Impulsive synchronization of two coupled complex networks with time-delayed dynamical nodes*

Wang Shu-Guo(王树国)[a)b)†] and Yao Hong-Xing(姚洪兴)[a)]

[a)] *Faculty of Science, Jiangsu University, Zhenjiang* 212013, *China*

[b)] *Department of Mathematics and Physics, Changzhou Campus, Hohai University, Changzhou* 213022, *China*

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In this paper, we investigate the impulsive synchronization between two coupled complex networks with time-delayed dynamical nodes. Based on the Lyapunov stability, the linear feedback control and the impulsive control theories, the linear feedback and the impulsive controllers are designed separately. By using the generalized Barbalat's lemma, the global asymptotic impulsive synchronization of the drive–response complex networks is derived and some corresponding sufficient conditions are also obtained. Numerical examples are presented to verify the effectiveness and the correctness of the synchronization criteria.

Keywords: complex dynamical networks, impulsive synchronization, time-delayed nodes, generalized Barbalat lemma

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1. Introduction

Complex networks widely exist in the real world, including food-webs, ecosystems, metabolic pathways, the Internet, the world wide web, social networks and global economic markets. Since the discovery of the small-world effect[1] and the scale-free feature[2] of complex networks, the analysis and the control of the dynamical behaviour in complex networks have become a very hot topic in various disciplines.[3,4] In general, a complex network is a large set of interconnected communicating and interacting nodes, and a node is a fundamental unit having specific contents and exhibiting dynamical behaviour.

In the past few years, the control and synchronization problem of the complex networks has been extensively investigated in various fields due to its many potential applications.[1−7] Many effective methods, including the pinning control,[8−11] the adaptive control,[12−14] and the impulsive control[15−26], have been used to study the above problem. The synchronization in a network as mentioned above is called inner synchronization,[27] i.e., the synchronization of all the nodes within a network, which has been investigated recently.[5−26] As a matter of fact, there are other kinds of network synchronization in the real world, for example, the synchronization between two or more complex networks regardless of the synchronization of the inner network, called outer synchronization.[27] Therefore, the synchronization between different networks is an interesting and challenging work. Recently, Li *et al.*[27] pioneered the study of the outer synchronization between two unidirectionally coupled complex networks and derived analytically a criterion for them having the identical topological structure. Tang *et al.*[28] discussed the synchronization between two complex dynamical networks with nonidentical topological structures by using the adaptive control method. Li *et al.*[29] studied the synchronization between two networks with different topology structures and different dynamical behaviour with the open-plus-closed-loop method. Li *et al.*[30] discussed the outer synchronization of a coupled discrete-time network. Wu *et al.*[31] studied the problem of the generalized outer synchronization between two complex dynamical networks with different topologies and diverse node dynamics. The adaptive-impulsive synchronization between two complex networks with non-delayed and delayed coupling was discussed in Ref. [32], however, the delays in the dynamical nodes were ignored. In fact, time delays are ubiquitous in natural and artificial systems. One can

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†Corresponding author. E-mail: wsg97@163.com

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find numerous examples in the real world, which are characterized by delayed differential equations having time delays in the dynamical nodes.[33,34] In most literature, time delays in the couplings are considered. However, time delays in the dynamical nodes, which are more complicated, are still relatively unexplored.

To the best of our knowledge, Barbalat's lemma is an effective tool to analyse the stability of the adaptive controlled dynamical system.[35,36] Barbalat's lemma requires that the function is differentiable in the interval $[0,+\infty)$, thus, it is not suitable for an impulsive dynamical system. Zhang *et al.*[37] proved the generalized Barbalat's lemma, in which the function was right continuous on some certain points and investigated the adaptive impulsive synchronization for a non-autonomous chaotic system by using the generalized Barbalat's lemma. Motivated by the above discussions and inspired by Ref. [37], we discuss the problem of the impulsive synchronization between two coupled dynamical networks with identical time-delayed dynamical nodes by using the generalized Barbalat's lemma in the present paper.

