Hölder continuity of generalized chaos synchronization in complex networks*

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A complex network consisting of chaotic systems is considered and the existence of the Hölder continuous generalized synchronization in the network is studied. First, we divide nodes of the network into two parts according to their dynamical behaviour. Then, based on the Schauder fixed point theorem, sufficient conditions for the existence of the generalized synchronization between them are derived. Moreover, the results are theoretically proved. Numerical simulations validate the theory.

Keywords: generalized synchronization, Hölder continuity, Schauder fixed point, chaotic system

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1. Introduction

Chaos synchronization, including complete synchronization (CS), phase synchronization (PS), lag synchronization (LS), anticipated synchronization (AS) and generalized synchronization (GS), has been extensively studied over the past 20 years.[1−9] Given two dynamical systems, X and Y, if there exists a manifold $S = \{(X, Y)|X = \Phi(Y)\}$, which includes at least one attractor, then X and Y carry out the GS, and Φ is called the GS map. The GS therein is an interesting and more important topic, which extends the notion of the CS and includes many synchronization phenomena observed in laboratory experiments.[10−14] In 1995, Rulkov *et al.* first described the GS phenomenon and presented the idea of mutual false nearest neighbors to detect the GS.[15] In 1996, Abarbanel *et al.* suggested an auxiliary system method to study the GS in driving-response systems,[16] and the theory about that method was given in Ref. [17]. In 2005, Hramov *et al.* proposed a modified system approach to study the GS.[18] Recently, based on the modified system method, we have presented some theoretical results about the GS in Refs. [19]–[25]. However, only two or three systems were considered in above references.

In this paper, we will further study the GS of chaotic systems in complex networks. Networks exist everywhere, for example, each person is a unit of a network of social relationships. Therefore the research of the synchronization in complex networks is most valuable. Some references, such as Refs. [26]–[29], have presented some related results. However, for the GS, only numerical simulations were given in many available literatures.[30] This paper will give a strict proof of the existence of the Hölder continuous GS in networks. Additionally, the network model considered here is very universal.

2. Hölder continuity of GS

For a function $\boldsymbol{f}(\boldsymbol{x})$, if it satisfies $|\boldsymbol{f}(\boldsymbol{x}_1) - \boldsymbol{f}(\boldsymbol{x}_2)| \leq L|\boldsymbol{x}_1 - \boldsymbol{x}_2|^\alpha$, where $|\cdot|$ denotes the Euclidean norm, $0 < \alpha < 1$, $L > 0$, then it is considered to have α order Hölder continuity.

Consider a complex network consisting of N chaotic systems described by

$$\frac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \boldsymbol{A}_i\boldsymbol{x}_i + \boldsymbol{f}_i(\boldsymbol{x}_i) - \boldsymbol{K}_i\sum_{j=1}^{N} c_{ij}\boldsymbol{x}_j, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $\boldsymbol{x}_i \in \mathbb{R}^n$, $\boldsymbol{A}_i$ denote $n \times n$ constant matrices, $\boldsymbol{f}_i(\boldsymbol{x}_i)$ are smooth maps, $\boldsymbol{K}_i \in \mathbb{R}^{n\times n}$ represent the coupling strength and $\boldsymbol{C} = (c_{ij}) \in \mathbb{R}^{N\times N}$ is the linking matrix, i.e., if there is a connection between nodes

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i and j ($i \neq j$), then $c_{ij} = c_{ji} = -1$, otherwise $c_{ij} = c_{ji} = 0$, and $c_{ii} = -\sum_{j=1,j\neq i}^{N} c_{ij} = -\sum_{j=1,j\neq i}^{N} c_{ji}$. We divide the nodes of network (1) into two parts, X and Y, and consider the GS between them. Without loss of generality, suppose that X consists of the first p nodes of network (1) and that the others compose Y. In order to simplify the discussion, we classify the GS between X and Y into several types according to their modified system dynamical behaviour. For each node of network (1), the modified system can be written as

$$\frac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \boldsymbol{A}_i\boldsymbol{x}_i + \boldsymbol{f}_i(\boldsymbol{x}_i) - \boldsymbol{K}_i c_{ii}\boldsymbol{x}_i, \quad i = 1, 2, \ldots, N. \tag{2}$$

We mainly investigate two types of GS in the following sections.

