Adaptive lag synchronization and parameter identification of fractional order chaotic systems*

Zhang Ruo-Xun(张若洵)[a)b)] and Yang Shi-Ping(杨世平)[a)†]

[a)] College of Physics Science and Information Engineering, Hebei Normal University, Shijiazhuang 050016, China
[b)] College of Elementary Education, Xingtai University, Xingtai 054001, China

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This paper proposes a simple scheme for the lag synchronization and the parameter identification of fractional order chaotic systems based on the new stability theory. The lag synchronization is achieved and the unknown parameters are identified by using the adaptive lag laws. Moreover, the scheme is analytical and is simple to implement in practice. The well-known fractional order chaotic Lü system is used to illustrate the validity of this theoretic method.

Keywords: adaptive lag synchronization, parameter identification, fractional order chaotic system

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1. Introduction

Even though fractional calculus is a mathematical topic with a history of more than 300 years, its application to physics and engineering has attracted a great deal of attention only in recent years. Most recently, many authors have begun to investigate the chaotic dynamics and the synchronization of fractional dynamical systems.[1−17]

It is well known that lag synchronization has important technological implications.[18] Generally, time delay is inevitable in practical engineering applications. For example, in a telephone communication system, the voice heard on the receiver side at time t is usually the voice from the transmitter side at time $t-\tau$. So, in many cases, it is more reasonable to require the slave system to synchronize the master system with a time delay τ. In recent years, lag synchronization has attracted a great deal of attention and has been widely studied for integral order systems.[19−23] There is still no report on the lag synchronization of fractional order chaotic systems. Motivated by the existing works, we discuss the adaptive lag synchronization and the parameter identification of fractional order chaotic systems in this paper. Numerical simulations are presented to show the effectiveness of the proposed scheme.

2. Theoretical analyses

Consider a class of uncertain fractional order chaotic systems described by

$$\frac{\mathrm{d}^{\alpha}\boldsymbol{x}}{\mathrm{d}t^{\alpha}} = \boldsymbol{F}(\boldsymbol{x})\boldsymbol{\theta} + \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \qquad (1)$$

where $0 < \alpha < 1$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$ is the state vector of the system, $\boldsymbol{f}: \mathbf{R}^n \to \mathbf{R}^n$ is a continuous vector function, $\boldsymbol{F}: \mathbf{R}^n \to \mathbf{R}^{n\times p}$ is a function matrix, $\boldsymbol{\theta} \in \mathbf{R}^p$ is an unknown parameter vector and $\boldsymbol{A} \in \mathbf{R}^{n\times n}$ is an unknown constant matrix.

Let system (1) be the drive system, then the controlled response system is given by

$$\frac{\mathrm{d}^{\alpha}\boldsymbol{y}}{\mathrm{d}t^{\alpha}} = \boldsymbol{F}(\boldsymbol{y})\hat{\boldsymbol{\theta}} + \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{u} = \hat{\boldsymbol{A}}\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{u}, \quad (2)$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in \mathbf{R}^n$ is the state vector, $\hat{\boldsymbol{\theta}} \in \mathbf{R}^p$ represents the estimate vector of unknown parameter vector $\boldsymbol{\theta}$, $\boldsymbol{u} \in \mathbf{R}^n$ is a controller to be determined and $\hat{\boldsymbol{A}}$ $in\mathbf{R}^{n\times n}$ is the estimated matrix of unknown matrix $\boldsymbol{A}$.

We now assume that the following hypothesis pertains to drive system (1) and response system (2).

Hypothesis 1 (H1) For a bounded matrix $\boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}$ with elements depending on $\boldsymbol{x}$ and $\boldsymbol{y}$,

$$\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y} - \boldsymbol{x}).$$

*Project supported by the Natural Science Foundation of Hebei Province, China (Grant No. A2010000343).
†Corresponding author. E-mail: yangship@mail.hebtu.edu.cn; xtzhrx@126.com
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Remark 1 It is important to note that many known fractional order chaotic systems can be described by H1, such as the fractional order Lorenz, Chen, Lü and Rössler chaotic system, the hyperchaotic Chen, Rössler and Lorenz systems.