2. Model description and preliminaries

In this paper, a complex dynamical network consisting of N identical time-delayed nodes with non-delayed or delayed coupling is considered, which is characterized by

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) &= \boldsymbol{f}(t,\boldsymbol{x}_i(t),\boldsymbol{x}_i(t-\tau_1))+\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\\ &\quad+\sum_{j=1}^{N} c_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\tau_2),\end{aligned}\tag{1}$$

where $i=1,2,\ldots,N$, $\boldsymbol{x}_i(t)=(x_{i1}(t),x_{i2}(t),\ldots,x_{in}(t))^{\mathrm{T}}\in\mathbf{R}^n$ is the state vector of the i-th node, $\boldsymbol{f}:\mathbf{R}\times\mathbf{R}^n\to\mathbf{R}^n$ is a smooth nonlinear function, τ_1 is the time delay in the dynamical nodes, τ_2 is the time delay of the coupling, $\boldsymbol{\Gamma}_1\in\mathbf{R}^{n\times n}$ and $\boldsymbol{\Gamma}_2\in\mathbf{R}^{n\times n}$ are the inner-coupling matrices, $\boldsymbol{B}=(b_{ij})_{N\times N}\in\mathbf{R}^{N\times N}$ and $\boldsymbol{C}=(c_{ij})_{N\times N}\in\mathbf{R}^{N\times N}$ are the weight configuration matrices. If there is a connection from node i to node j $(j\neq i)$, then coupling $b_{ij}>0$, $c_{ij}>0$; otherwise, $b_{ij}=c_{ij}=0$. Particularly, the weight configuration matrix is not assumed to be symmetric, irreducible or dissipative, and the inner coupling matrix need not to be symmetric.

Before proceeding, we give some necessary preliminaries to derive the main results of this paper.

Assumption 1 (A1) For the vector valued function $\boldsymbol{f}(t,\boldsymbol{x}_i(t),\boldsymbol{x}_i(t-\tau_1))$, assuming that there exist positive constants $\gamma_1>0$ and $\gamma_2>0$, such that $\boldsymbol{f}$ satisfies the semi-Lipschitz condition[38]

$$\begin{aligned}&(\boldsymbol{x}_i(t)-\boldsymbol{y}_i(t))^{\mathrm{T}}(f(t,\boldsymbol{x}_i(t),\boldsymbol{x}_i(t-\tau_1))\\ &-f(t,\boldsymbol{y}_i(t),\boldsymbol{y}_i(t-\tau_1)))\\ &\le\gamma_1(\boldsymbol{x}_i(t)-\boldsymbol{y}_i(t))^{\mathrm{T}}(\boldsymbol{x}_i(t)-\boldsymbol{y}_i(t))+\gamma_2(\boldsymbol{x}_i(t-\tau_1)\\ &-\boldsymbol{y}_i(t-\tau_1))^{\mathrm{T}}(\boldsymbol{x}_i(t-\tau_1)-\boldsymbol{y}_i(t-\tau_1)),\end{aligned}$$

for all $\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n$ and $t\ge0$.

Lemma 1 (Generalized Barbalat's Lemma[37]) The sequence $\{x_k\}$ satisfies $0\le x_0<\cdots<x_{k-1}<x_k<\cdots$, $x_k\to\infty$ as $k\to\infty$. There exists a positive constant λ such that $\lambda=\inf\{x_k-x_{k-1}\}>0$, $k=1,2\ldots$ The function $f(x)$ is defined in the interval $[0,+\infty)$ and is differentiable in the interval $[x_{k-1},x_k)$. Suppose that $f(x)$ and $f'(x)$ are uniform bounded in the interval $[x_{k-1},x_k)$, that is, there exist two positive constants M_0 and M_1 such that for every $x\in[x_{k-1},x_k)$ both $|f(x)|\le M_0$ and $|f'(x)|\le M_1$ hold. If the improper integral $\int_{x_0}^{+\infty}f(x)\mathrm{d}x$ converges, then $\lim_{x\to+\infty}f(x)=0$.