2.1. The first kind of Hölder continuous GS

In Eq. (2), we assume that the modified systems of Y are chaotic and X has equilibrium points $\boldsymbol{x}_i = \boldsymbol{0}$ ($i = 1, 2, \ldots, p$) (If $\boldsymbol{x}_i \neq \boldsymbol{0}$, we can use the linear transformation to make $\boldsymbol{0}$ to be their equilibrium points). Then the first p systems in Eq. (2) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} &= \boldsymbol{A}_i\boldsymbol{x}_i - \boldsymbol{K}_i c_{ii}\boldsymbol{x}_i + \boldsymbol{f}_i(\boldsymbol{x}_i) + \boldsymbol{f}_i'(\boldsymbol{0})\boldsymbol{x}_i - \boldsymbol{f}_i'(\boldsymbol{0})\boldsymbol{x}_i \\ &= \boldsymbol{B}_i\boldsymbol{x}_i + \boldsymbol{F}_i(\boldsymbol{x}_i), \quad i = 1, 2, \ldots, p, \end{aligned} \tag{3}$$

where $\boldsymbol{B}_i = \boldsymbol{A}_i - \boldsymbol{K}_i c_{ii} + \boldsymbol{f}_i'(\boldsymbol{0})$, $\boldsymbol{F}_i(\boldsymbol{x}_i) = \boldsymbol{f}_i(\boldsymbol{x}_i) - \boldsymbol{f}_i'(\boldsymbol{0})\boldsymbol{x}_i$. In addition, the real parts of all the eigenvalues of matrix $\boldsymbol{B}_i$ are negative.

For clarity, we introduce $\boldsymbol{y}_i$ ($i = p+1, p+2, \ldots, N$) to replace $\boldsymbol{x}_i$ ($i = p+1, p+2, \ldots, N$). Now, network (1) can be rewritten as

$$\begin{aligned} Y:\ \frac{\mathrm{d}\boldsymbol{y}_i}{\mathrm{d}t} &= \boldsymbol{A}_i\boldsymbol{y}_i + \boldsymbol{f}_i(\boldsymbol{y}_i) - \boldsymbol{K}_i\sum_{j=p+1}^{N} c_{ij}\boldsymbol{y}_j \\ &\quad - \boldsymbol{K}_i\sum_{j=1}^{p} c_{ij}\boldsymbol{x}_j, \\ &\quad i = p+1, p+2, \ldots, N, \end{aligned} \tag{4}$$

$$\begin{aligned} X:\ \frac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} &= \boldsymbol{B}_i\boldsymbol{x}_i + \boldsymbol{F}_i(\boldsymbol{x}_i) - \boldsymbol{K}_i\sum_{j=1,j\neq i}^{p} c_{ij}\boldsymbol{x}_j \\ &\quad - \boldsymbol{K}_i\sum_{j=p+1}^{N} c_{ij}\boldsymbol{y}_j, \quad i = 1, 2, \ldots, p. \end{aligned} \tag{5}$$

Suppose $\boldsymbol{f}_i$ in network (1) satisfies the Lipschitz condition, i.e., $|\boldsymbol{f}_i(\boldsymbol{u}_i) - \boldsymbol{f}_i(\boldsymbol{u}_i')| \leq L|\boldsymbol{u}_i - \boldsymbol{u}_i'|$, where L is a positive constant, $|\cdot|$ denotes the vector norm; $\| \mathrm{e}^{(\boldsymbol{A}_i - \boldsymbol{K}_i c_{ii})t} \| \leq M_1 \mathrm{e}^{\zeta t}$, where $M_1 > 0$, $\zeta > 0$, $t > 0$, $\|\cdot\|$ denotes the matrix or the operator norm. When $t \leq \tau$, $\boldsymbol{y}_i(\tau) = \boldsymbol{\eta}_i$, the solutions $\boldsymbol{y}_i(t) = \boldsymbol{y}_i(t; \tau, \boldsymbol{\eta}_i)$ ($i = p+1, p+2, \ldots, N$) of Eq. (4) exist for any continuous function $\chi: (-\infty, \tau] \to U \subset \mathbb{R}^n$ on $(-\infty, \tau]$. And for any $\boldsymbol{\eta}_i$, $\boldsymbol{\eta}_i'$, $\boldsymbol{x}_j$, $\boldsymbol{x}_j'$ ($j = 1, 2, \ldots, p$), the following inequality is satisfied:

$$\begin{aligned} &|\boldsymbol{y}_i(t; \tau, \boldsymbol{\eta}_i) - \boldsymbol{y}_i'(t; \tau, \boldsymbol{\eta}_i')| \\ &\leq M_1 \mathrm{e}^{\zeta(\tau - t)}|\boldsymbol{\eta}_i - \boldsymbol{\eta}_i'| + M_1\int_t^{\tau} \mathrm{e}^{\zeta(s-t)}\Big[L|\boldsymbol{y}_i - \boldsymbol{y}_i'| \\ &\quad + \| \boldsymbol{K}_i \| \sum_{j=1}^{p} |c_{ij}(\boldsymbol{x}_j - \boldsymbol{x}_j')| \\ &\quad + \| \boldsymbol{K}_i \| \sum_{j=p+1}^{N} |c_{ij}(\boldsymbol{y}_j - \boldsymbol{y}_j')|\Big]\mathrm{d}s. \end{aligned} \tag{6}$$

The proof of inequality (6) is reported in Appendix A.