With H1, the error system for Eqs. (1) and (2) is

$$\frac{\mathrm{d}^\alpha \boldsymbol{e}}{\mathrm{d}t^\alpha} = (\hat{\boldsymbol{A}} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}})\boldsymbol{e} + \boldsymbol{F}(\boldsymbol{x})\boldsymbol{e}_{\boldsymbol{\theta}} + \boldsymbol{u}, \tag{3}$$

where $\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{x}(t-\tau)$, and $\boldsymbol{e}_{\boldsymbol{\theta}}(t) = \hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}$.

Theorem 1 If H1 holds, then error system (3) is asymptotically stable under the adaptive controller

$$\boldsymbol{u} = -k(t)\boldsymbol{e}(t) \tag{4}$$

and the updating law

$$\begin{aligned} \frac{\mathrm{d}^\alpha k(t)}{\mathrm{d}t^\alpha} &= \sigma \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t), \\ \frac{\mathrm{d}^\alpha \hat{\boldsymbol{\theta}}(t)}{\mathrm{d}t^\alpha} &= -\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x}(t-\tau))\boldsymbol{e}(t), \end{aligned} \tag{5}$$

where σ is a positive constant.

Proof Since $\hat{\boldsymbol{A}}$ is an unknown constant matrix and $\boldsymbol{M}_{x,y}$ is a bounded matrix with elements depending on $\boldsymbol{x}$ and $\boldsymbol{y}$, there exist two nonnegative constants, β and γ, such that $||\hat{\boldsymbol{A}}||_2 \leq \beta$ and $||\boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}||_2 \leq \gamma$. It follows that $\left\|(\hat{\boldsymbol{A}} + \hat{\boldsymbol{A}}^{\mathrm{T}})/2\right\|_2 \leq \beta$ and $\left\|(\boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}^{\mathrm{T}})/2\right\|_2 \leq \gamma$.

Substituting Eq. (4) into Eq. (3), we can obtain

$$\begin{aligned} &\frac{\mathrm{d}^\alpha \boldsymbol{e}(t)}{\mathrm{d}t^\alpha} \\ &= (\hat{\boldsymbol{A}} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}})\boldsymbol{e}(t) + \boldsymbol{F}(\boldsymbol{x}(t-\tau))\boldsymbol{e}_{\boldsymbol{\theta}}(t) - k(t)\boldsymbol{e}(t) \\ &= (\hat{\boldsymbol{A}} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}})\boldsymbol{e}(t) + \boldsymbol{F}(\boldsymbol{x}(t-\tau))\boldsymbol{e}_{\boldsymbol{\theta}}(t) - k^*\boldsymbol{e}(t) \\ &\quad - (k(t) - k^*)\boldsymbol{e}(t) \\ &= (\hat{\boldsymbol{A}} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}})\boldsymbol{e}(t) + \boldsymbol{F}(\boldsymbol{x}(t-\tau))\boldsymbol{e}_{\boldsymbol{\theta}}(t) \\ &\quad - k^*\boldsymbol{e}(t) - e_k(t)\boldsymbol{e}(t), \end{aligned} \tag{6}$$

where $e_k(t) = k(t) - k^*$ and $k^* > \beta + \gamma$ is a positive constant.