Lemma 2 For any vectors $\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^n$ and a positive definite matrix $\boldsymbol{Q}\in\mathbf{R}^{n\times n}$, the following matrix inequality holds, $2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}\le\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}+\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{y}$.

3. Synchronization criteria

In this section, we will make drive–response complex dynamical networks with non-delayed and delayed coupling achieve impulsive synchronization by employing the linear feedback control and the impulsive control methods.

We refer to model (1) as the drive network. The response network can be rewritten in the following form of impulsive differential equation:

$$\begin{aligned}\dot{\boldsymbol{y}}_i &= \boldsymbol{f}(t,\boldsymbol{y}_i(t),\boldsymbol{y}_i(t-\tau_1))+\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_1\boldsymbol{y}_j(t)\\ &\quad+\sum_{j=1}^{N} c_{ij}\boldsymbol{\Gamma}_2\boldsymbol{y}_j(t-\tau_2)+\boldsymbol{u}_i,\quad t\neq t_k,\\ \Delta\boldsymbol{y}_i &= \boldsymbol{y}_i(t_k^+)-\boldsymbol{y}_i(t_k^-)=\boldsymbol{B}_{ik}(\boldsymbol{y}_i-\boldsymbol{x}_i)\\ &\quad t=t_k,\quad k=1,2\ldots\end{aligned}\tag{2}$$

where $\boldsymbol{y}_i(t)=(y_{i1}(t),y_{i2}(t),\ldots,y_{in}(t))^{\mathrm{T}}\in\mathbf{R}^n(i=1,2,\ldots,N)$ is the response state vector of the i-th node and $\boldsymbol{u}_i$ are the linear controllers to be designed later.

Define outer synchronization error $\boldsymbol{e}_i(t) = \boldsymbol{y}_i(t) - \boldsymbol{x}_i(t)$, the error dynamics system is obtained as

$$\begin{aligned}\dot{\boldsymbol{e}}_i &= \boldsymbol{f}(t, \boldsymbol{y}_i(t), \boldsymbol{y}_i(t-\tau_1)) - \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1)) \\ &\quad + l\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t) + \sum_{j=1}^{N} c_{ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\tau_2) \\ &\quad + \boldsymbol{u}_i, \quad t \neq t_k, \\ \Delta\boldsymbol{e}_i &= \boldsymbol{B}_{ik}\boldsymbol{e}_i, \quad t = t_k, \quad k = 1, 2\ldots \end{aligned} \tag{3}$$

The linear controllers $\boldsymbol{u}_i$ are designed as

$$\boldsymbol{u}_i = -d_i\boldsymbol{e}_i(t)\,, \quad i = 1, 2\ldots N, \tag{4}$$

where the feedback gain d_i are constants. We choose the impulsive controller $\boldsymbol{B}_{ik}$ to be an $n \times n$ constant matrix. Then, the objective of this paper is to design linear feedback controller $\boldsymbol{u}_i$ and impulsive controller $\boldsymbol{B}_{ik}$, such that the state of drive dynamical network (1) synchronizes with the state of response dynamical network (2), that is, $\lim_{t\to\infty} \|\boldsymbol{e}_i(t)\| = 0$.

Theorem 1 Suppose that A1 holds. There exists a positive constant $\lambda > 0$ such that $\lambda = \inf\{t_k - t_{k-1}\}$, $k = 1, 2\ldots$ Then drive networks (1) and response networks (2) can achieve impulsive synchronization if the following conditions are satisfied: (i) $d = \min_{i=1}^{N}(d_i) = \gamma_1 + \gamma_2 + \lambda_{\max}(\boldsymbol{Q}) + 1/2\lambda_{\max}(\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}}) + 1/2 + \rho$; (ii) $\lambda_{\max}\left((\boldsymbol{I} + \boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I} + \boldsymbol{B}_{ik})\right) < 1$, $(k = 1, 2, \ldots)$, where $\rho > 0$ is a constant, $\boldsymbol{Q} = \boldsymbol{B}\otimes\boldsymbol{\Gamma}_1$, $\boldsymbol{P} = \boldsymbol{C}\otimes\boldsymbol{\Gamma}_2$, and $\boldsymbol{I} \in \mathbf{R}^{n\times n}$ denotes the identity matrix. The $\otimes$ represents the Kronecker product. The $\lambda_{\max}$ gives the corresponding maximum eigenvalue.