Now, we present sufficient conditions for the existence of the first kind of Hölder continuous GS manifold as theorem 1.

Theorem 1 In systems (4) and (5), $\boldsymbol{f}_i: \mathbb{R}^n \to \mathbb{R}^n$ ($i = 1, 2, \ldots, N$) and $F_l: \mathbb{R}^n \to \mathbb{R}^n$ ($l = 1, 2, \ldots, p$) are smooth functions. We assume that the following conditions are satisfied:

$$\begin{aligned} &|\boldsymbol{F}_l(\boldsymbol{x}_l) - \boldsymbol{K}_l\sum_{j=1,j\neq l}^{p} c_{lj}\boldsymbol{x}_j - \boldsymbol{K}_l\sum_{j=p+1}^{N} c_{lj}\boldsymbol{y}_j - \boldsymbol{F}_l(\boldsymbol{x}_l') \\ &\quad + \boldsymbol{K}_l\sum_{j=1,j\neq l}^{p} c_{lj}\boldsymbol{x}_j' + \boldsymbol{K}_l\sum_{j=p+1}^{N} c_{lj}\boldsymbol{y}_j'| \\ &\leq \lambda M_\alpha\Big[\sum_{j=1}^{p}|\boldsymbol{x}_j - \boldsymbol{x}_j'| + \sum_{j=p+1}^{N}|\boldsymbol{y}_j - \boldsymbol{y}_j'|^\alpha\Big], \\ &\quad 0 < \alpha < 1, \end{aligned} \tag{7}$$

$$|\boldsymbol{F}_l(\boldsymbol{x}_l) - \boldsymbol{K}_l\sum_{j=1,j\neq l}^{p} c_{lj}\boldsymbol{x}_j - \boldsymbol{K}_l\sum_{j=p+1}^{N} c_{lj}\boldsymbol{y}_j| \leq \tilde{N}, \tag{8}$$

$$|\mathrm{e}^{\boldsymbol{B}_l t}| \leq M\mathrm{e}^{-\beta t}, \tag{9}$$

the solutions of systems (4) and (5) exist on $(-\infty, \tau)$, and inequality (6) holds for any $\boldsymbol{\eta}_i$, $\boldsymbol{\eta}_i'$. Additionally,

$$|\boldsymbol{x}_l| \leq D, \quad \frac{M\tilde{N}}{\beta} \leq D, \quad \frac{\lambda M M_\alpha\sqrt{2p}}{\beta} + \frac{\lambda M M_\alpha(1+\Delta p)\sqrt{p}[M_1(N-p-1)\parallel \boldsymbol{K} \parallel c\sqrt{2}/\zeta]^\alpha}{\beta - \alpha\zeta - \alpha M_1[L + \parallel \boldsymbol{K} \parallel c(N-p-1)(\Delta p+1)]} < \Delta,$$

$$\frac{\lambda M M_\alpha(1+\Delta p)\sqrt{p}(M_1)^\alpha}{\beta - \alpha\zeta - \alpha M_1[L + \parallel \boldsymbol{K} \parallel c(N-p-1)(\Delta p+1)]} \leq \Delta,$$

where $|c_{ij}| \leq c$, $\| \boldsymbol{K}_i \| \leq \| \boldsymbol{K} \|$, λ, M, M_1, $\tilde{N}$, β, ζ, D and Δ are non-negative constants. Then there exists a Hölder continuous GS manifold between systems (4) and (5),

$$S = \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p, \boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N) | (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p) = \sigma(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N), -\infty < t < +\infty\}.$$

Moreover,

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p),$$
$$|\sigma_l(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N)| \leq D,$$

and

$$|\sigma_l(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N) - \sigma_l(\boldsymbol{y}'_{p+1}, \boldsymbol{y}'_{p+2}, \ldots, \boldsymbol{y}'_N)| \leq \Delta \Big(\sum_{i=p+1}^{N} |\boldsymbol{y}_i - \boldsymbol{y}'_i| \Big)^{\alpha}$$

are satisfied.