From Eqs. (5) and (6), we have

$$\begin{aligned} &H(\boldsymbol{e}(t), e_k(t), \boldsymbol{e}_{\boldsymbol{\theta}}(t)) \\ &= \left(\boldsymbol{e}^{\mathrm{T}}(t) \quad \frac{1}{\sqrt{\sigma}}e_k(t) \quad \boldsymbol{e}_{\boldsymbol{\theta}}^{\mathrm{T}}(t)\right) \begin{pmatrix} \dfrac{\mathrm{d}^\alpha \boldsymbol{e}(t)}{\mathrm{d}t^\alpha} \\ \dfrac{1}{\sqrt{\sigma}}\dfrac{\mathrm{d}^\alpha e_k(t)}{\mathrm{d}t^\alpha} \\ \dfrac{\mathrm{d}^\alpha \boldsymbol{e}_\theta(t)}{\mathrm{d}t^\alpha} \end{pmatrix} \\ &= \left(\boldsymbol{e}^{\mathrm{T}}(t) \quad e_k(t) \quad \boldsymbol{e}_{\boldsymbol{\theta}}^{\mathrm{T}}(t)\right) \begin{pmatrix} \dfrac{\mathrm{d}^\alpha \boldsymbol{e}(t)}{\mathrm{d}t^\alpha} \\ \dfrac{1}{\sigma}\dfrac{\mathrm{d}^\alpha k(t)}{\mathrm{d}t^\alpha} \\ \dfrac{\mathrm{d}^\alpha \hat{\boldsymbol{\theta}}(t)}{\mathrm{d}t^\alpha} \end{pmatrix} \\ &= \left(\boldsymbol{e}^{\mathrm{T}}(t) \ e_k(t) \ \boldsymbol{e}_{\boldsymbol{\theta}}^{\mathrm{T}}(t)\right) \\ &\quad \times \begin{pmatrix} \hat{\boldsymbol{A}} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}} - k^*\boldsymbol{I} & -\boldsymbol{e}(t) & \boldsymbol{F}(\boldsymbol{x}(t-\tau)) \\ \boldsymbol{e}^{\mathrm{T}}(t) & 0 & 0 \\ -\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x}(t-\tau)) & 0 & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} \boldsymbol{e}(t) \\ e_k(t) \\ \boldsymbol{e}_{\boldsymbol{\theta}}(t) \end{pmatrix} \\ &= \boldsymbol{e}^{\mathrm{T}}(\hat{\boldsymbol{A}} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}} - k^*\boldsymbol{I})\boldsymbol{e} + \boldsymbol{e}_k\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} \\ &\quad - \boldsymbol{e}_{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x}(t-\tau))\boldsymbol{e} - \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}e_k + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{x}(t-\tau))\boldsymbol{e}_{\boldsymbol{\theta}} \\ &= \boldsymbol{e}^{\mathrm{T}}(t)(\hat{\boldsymbol{A}} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}} - k^*\boldsymbol{I})\boldsymbol{e}(t) \\ &= \boldsymbol{e}^{\mathrm{T}}(t)\left(\frac{\hat{\boldsymbol{A}} + \hat{\boldsymbol{A}}^{\mathrm{T}}}{2} + \frac{\boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}^{\mathrm{T}}}{2} - k^*\boldsymbol{I}\right)\boldsymbol{e}(t) \\ &\leq (\beta + \gamma - k^*)\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) \leq 0. \end{aligned}$$

According to the new stability theory of fractional order systems,[14] error system (3) is asymptotically stable under controller (4) and updating law (5). That is $\boldsymbol{e}(t) \to 0$, $e_k(t) \to 0$, and $\boldsymbol{e}_\theta(t) \to 0$, as $t \to +\infty$. Then, the lag synchronization and the parameter identification of the coupled identical fractional order chaotic systems is achieved simultaneously. The proof is completed.

Remark 2 The control strength $k(t)$ is automatically adapted to a suitable strength k^* depending on the initial values, which is significantly different from the usual linear feedback.

Remark 3 The controller $k(t)e(t)$ can include only one feedback term $e_i(t)$ for some chaotic systems.

3. Example

In this section, one illustrative example about the well-known fractional order Lü chaotic system[3] is given to show the effectiveness of the above-mentioned scheme. The fractional order Lü system can be described by the equations

$$\begin{aligned} \frac{\mathrm{d}^\alpha x_1}{\mathrm{d}t^\alpha} &= a(x_2 - x_1), \\ \frac{\mathrm{d}^\alpha x_2}{\mathrm{d}t^\alpha} &= cx_2 - x_1x_3, \end{aligned}$$

