{aligned}\psi_{p,m}^{\mathrm{o}}&=s_{pm}^{\mathrm{o}}\frac{\cos^{p/2}(az)}{w^{p/2+1}(z)}S_p^m(\mathrm{i}\xi,\epsilon)S_p^m(\eta,\epsilon)\\&\quad\times\exp\left[-\frac{r^2}{w^2(z)}+\frac{\mathrm{i}kr^2}{2R(z)}\right.\\&\quad\left.+\mathrm{i}\left(1+\frac{p}{2}\right)\theta(z)\right],\end{aligned} \quad (9)$$

where the superscript indices e and o denote the even and odd modes, respectively, and c_{pm}^{e} and s_{pm}^{o} are normalization constants that are determined by $\int_{-\infty}^{+\infty} |\psi_{pm}^{\mathrm{e,o}}|^2 \mathrm{d}S = P_0$, with $\mathrm{d}S$ being the area differential element across the transverse plane. At any z plane, the $\psi_{p,m}^{\mathrm{e,o}}$ and the adjoint form $\widehat{\psi_{p',m'}}^{\mathrm{e',o'}}$ constitute a biorthogonal set

$$\int_{-\infty}^{+\infty} \psi_{p,m}^{\mathrm{e,o}} \widehat{\psi_{p',m'}}^{\mathrm{e',o'}} \mathrm{d}S = \delta_{\mathrm{ee'},\mathrm{oo'}} \delta_{pp'} \delta_{mm'},$$

where δ is the Kronecker delta function and

$$R(z) = \frac{\tan(az) + a^2 z_{\mathrm{R}}^2 \cot(az)}{a(1 - a^2 z_{\mathrm{R}}^2)}$$

is the radius of curvature.

3. Discussion

We have derived the complete exact elegant IG solutions of Eq. (1). As a novel class of solutions, there are some evident differences from standard IG beams. First of all, we look carefully at the phase. For standard IG beams, the transverse phase only appears in the term of the Gaussian function and when the input condition is proper, the transverse phase is a constant, which is displayed by the third row of Figs. 1 and 2. Generally, as IG beams propagate along the z axis, their transverse phase changes obviously. The phenomenon is shown by the third row of Figs. 3 and 4. For elegant IG beams, the Ince-polynomials contribute an additional complex transverse phase variation because of the complex elliptical variations (ξ, η), which always exists except at $z = 0$ and forces the total transverse phase not to be a constant. The last row of Figs. 1–4 show the changes of the transverse phase of elegant IG beams. We also find that the relationship of the longitudinal phase and the order p of the Ince-polynomial is different for the two kinds of beams: $kz + (p+1)\theta$ for standard IG beams and $kz + (p/2+1)\theta$ for elegant IG beams.