Proof We construct the Lyapunov function as

$$\begin{aligned}V(t) &= \frac{1}{2}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) + \gamma_2\int_{t-\tau_1}^{t}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(\theta)\boldsymbol{e}_i(\theta)\,\mathrm{d}\theta \\ &\quad + \frac{1}{2}\int_{t-\tau_2}^{t}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(\theta)\boldsymbol{e}_i(\theta)\,\mathrm{d}\theta.\end{aligned}$$

For $t \in [t_{k-1}, t_k)$, by A1, the Dini derivative of $V(t)$ along the trajectories of Eq. (3) is

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)[\boldsymbol{f}(t, \boldsymbol{y}_i(t), \boldsymbol{y}_i(t-\tau_1)) - \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1))] + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\tau_2) \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t) - \sum_{i=1}^{N} d_i\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) + \left(\gamma_2 + \frac{1}{2}\right)\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) - \gamma_2\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1)\boldsymbol{e}_i(t-\tau_1) \\ &\quad - \frac{1}{2}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_2)\boldsymbol{e}_i(t-\tau_2) \\ &\leq \gamma_1\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) + \gamma_2\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1)\boldsymbol{e}_i(t-\tau_1) - \sum_{i=1}^{N} d\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) + \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{B}\otimes\boldsymbol{\Gamma}_1)\boldsymbol{e} + \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{C}\otimes\boldsymbol{\Gamma}_2)\boldsymbol{e}(t-\tau_2) \\ &\quad + \left(\gamma_2 + \frac{1}{2}\right)\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) - \gamma_2\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1)\boldsymbol{e}_i(t-\tau_1) - \frac{1}{2}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_2)\boldsymbol{e}_i(t-\tau_2).\end{aligned}$$

Let $\boldsymbol{e}(t) = (e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t))^{\mathrm{T}} \in \mathbf{R}^{nN}$, $\boldsymbol{Q} = (\boldsymbol{B}\otimes\boldsymbol{\Gamma}_1)$, $\boldsymbol{P} = (\boldsymbol{C}\otimes\boldsymbol{\Gamma}_2)$, then according to lemma 2, we have

$$\begin{aligned}\dot{V}(t) &\leq \gamma_1\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) + \gamma_2\boldsymbol{e}^{\mathrm{T}}(t-\tau_1)\boldsymbol{e}(t-\tau_1) + \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}(t) + \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{e}(t) - d\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) \\ &\quad + \left(\gamma_2 + \frac{1}{2}\right)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) - \gamma_2\boldsymbol{e}^{\mathrm{T}}(t-\tau_1)\boldsymbol{e}(t-\tau_1) \\ &\leq \left(\gamma_1 + \gamma_2 + \lambda_{\max}(\boldsymbol{Q}) + \frac{1}{2}\lambda_{\max}(\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}}) + \frac{1}{2} - d\right)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) \\ &\leq -\rho\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) \leq 0.\end{aligned} \tag{5}$$

On the other hand, for $t = t_k$ and $t = t_k^-$, we have

$$V(t_k) = \frac{1}{2}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t_k)\boldsymbol{e}_i(t_k) + \gamma_2\int_{t_k-\tau_1}^{t_k}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(\theta)\boldsymbol{e}_i(\theta)\,\mathrm{d}\theta + \frac{1}{2}\int_{t_k-\tau_2}^{t_k}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(\theta)\boldsymbol{e}_i(\theta)\,\mathrm{d}\theta$$