Proof Suppose σ is a Hölder continuous (denoted by $\sigma \in F_{D,\Delta}$) if and only if

$$|\sigma_l(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N)| \leq D,$$
$$|\sigma_l(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N) - \sigma_l(\boldsymbol{y}'_{p+1}, \boldsymbol{y}'_{p+2}, \ldots, \boldsymbol{y}'_N)| \leq \Delta \Big(\sum_{i=p+1}^{N} |\boldsymbol{y}_i - \boldsymbol{y}'_i| \Big)^{\alpha}. \quad (10)$$

Define the distance

$$\| \sigma - \sigma' \| = \mathrm{Sup}_{\boldsymbol{y}_i \in \mathbb{R}^n} \{ \sum_{l=1}^{p} |\sigma_l(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N) - \sigma_l(\boldsymbol{y}'_{p+1}, \boldsymbol{y}'_{p+2}, \ldots, \boldsymbol{y}'_N)|^2 \}^{1/2},$$

it is easy to verify that $F_{D,\Delta}$ is a complete metric space. Now we define a Hölder space $C^{0,\alpha}$, whose norm and distance are described as:

$$\|\sigma\|_{0,\alpha} = \mathrm{Sup}_{\boldsymbol{y}} |\sigma(\boldsymbol{y})| + \mathrm{Sup}_{\boldsymbol{y} \neq \boldsymbol{y}'} \frac{|\sigma(\boldsymbol{y}) - \sigma(\boldsymbol{y}')|}{|\boldsymbol{y} - \boldsymbol{y}'|^{\alpha}}, \quad (11)$$

$$\|\sigma_1 - \sigma_2\|_{0,\alpha} = \mathrm{Sup}_{\boldsymbol{y}} |\sigma_1(\boldsymbol{y}) - \sigma_2(\boldsymbol{y})| + \mathrm{Sup}_{\boldsymbol{y} \neq \boldsymbol{y}'} \frac{|[\sigma_1(\boldsymbol{y}) - \sigma_2(\boldsymbol{y})] - [\sigma_1(\boldsymbol{y}') - \sigma_2(\boldsymbol{y}')]|}{|\boldsymbol{y} - \boldsymbol{y}'|^{\alpha}}, \quad (12)$$

where $\boldsymbol{y} = (\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N)$. For any $\sigma \in F_{D,\Delta}$, $\|\sigma\|_{0,\alpha} \leq \Delta + D$, i.e., $F_{D,\Delta}$ is a close ball in the space $C^{0,\alpha}$.

The operator G is defined as

$$G_l(\sigma)(\tau, \boldsymbol{\eta}_i) = \int_{-\infty}^{\tau} \mathrm{e}^{\boldsymbol{B}_l(\tau - s)} [\boldsymbol{F}_l(\sigma_l(\boldsymbol{y}_{p+1}(s), \boldsymbol{y}_{p+2}(s), \ldots, \boldsymbol{y}_N(s))) - \boldsymbol{K}_l \sum_{j=1, j \neq l}^{p} c_{lj} \boldsymbol{x}_j(s) - \boldsymbol{K}_l \sum_{j=p+1}^{N} c_{lj} \boldsymbol{y}_j(s)] \mathrm{d}s. \quad (13)$$

To make use of the Schauder fixed point theorem in Appendix A, we will prove the following three results.

(a) The operator G maps $F_{D,\Delta}$ to $F_{D,\Delta}$.

In fact, $|G_l(\sigma)(\tau, \boldsymbol{\eta}_i)| \leq \int_{-\infty}^{\tau} \mathrm{e}^{-\beta(\tau - s)} M \tilde{N} \mathrm{d}s = M\tilde{N}/\beta \leq D$, if σ and σ' are two maps satisfying the inequality (10), $\boldsymbol{\eta}_i$ and $\boldsymbol{\eta}'_i \in \mathbb{R}^n$. Let $\boldsymbol{y}_i(t) = \boldsymbol{y}_i(t; \tau, \boldsymbol{\eta}_i)$, $\boldsymbol{y}'_i(t) = \boldsymbol{y}'_i(t; \tau, \boldsymbol{\eta}'_i)$, based on the inequality (6), we have