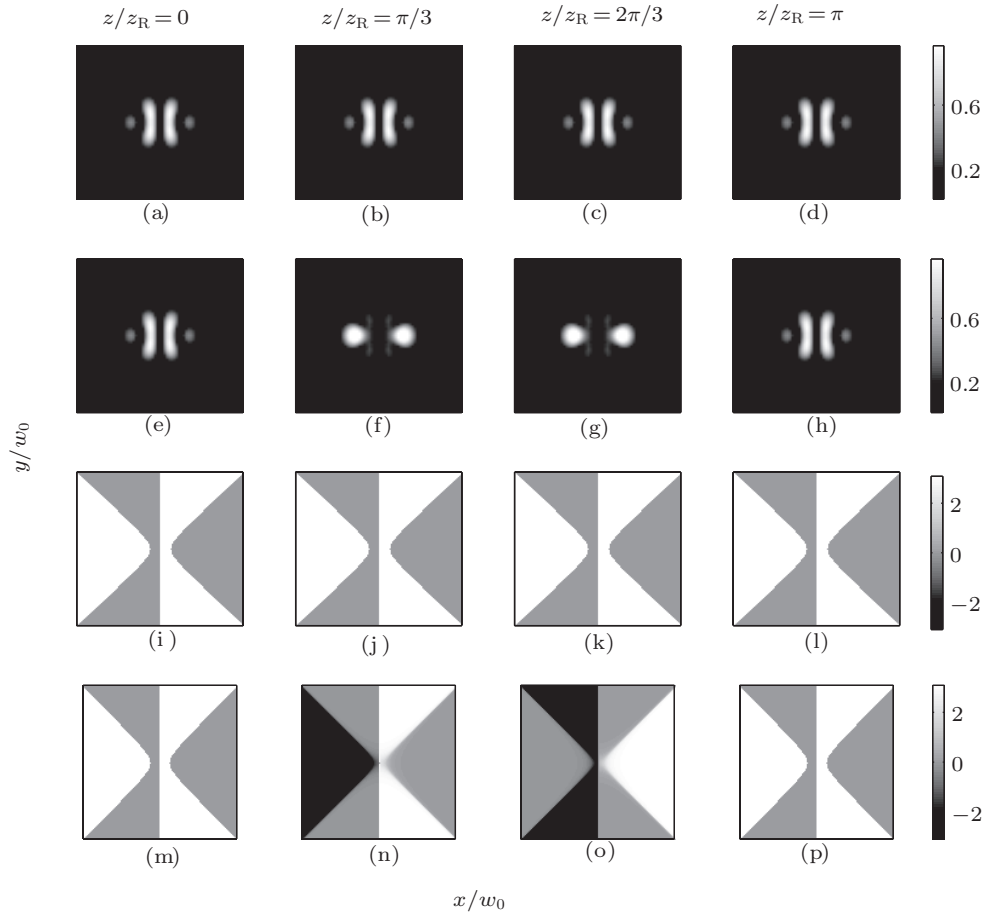


Fig. 1. Propagation dynamics of even-mode standard and elegant IG beams with $p = 3$, $m = 3$, $\epsilon = 2$, $az_{\mathrm{R}} = 1$. The first and third rows represent the evolution of transverse intensity distribution and transverse phase of standard IG beams, respectively, for several values of the normalized propagation distance z/z_{R}, and the second and last rows represent the elegant IG beams.

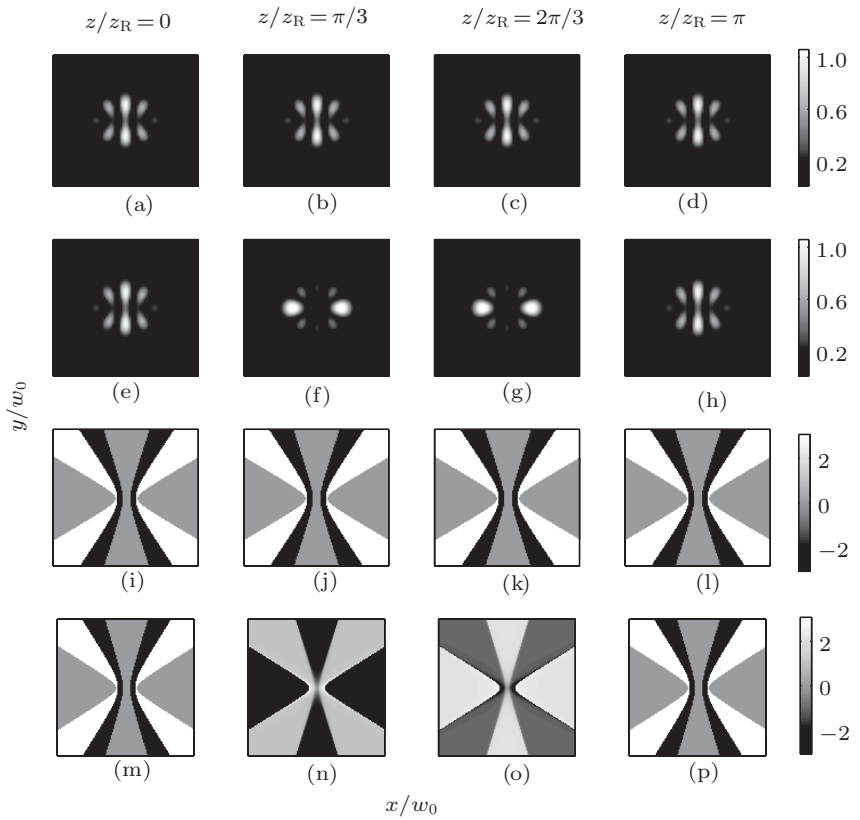


Fig. 2. Propagation dynamics of even-mode standard and elegant IG beams with $p = 4$, $m = 4$, $\epsilon = 2$, $az_R = 1$. Each row represents the same content as the counterpart of Fig. 1.