and

$$\begin{aligned} & V(t_k^-) \\ &= \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t_k^-)\boldsymbol{e}_i(t_k^-) + \gamma_2 \int_{t_k^- - \tau_1}^{t_k^-} \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(\theta)\boldsymbol{e}_i(\theta)\,\mathrm{d}\theta \\ &\quad + \frac{1}{2}\int_{t_k^- - \tau_2}^{t_k^-} \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(\theta)\boldsymbol{e}_i(\theta)\,\mathrm{d}\theta. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \Delta V(t_k) &= \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t_k)\boldsymbol{e}_i(t_k) - \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t_k^-)\boldsymbol{e}_i(t_k^-) \\ &= \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t_k^-)(\boldsymbol{I}+\boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})\boldsymbol{e}_i(t_k^-) \\ &\quad - \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t_k^-)\boldsymbol{e}_i(t_k^-) \\ &\le \frac{1}{2}\sum_{i=1}^{N} \left[\lambda_{\max}((\boldsymbol{I}+\boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})) - 1\right] \\ &\quad \times \boldsymbol{e}_i^{\mathrm{T}}(t_k^-)\boldsymbol{e}_i(t_k^-) \\ &\le 0, \quad k = 1, 2, \ldots \end{aligned} \tag{6}$$

Therefore, we have $\dot{V}(t) \le 0$, thus $0 \le V(t) \le V(0)$ for any $t \ge 0$, that is, $V(t)$ is bounded. According to the Cauchy rule, for every $\varepsilon > 0$, there exists a constant $M > 0$, such that when $t'' > t' > M$, $\quad V(t') - V(t'') < \varepsilon$. Then, we have

$$\begin{aligned} & \rho \int_{t'}^{t''} \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\,\mathrm{d}t \\ &= \rho\Bigg(\int_{t'}^{t_{k_1}} \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\,\mathrm{d}t \\ &\quad + \int_{t_{k_1}}^{t_{k_2}} \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\,\mathrm{d}t + \cdots \\ &\quad + \int_{t_{k_s}}^{t''} \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\,\mathrm{d}t \Bigg) \\ &\le -\Bigg(\int_{t'}^{t_{k_1}} D^+V\,\mathrm{d}t + \int_{t_{k_1}}^{t_{k_2}} D^+V\,\mathrm{d}t + \cdots \\ &\quad + \int_{t_{k_s}}^{t''} D^+V\,\mathrm{d}t \Bigg) \\ &= V(t') - V(t_{k_1}^-) + V(t_{k_1}) - V(t_{k_2}^-) + \cdots \\ &\quad + V(t_{k_s}) - V(t'') \\ &= V(t') - V(t'') < \varepsilon. \end{aligned}$$

By the Cauchy rule, $\rho\int_0^{+\infty}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\,\mathrm{d}t$ is convergent, that is, $\int_0^{+\infty}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\,\mathrm{d}t$ is convergent. From expressions (3), (5) and (6), we obtain that $\sum_{i=1}^{N}\boldsymbol{e}_i(t)$ and $\sum_{i=1}^{N}\dot{\boldsymbol{e}}_i(t)$ are uniformly bounded in the interval $[t_{k-1}, t_k)$, $k = 1, 2\ldots$. Therefore, $\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)$ and $\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\dot{\boldsymbol{e}}_i(t)$ are uniformly bounded in the interval $[t_{k-1}, t_k)$. According to lemma 1, we have $\lim_{t\to+\infty}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) = 0$, i.e., $\lim_{t\to+\infty}\boldsymbol{e}_i(t) = 0$. This implies that error system (3) is globally asymptotically stable about zero. Therefore, the impulsive synchronization between dynamical networks (1) and (2) is achieved. The proof is completed.

When the weight configuration matrix $\boldsymbol{B} = 0$ or $\boldsymbol{C} = 0$, we have the following corollaries.