$$\begin{aligned}
&|\boldsymbol{y}_i(t; \tau, \boldsymbol{\eta}_i) - \boldsymbol{y}'_i(t; \tau, \boldsymbol{\eta}'_i)| \\
&\leq M_1 \mathrm{e}^{\zeta(\tau - t)} |\boldsymbol{\eta}_i - \boldsymbol{\eta}'_i| + M_1 \int_t^{\tau} \mathrm{e}^{\zeta(s - t)} \{ L |\boldsymbol{y}_i - \boldsymbol{y}'_i| \\
&\quad + \| \boldsymbol{K} \| \sum_{j=1}^{p} |c_{ij}(\boldsymbol{x}_j - \boldsymbol{x}'_j)| \\
&\quad + \| \boldsymbol{K} \| \sum_{j=p+1}^{N} |c_{ij}(\boldsymbol{y}_j - \boldsymbol{y}'_j)| \} \mathrm{d}s \\
&\leq M_1 \mathrm{e}^{\zeta(\tau - t)} |\boldsymbol{\eta}_i - \boldsymbol{\eta}'_i| + M_1 \mathrm{e}^{\zeta(\tau - t)} \| \\
&\quad \times \boldsymbol{K} \| c\sqrt{2} \| \sigma - \sigma' \| / \zeta \\
&\quad + M_1 \int_t^{\tau} \mathrm{e}^{\zeta(s - t)} \{ L |\boldsymbol{y}_i - \boldsymbol{y}'_i| \\
&\quad + \| \boldsymbol{K} \| [c \Delta p \sum_{j=p+1}^{N} |\boldsymbol{y}_j - \boldsymbol{y}'_j|] \\
&\quad + \| \boldsymbol{K} \| \sum_{j=p+1}^{N} c |\boldsymbol{y}_j - \boldsymbol{y}'_j| \} \mathrm{d}s,
\end{aligned}$$

and then

$$\begin{aligned}
&\sum_{i=p+1}^{N} \mathrm{e}^{\zeta(t - \tau)} |\boldsymbol{y}_i(t; \tau, \boldsymbol{\eta}_i) - \boldsymbol{y}'_i(t; \tau, \boldsymbol{\eta}'_i)| \\
&\leq \sum_{i=p+1}^{N} M_1 |\boldsymbol{\eta}_i - \boldsymbol{\eta}'_i| + M_1 (N - P - 1) \\
&\quad \times \| \boldsymbol{K} c \| \sqrt{2} \| \sigma - \sigma' \| / \zeta \\
&\quad + M_1 \int_t^{\tau} \mathrm{e}^{\zeta(s - \tau)} \Big\{ L \sum_{i=p+1}^{N} |\boldsymbol{y}_i - \boldsymbol{y}'_i| \\
&\quad + \| \boldsymbol{K} \| c (N - p - 1) \Big[\Delta p \sum_{j=p+1}^{N} |\boldsymbol{y}_j - \boldsymbol{y}'_j| \\
&\quad + \sum_{j=p+1}^{N} |\boldsymbol{y}_j - \boldsymbol{y}'_j| \Big] \Big\} \mathrm{d}s.
\end{aligned}$$

Based on Lemma 1 in Appendix A, the follow inequality is satisfied:

$$\begin{aligned}
&\sum_{i=p+1}^{N} \mathrm{e}^{\zeta(t-\tau)}|\boldsymbol{y}_i(t;\tau,\boldsymbol{\eta}_i)-\boldsymbol{y}'_i(t;\tau,\boldsymbol{\eta}'_i)|\\
&\le \Bigg[\sum_{i=p+1}^{N} M_1|\boldsymbol{\eta}_i-\boldsymbol{\eta}'_i|+M_1(N-P-1)\\
&\quad\times \parallel \boldsymbol{K}c \parallel \sqrt{2}\parallel \sigma-\sigma' \parallel /\zeta\Bigg]\\
&\quad\times \exp\{M_1[L+\parallel \boldsymbol{K} \parallel c(N-p-1)\\
&\quad\times(\Delta p+1)](\tau-t)\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Bigg[\sum_{i=p+1}^{N}|\boldsymbol{y}_i(t;\tau,\boldsymbol{\eta}_i)-\boldsymbol{y}'_i(t;\tau,\boldsymbol{\eta}'_i)|\Bigg]^{\alpha}\\
&\le \Bigg[\sum_{i=p+1}^{N} M_1|\boldsymbol{\eta}_i-\boldsymbol{\eta}='_i|+M_1(N-P-1)\\
&\quad\times \parallel \boldsymbol{K}c \parallel \sqrt{2}\parallel \sigma-\sigma' \parallel /\zeta\Bigg]^{\alpha}\\
&\quad\times \exp\{\alpha M_1[L+\parallel \boldsymbol{K} \parallel c(N-p-1)\\
&\quad\times(\Delta p+1)]+\alpha\zeta\}(\tau-t).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&|G_l(\sigma)(\tau,\boldsymbol{\eta}_i)-G_l(\sigma')(\tau,\boldsymbol{\eta}'_i)|\\
&=\Bigg|\int_{-\infty}^{\tau}\mathrm{e}^{\boldsymbol{B}_l(\tau-s)}[\boldsymbol{F}_l(\sigma_l(\boldsymbol{y}_{p+1},\boldsymbol{y}_{p+2},\ldots,\boldsymbol{y}_N))\\
&\quad-\boldsymbol{K}_l\sum_{j=1,j\ne l}^{p}c_{lj}\boldsymbol{x}_j-\boldsymbol{K}_l\sum_{j=p+1}^{N}c_{lj}\boldsymbol{y}_j\\
&\quad-F_l(\sigma'_l(\boldsymbol{y}'_{p+1},\boldsymbol{y}'_{p+2},\ldots,\boldsymbol{y}'_N))\\
&\quad+\boldsymbol{K}_l\sum_{j=1,j\ne l}^{p}c_{lj}\boldsymbol{x}'_j+\boldsymbol{K}_l\sum_{j=p+1}^{N}c_{lj}\boldsymbol{y}'_j\,\mathrm{d}s\Bigg|\\
&\le M\lambda M_{\alpha}\Bigg|\int_{-\infty}^{\tau}\mathrm{e}^{-\beta(\tau-s)}\Bigg\{\sum_{j=1}^{p}[\sigma_j(\boldsymbol{y}_{p+1},\boldsymbol{y}_{p+2},\ldots,\boldsymbol{y}_N)\\
&\quad-\sigma'_j(\boldsymbol{y}'_{p+1},\boldsymbol{y}'_{p+2},\ldots,\boldsymbol{y}'_N)]\\
&\quad+\Bigg(\sum_{j=p+1}^{N}|\boldsymbol{y}_j-\boldsymbol{y}'_j|\Bigg)^{\alpha}\Bigg\}\mathrm{d}s\Bigg|\\
&\le \frac{\lambda MM_{\alpha}\sqrt{2}}{\beta}\parallel \sigma-\sigma' \parallel\\
&\quad+\frac{\lambda MM_{\alpha}(1+\Delta p)}{\beta-\alpha\zeta-\alpha M_1[L+\parallel \boldsymbol{K} \parallel c(N-p-1)(\Delta p+1)]}\\
&\quad\times[M_1\sum_{j=p+1}^{N}|\boldsymbol{\eta}_j-\boldsymbol{\eta}'_j|\\
&\quad+M_1(N-P-1)\parallel \boldsymbol{K}c \parallel \sqrt{2}\parallel \sigma-\sigma' \parallel /\zeta]^{\alpha}.
\end{aligned}$$