$$\frac{\mathrm{d}^\alpha x_3}{\mathrm{d}t^\alpha} = x_1x_2 - bx_3, \tag{7}$$

It can be presented in the form of

$$\begin{aligned}\begin{pmatrix}\frac{\mathrm{d}^\alpha x_1}{\mathrm{d}t^\alpha}\\ \frac{\mathrm{d}^\alpha x_2}{\mathrm{d}t^\alpha}\\ \frac{\mathrm{d}^\alpha x_3}{\mathrm{d}t^\alpha}\end{pmatrix} &= \begin{pmatrix} x_2 - x_1 & 0 & 0\\ 0 & 0 & x_2\\ 0 & -x_3 & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\end{pmatrix}\\ &\quad + \begin{pmatrix} 0\\ -x_1x_3\\ x_1x_2\end{pmatrix}\\ &= \begin{pmatrix} -a & a & 0\\ 0 & c & 0\\ 0 & 0 & -b\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}\\ &\quad + \begin{pmatrix} 0\\ -x_1x_3\\ x_1x_2\end{pmatrix}.\end{aligned} \tag{8}$$

Similarly, the response system becomes

$$\begin{aligned}\begin{pmatrix}\frac{\mathrm{d}^\alpha y_1}{\mathrm{d}t^\alpha}\\ \frac{\mathrm{d}^\alpha y_2}{\mathrm{d}t^\alpha}\\ \frac{\mathrm{d}^\alpha y_3}{\mathrm{d}t^\alpha}\end{pmatrix} &= \begin{pmatrix} y_2 - y_1 & 0 & 0\\ 0 & 0 & y_2\\ 0 & -y_3 & 0\end{pmatrix}\begin{pmatrix} \hat{a}\\ \hat{b}\\ \hat{c}\end{pmatrix}\\ &\quad + \begin{pmatrix} 0\\ -y_1y_3\\ y_1y_2\end{pmatrix} + \begin{pmatrix} u_1\\ u_2\\ u_3\end{pmatrix},\end{aligned} \tag{9}$$

where a, b, and c are unknown positive parameters, $\hat{a}(t)$, $\hat{b}(t)$, and $\hat{c}(t)$ are the estimates of a, b and c, respectively. The $\boldsymbol{u} = (u_1,\ u_2,\ u_3)^{\mathrm{T}}$ is the controller vector. We define the parameter errors as $e_a(t) = \hat{a}(t) - a, e_b(t) = \hat{b}(t) - b, e_c(t) = \hat{c}(t) - c$ and the errors in the variables as $e_i(t) = y_i(t) - x_i(t-\tau),\ (i = 1, 2, 3)$. Subtracting Eq. (8) from Eq. (19) yields the error dynamical system between Eqs. (8) and (9),

$$\begin{aligned}&\begin{pmatrix}\frac{\mathrm{d}^\alpha e_1(t)}{\mathrm{d}t^\alpha}\\ \frac{\mathrm{d}^\alpha e_2(t)}{\mathrm{d}t^\alpha}\\ \frac{\mathrm{d}^\alpha e_3(t)}{\mathrm{d}t^\alpha}\end{pmatrix}\\ &= \begin{pmatrix} -\hat{a}(t) & \hat{a}(t) & 0\\ -y_3(t) & \hat{c} & -x_1(t-\tau)\\ y_2(t) & x_1(t-\tau) & -\hat{b}\end{pmatrix}\begin{pmatrix} e_1(t)\\ e_2(t)\\ e_3(t)\end{pmatrix}\\ &\quad + \begin{pmatrix} x_2(t-\tau) - x_1(t-\tau) & 0 & 0\\ 0 & 0 & x_2(t-\tau)\\ 0 & -x_3(t-\tau) & 0\end{pmatrix}\\ &\quad \times \begin{pmatrix} e_a(t)\\ e_b(t)\\ e_c(t)\end{pmatrix} + \begin{pmatrix} u_1\\ u_2\\ u_3\end{pmatrix}.\end{aligned} \tag{10}$$