Corollary 1 When $\boldsymbol{C} = 0$, network (1) is transformed into

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1)) + \sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t). \tag{7}$$

According to the assumption given in theorem 1, the drive-response complex networks can also achieve synchronization, where (i) $d = \min_{i=1}^{N}(d_i) = \gamma_1 + \gamma_2 + \lambda_{\max}(\boldsymbol{Q}) + \rho$, (ii) $\lambda_{\max}\left((\boldsymbol{I}+\boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})\right) < 1, \quad k = 1, 2, \ldots$

Corollary 2 When $\boldsymbol{B} = 0$, network (1) is transformed into

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1)) + \sum_{j=1}^{N} c_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\tau_2), \tag{8}$$

then the drive–response complex networks can also achieve synchronization, where (i) $d = \min_{i=1}^{N}(d_i) = \gamma_1 + \gamma_2 + 1/2\lambda_{\max}(\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}}) + 1/2 + \rho$, (ii) $\lambda_{\max}\left((\boldsymbol{I}+\boldsymbol{B}_{ik})^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{B}_{ik})\right) < 1, \quad k = 1, 2, \ldots$

Remark 1 The synchronization criteria given in theorem 1 and the corollaries are independent of coupling delay. Moreover, delay τ_1 should ensure vector valued function $\boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1))$ is bounded. Otherwise, the A1 is not satisfied, so the synchronization cannot be realized.

Remark 2 Barbalat's lemma is an effective tool to analyse the stability of the adaptive controlled dynamical system.[35,36] The lemma also requires function $f(x)$ to be differentiable in the interval $[0, \infty)$, thus it is not suitable for impulsive dynamical systems. However, in the generalized Barbalat's lemma, function $f(x)$ is only right continuous at some certain points, so it can be used to study the impulsive dynamical systems.

Remark 3 Compared with the other control methods in the literature, the impulsive controller is relatively simple and is easy to implement. In the impulsive synchronization method, the synchronization

impulses are sent only to the response network at discrete impulsive instances, which can reduce the information redundancy in the transmitted signal, increase the robustness and reduce the control cost.

4. Numerical simulations

In this section, numerical simulations are given to verify and demonstrate the effectiveness of the proposed method for impulsively synchronizing two coupled complex networks with time-delayed dynamical nodes. The time-delayed Chua oscillator is considered as the node dynamic.

The time-delayed Chua chaotic system[39] is described by $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}(t-\tau_1))$, where $\boldsymbol{x} = (x_1, x_2, x_3)^{\mathrm{T}}$, matrix $\boldsymbol{A}$ and function $\boldsymbol{f}$ respectively satisfy

$$\boldsymbol{A} = \begin{pmatrix} -\alpha(1+m_2) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\omega \end{pmatrix}$$

and

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}(t-\tau_1)) = \begin{pmatrix} -\alpha g(x_1) \\ 0 \\ -0.2\beta \sin(0.5 x_1(t-\tau_1)) \end{pmatrix}, \quad (9)$$

where $g(x_1) = -0.5\alpha(m_1 - m_2)(|x_1 + 1| - |x_1 - 1|)$, parameters $(\alpha, \beta, \omega, m_1, m_2)$ are chosen to be $(10, 19.53, 0.1636, -1.4325, 0.7831)$, and time delay $\tau_1 = 0.02$. Then, we obtain $|g(y_1) - g(x_1)| \leq |\alpha(m_1 - m_2)|\,|y_1 - x_1|$.

To satisfy A1, considering that we can always find $\eta > 0$ such that $|xy| \leq \eta x^2/2 + y^2/2\eta$, then we have[18,40]

$$\begin{aligned} & \boldsymbol{e}_i^{\mathrm{T}}(t)[\boldsymbol{f}(t, \boldsymbol{y}_i(t), \boldsymbol{y}_i(t-\tau_1)) \\ & - \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1))] \\ \leq\ & \lambda_{\max}(\tilde{\boldsymbol{A}})\boldsymbol{e}_i^{\mathrm{T}}\boldsymbol{e}_i + \beta/20\eta e_{i1}^2(t-\tau_1) \\ \leq\ & \gamma_1 \boldsymbol{e}_i^{\mathrm{T}}\boldsymbol{e}_i + \gamma_2 \boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1)\boldsymbol{e}_i(t-\tau_1), \end{aligned}$$

where $\tilde{\boldsymbol{A}} = ((\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})/2 + \mathrm{diag}(|\alpha(m_1 - m_2)|),\ 0,\ \eta\beta/20)$, γ_1 and γ_2 can be determined by choosing appropriate constant $\eta > 0$.