If $\parallel \sigma-\sigma' \parallel\le 1$, then $\parallel \sigma-\sigma' \parallel\le\parallel \sigma-\sigma' \parallel^{\alpha}$, where $0<\alpha<1$, and we have

$$\begin{aligned}
&\parallel G_l(\sigma)-G_l(\sigma') \parallel\\
&\le \frac{\lambda MM_{\alpha}\sqrt{2}}{\beta}\parallel \sigma-\sigma' \parallel^{\alpha}\\
&\quad+\frac{\lambda MM_{\alpha}(1+\Delta p)}{\beta-\alpha\zeta-\alpha M_1[L+\parallel \boldsymbol{K} \parallel c(N-p-1)(\Delta p+1)]}\\
&\quad\times\Bigg[M_1\sum_{j=p+1}^{N}|\boldsymbol{\eta}_j-\boldsymbol{\eta}'_j|\\
&\quad+M_1(N-P-1)\parallel \boldsymbol{K}c \parallel \sqrt{2}\parallel \sigma-\sigma' \parallel /\zeta\Bigg]^{\alpha}.
\end{aligned}$$

Then

$$\begin{aligned}
&\parallel G(\sigma)(\boldsymbol{\eta})-G(\sigma')(\boldsymbol{\eta}') \parallel\\
&\le\Bigg[\sum_{l=1}^{p}\parallel G_l(\sigma)(\boldsymbol{\eta})-G_l(\sigma')(\boldsymbol{\eta}')\parallel^2\Bigg]^{1/2}\\
&\le\frac{\lambda MM_{\alpha}(1+\Delta p)\sqrt{p}(M_1\sum_{j=p+1}^{N}|\boldsymbol{\eta}_j-\boldsymbol{\eta}'_j|)^{\alpha}}{\beta-\alpha\zeta-\alpha M_1[L+\parallel \boldsymbol{K} \parallel c(N-p-1)(\Delta p+1)]}.
\end{aligned}\tag{14}$$

Thus the operator G is a bounded map on $F_{D,\Delta}$. Based on the Schauder fixed point theorem, we have

$$\begin{aligned}
&\|G(\sigma)\|_{0,\alpha}\\
&=\mathrm{Sup}_{\boldsymbol{y}}|G(\sigma)(\boldsymbol{y})|+\mathrm{Sup}_{\boldsymbol{y}\ne\boldsymbol{y}'}\frac{|G(\sigma)(\boldsymbol{y})-G(\sigma)(\boldsymbol{y}')|}{|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha}}\\
&\le D+\Delta,
\end{aligned}$$

i.e., G maps $F_{D,\Delta}$ to $F_{D,\Delta}$.