Let the adaptive controller be

$$u_1 = u_3 = 0, u_2 = -k(t)e_2(t), \tag{11}$$

and the updating law be

$$\begin{aligned}&\frac{\mathrm{d}^\alpha k(t)}{\mathrm{d}t^\alpha} = \sigma e_2^2(t),\\ &\frac{\mathrm{d}^\alpha e_a(t)}{\mathrm{d}t^\alpha} = -(x_2(t-\tau) - x_1(t-\tau))e_1(t),\\ &\frac{\mathrm{d}^\alpha e_b(t)}{\mathrm{d}t^\alpha} = x_3(t-\tau)e_3(t),\\ &\frac{\mathrm{d}^\alpha e_c(t)}{\mathrm{d}t^\alpha} = -x_2(t-\tau)e_2(t).\end{aligned}$$

In numerical simulations,[24] we choose $\alpha = 0.90$, unknown parameter vector $(a,\ b,\ c)^{\mathrm{T}} = (35,\ 3,\ 28)^{\mathrm{T}}$, so the fractional order Lü system has a chaotic attractor. The initial conditions are $\boldsymbol{x}(0) = (2, -2, 1)^{\mathrm{T}}$, $y(0) = (-2, 2, -1)^{\mathrm{T}}$, $k(0) = 0$, $(\hat{a}(0), \hat{b}(0), \hat{c}(0))^{\mathrm{T}} = (20, 1, 15)^{\mathrm{T}}$ and $\tau = 0.2$. The numerical results are illustrated in Figs. 1 and 2. From Fig. 1, we can see that the errors converge to zero as time increases. It shows that the trajectories of the response system asymptotically converge to those of the drive system. Figure 2 shows that estimates that $\hat{a}$, $\hat{b}$ and $\hat{c}$ of the unknown parameters converge to $a = 35$, $b = 3$ and $c = 28$, respectively, as $t \to \infty$.

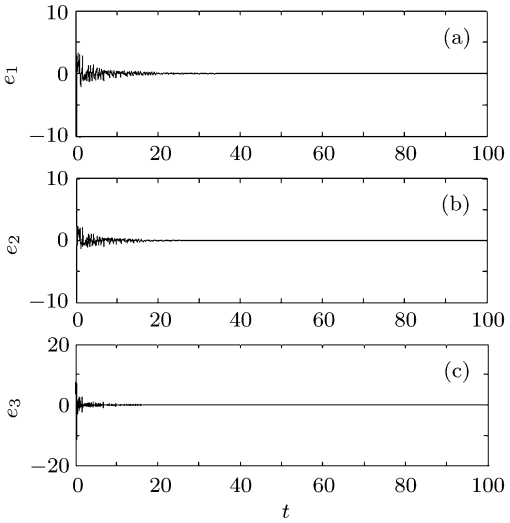


Fig. 1. Synchronization errors of two fractional order Lü chaotic systems with unknown parameters. (a) e_1, (b) e_2, and (c) e_3.

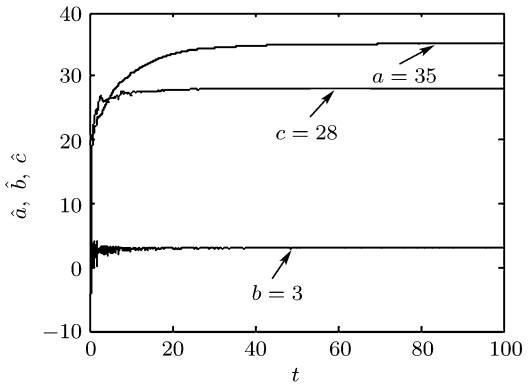


Fig. 2. Change of parameters $\hat{a}$, $\hat{b}$ and $\hat{c}$ of response system with time t.

4. Conclusion

In this paper, based on the new stability theory of fractional order systems,[7] a systematic adaptive scheme has been developed to investigate the lag synchronization of fractional order chaotic systems and identify the unknown parameters simultaneously for the first time. In comparison with the previous methods, the present controller not only is simpler but also can be generalized. Especially, the controller can include only one feedback term e_i for some chaotic systems. Therefore, it can be easily implemented in practice. The numerical simulation demonstrates the effectiveness and the validity of the controller. We also believe that such a simple adaptive controller will be very beneficial for the applications of fractional order chaos synchronization.

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