We consider the two coupled complex dynamical networks, (1) and (3), with coupling delays consisting of 5 identical time-delayed chaotic systems. We take the weight configuration coupling matrices to be

$$\boldsymbol{B} = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & 0 & -2 \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} -3 & 2 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

In the simulation, the equidistant impulsive interval is $t_k - t_{k-1} = 0.1$, $k = 1, 2, \ldots$, the impulsive control matrices are $\boldsymbol{B}_{ik} = \mathrm{diag}(-0.5, -0.5, -0.5)$, the time coupling delay is $\tau_2 = 1$ and $\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_2 = I_{3\times 3}$. The calculation yields $\gamma_1 = 11.3931$, $\gamma_2 = 0.9765$, $\lambda_{\max}(\boldsymbol{Q}) = 0.5545$, and $\lambda_{\max}(\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}}) = 19.4532$. Thus, according to theorem 1, we choose $d = 25$. Let $\|\boldsymbol{e}(t)\| = \sqrt{\sum_{i=1}^{N}[(y_{i1} - x_{i1})^2 + (y_{i2} - x_{i2})^2 + (y_{i3} - x_{i3})^2]}$ be the 2-norm of the total synchronization error at time t for $t \in [0, +\infty)$. Figure 1 displays error $\|\boldsymbol{e}(t)\|$ of drive–response complex networks with non-delayed and delayed coupling. The numerical result shows that the impulsive scheme for the drive–response complex network is effective, as described by theorem 1. Furthermore, when impulsive control matrices $\boldsymbol{B}_{ik} = \mathrm{diag}(-0.8, -0.8, -0.8)$, $d = 30$, and the other conditions are chosen as the above ones, the simulation result is shown in Fig. 2. From Figs. 1 and 2, we find that the control gains and the impulsive control matrices have an effect on the synchronization process of drive–response complex networks.

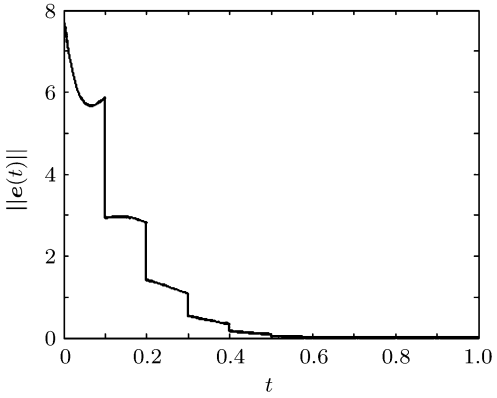


Fig. 1. Synchronization error $\|\boldsymbol{e}(t)\|$.

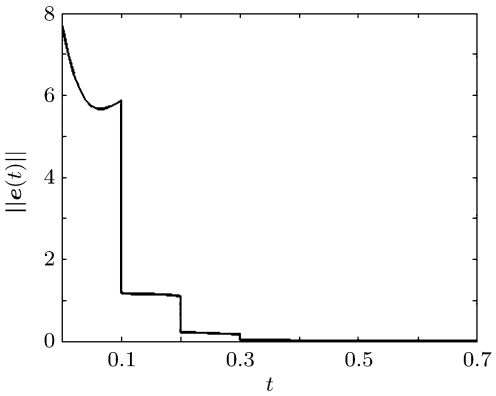


Fig. 2. Synchronization error $\|\boldsymbol{e}(t)\|$.

5. Conclusion

In this paper, we investigate the impulsive synchronization problem of two coupled dynamical networks with time-delayed dynamical nodes by employing linear feedback control and impulsive control strategies. Some criteria for the synchronization are derived and the linear feedback and the impulsive controllers are designed. By using the generalized Barbalat lemma, the drive–response complex network is proved to be globally asymptotic synchronous. Compared with the other control methods in the literature, this scheme is efficient and practical in dealing with the synchronization problem. As an application, numerical simulations of the delayed Chua oscillations are given to demonstrate the effectiveness and the practicability of the theoretical results.

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