(b) G is a continuous map.

For $\forall\varepsilon>0$, $\exists|\sigma_1(\boldsymbol{y})-\sigma_2(\boldsymbol{y})|<\varepsilon/3\Delta$, $|\sigma_1(\boldsymbol{y})-\sigma_2(\boldsymbol{y})|/|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha}<\varepsilon/3\Delta$, $|\sigma_1(\boldsymbol{y}')-\sigma_2(\boldsymbol{y}')|/|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha}<\varepsilon/3\Delta$, we have

$$\begin{aligned}
&\|G(\sigma_1)-G(\sigma_2)\|_{0,\alpha}\\
&=\mathrm{Sup}_{\boldsymbol{y}}|G(\sigma_1(\boldsymbol{y}))-G(\sigma_2(\boldsymbol{y}))|\\
&\quad+\mathrm{Sup}_{\boldsymbol{y}\ne\boldsymbol{y}'}\{|[G(\sigma_1(\boldsymbol{y}))-G(\sigma_2(\boldsymbol{y}))]\\
&\quad-[G(\sigma_1(\boldsymbol{y}'))-G(\sigma_2(\boldsymbol{y}'))]|/|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha}\}\\
&<\varepsilon.
\end{aligned}$$

(c) G maps bounded sets to compact sets in $C^{0,\alpha/2}\cap F_{D,\Delta}$.

According to inequality (14), $|G(\sigma)(\boldsymbol{y})-G(\sigma')(\boldsymbol{y}')|\le\Delta|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha/2}$. When $|\boldsymbol{y}-\boldsymbol{y}'|<\delta=(\varepsilon/\Delta)^{2/\alpha}$, the family of functions $\{G(\sigma(\boldsymbol{y}))\}$ with parameter $\boldsymbol{y}$ are uniformly bounded. By using the Arzela theorem in Appendix A, there exists a convergent function sequence $\{G(\sigma_n)\}$ in $\{G(\sigma)\}$ on $C^{0,\alpha}\cap F_{D,\Delta}$. Let $g_{mn}(\boldsymbol{y})=G(\sigma_m(\boldsymbol{y}))-G(\sigma_n(\boldsymbol{y}))$, we have

$$\begin{aligned}
&\frac{|g_{mn}(\boldsymbol{y})-g_{mn}(\boldsymbol{y}')|}{|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha/2}}\\
&=\Bigg[\frac{|G(\sigma_m(\boldsymbol{y}))-G(\sigma_n(\boldsymbol{y}))-G(\sigma_m(\boldsymbol{y}'))+G(\sigma_n(\boldsymbol{y}'))|}{|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha}}\Bigg]^{\frac{\alpha}{2\alpha}}
\end{aligned}$$

$\times |g_{mn}(\boldsymbol{y}) - g_{mn}(\boldsymbol{y}')|^{1-\alpha/2\alpha}$.

Since $C^{0,\alpha} \cap F_{D,\Delta}$ is bounded,

$$\frac{|G(\sigma_m(\boldsymbol{y}))-G(\sigma_n(\boldsymbol{y}))-G(\sigma_m(\boldsymbol{y}'))+G(\sigma_n(\boldsymbol{y}'))|}{|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha}}$$

is also bounded. Suppose the upper bound is $\tilde{K}$, then

$$\begin{aligned}
&\|g_{mn}\|_{0,\alpha/2} \\
&= \mathrm{Sup}_{\boldsymbol{y}}|g_{mn}(\boldsymbol{y})| + \mathrm{Sup}_{\boldsymbol{y}\neq\boldsymbol{y}'}\frac{|g_{mn}(\boldsymbol{y}) - g_{mn}(\boldsymbol{y}')|}{|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha/2}} \\
&\leq \mathrm{Sup}_{\boldsymbol{y}}|g_{mn}(\boldsymbol{y})| + \tilde{K}\mathrm{Sup}_{\boldsymbol{y}\neq\boldsymbol{y}'}|g_{mn}(\boldsymbol{y}) - g_{mn}(\boldsymbol{y}')| \\
&\leq \|g_{mn}\|_0 + 2\tilde{K}\|g_{mn}\|_0^{1/2}. \qquad (15)
\end{aligned}$$

It means that if $\{G(\sigma_n)\}$ is a Cauchy function sequence in $C^{0,\alpha}$, it is also a Cauchy function sequence in $C^{0,\alpha/2}$, then $\{G(\sigma_n)\}$ is convergent in $C^{0,\alpha/2} \cap F_{D,\Delta}$.