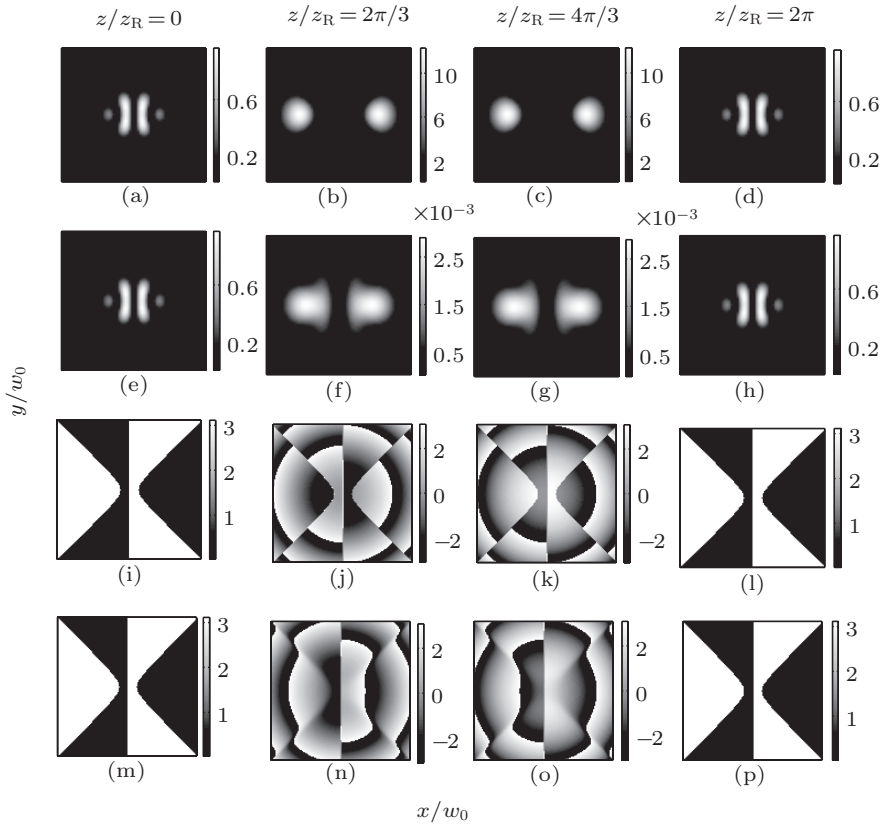


Fig. 3. All of the pictures are the same as those in Fig. 1 except $az_R = 1/4$.

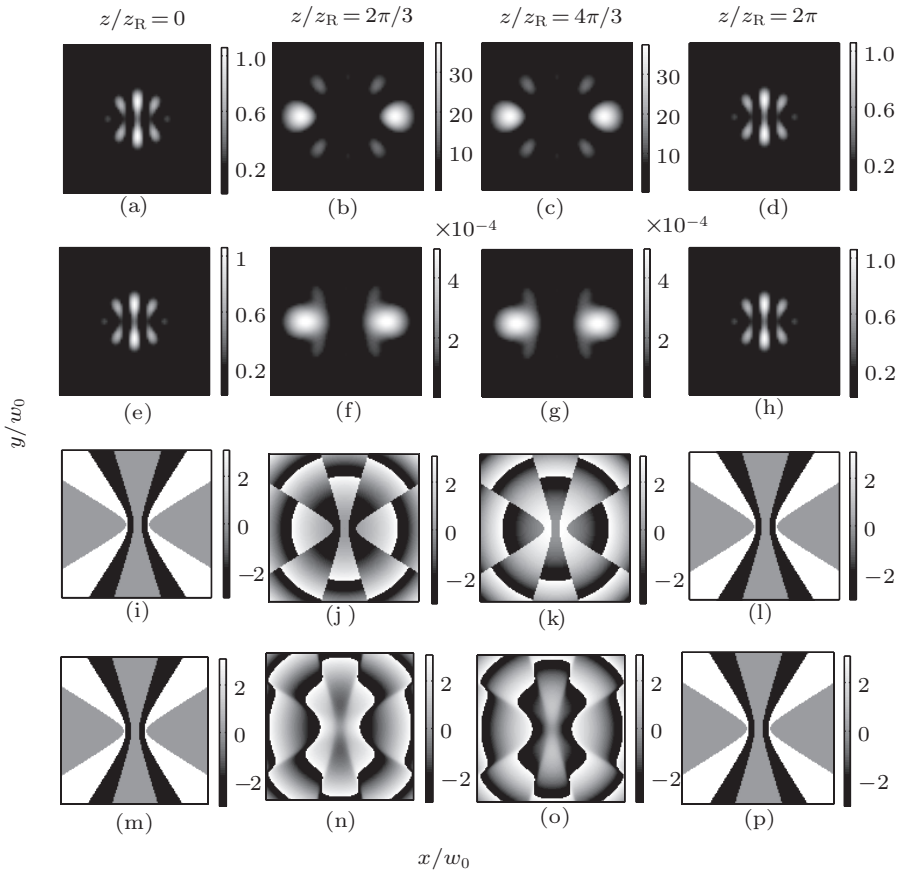


Fig. 4. All of the pictures are the same as those in Fig. 2 except $az_R = 1/4$.

Secondly, we compare the distributions of the transverse intensity of the two classes of beams. We have obtained that the standard IG beams are shape invariant with the width varying periodically along the longitudinal axis and if the beam widths of the input IG beams meet the condition of $w_0 = (2/ak)^{1/2}$ or $az_R = 1$, the beams will be in stable states, which are shown in the first row of Figs. 1 and 2. For the elegant IG beams, displayed in the second row of Figs. 1–4, no matter what the condition is, the transverse intensity distribution varies periodically.

Finally, we demonstrate the symmetry that is referred to in the abstract. The standard field expression can be divided into two parts: the Ince-polynomials with a purely real argument and the Gaussian function with complex parameters. This is lacking in symmetry in mathematics. However, elegant IG beams, composed of Ince-polynomials with complex arguments and the Gaussian function with complex parameters, display better symmetry.

4. Conclusion

The expressions of elegant IG beams in a quadratic-index medium have been given. The process of the derivation proves the existence of the elegant form solutions well. The elegant IG beams form a complete biorthogonal set of the solutions of the paraxial wave equation in the quadratic-index medium in elliptic coordinates, which offers a new way of superposition of paraxial fields.

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