According to the above results (a), (b), (c), and the Schauder fixed point theorem, the operator G has a fixed point $\sigma : \sigma = G(\sigma)$ in $C^{0,\alpha/2} \cap F_{D,\Delta}$, i.e., there exists a Hölder continuous GS manifold between systems (4) and (5): $S = \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p, \boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N) | (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p) = \sigma(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N), -\infty < t < +\infty\}$, which satisfies inequality (10).

In addition, the manifold S is invariant. Suppose $(\boldsymbol{x}_{10}, \boldsymbol{x}_{20}, \ldots, \boldsymbol{x}_{p0}, \boldsymbol{y}_{(p+1)0}, \boldsymbol{y}_{(p+2)0}, \ldots, \boldsymbol{y}_{N0}) \in S$, $\boldsymbol{x}_{l\boldsymbol{0}} = \sigma_l(\boldsymbol{y}_{(p+1)0}, \boldsymbol{y}_{(p+2)0}, \ldots, \boldsymbol{y}_{N0})$, for equation

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}_l}{\mathrm{d}t} &= \boldsymbol{B}_l\boldsymbol{x}_l + \boldsymbol{F}_l(\boldsymbol{x}_l) \\
&\quad - \boldsymbol{K}_l \sum_{j=1,j\neq l}^{p} c_{lj}\sigma_j(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N) \\
&\quad - \boldsymbol{K}_l \sum_{j=p+1}^{N} c_{lj}\boldsymbol{y}_j, \quad \boldsymbol{x}_l(t_0) = \boldsymbol{x}_{l\boldsymbol{0}}, \\
&\quad l = 1, 2, \ldots, p,
\end{aligned}$$

let $\boldsymbol{x}_l(t) = \sigma_l(\boldsymbol{y}_{p+1}(t), \boldsymbol{y}_{p+2}(t), \ldots, \boldsymbol{y}_N(t))$, there exists the unique solution

$$\begin{aligned}
\boldsymbol{x}_l(t) = \int_{-\infty}^{t} \mathrm{e}^{\boldsymbol{B}_l(t-s)}[&\boldsymbol{F}_l(\sigma_l(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N)) \\
&- \boldsymbol{K}_l \sum_{j=1,j\neq l}^{p} c_{lj}\sigma_j(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N) \\
&- \boldsymbol{K}_l \sum_{j=p+1}^{N} c_{lj}\boldsymbol{y}_j]\mathrm{d}s.
\end{aligned}$$

Moreover, $\boldsymbol{x}_l(t)$ is bounded when $t \to \infty$. It means that S is invariant.

2.2. The second kind of Hölder continuous GS

In this subsection, we assume that the first p systems of Eq. (2) have asymptotically stable periodic solutions $\bar{\boldsymbol{x}}_i(t) = \bar{\boldsymbol{x}}_i(t + T_i)$, $T_i > 0$. Let $\boldsymbol{x}(t) = (\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_p(t))$, $\bar{\boldsymbol{x}}(t) = (\bar{\boldsymbol{x}}_1(t + T_1), \bar{\boldsymbol{x}}_2(t + T_2), \ldots, \bar{\boldsymbol{x}}_p(t + T_p))$, and $\boldsymbol{e}_l(t) = \boldsymbol{x}_l(t) - \bar{\boldsymbol{x}}_l(t)$, $l = 1, 2, \ldots, p$, then

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{e}_l}{\mathrm{d}t} &= \boldsymbol{A}_l\boldsymbol{x}_l + \boldsymbol{f}_l(\boldsymbol{x}_l) - \boldsymbol{K}_l \sum_{j=1}^{p} c_{lj}\boldsymbol{x}_j \\
&\quad - [\boldsymbol{A}_l\bar{\boldsymbol{x}}_l + \boldsymbol{f}_l(\bar{\boldsymbol{x}}_l) - \boldsymbol{K}_l c_{ll}\bar{\boldsymbol{x}}_l] - \boldsymbol{K}_l \sum_{j=p+1}^{N} c_{lj}\boldsymbol{y}_j \\
&= \boldsymbol{A}_l(t)\boldsymbol{e}_l + \boldsymbol{F}_l(\boldsymbol{x}_l, \bar{\boldsymbol{x}}_l) - \boldsymbol{K}_l \sum_{j=1,j\neq l}^{p} c_{lj}(\bar{\boldsymbol{x}}_j + \boldsymbol{e}_j) \\
&\quad - \boldsymbol{K}_l \sum_{j=p+1}^{N} c_{lj}(\bar{\boldsymbol{y}}_j + \boldsymbol{e}_j), \qquad (16)
\end{aligned}$$

where $\boldsymbol{A}_l(t) = \boldsymbol{A}_l - \boldsymbol{K}_l c_{ll} + D\boldsymbol{f}_l(\bar{\boldsymbol{x}}_l)$, $\boldsymbol{F}_l(\boldsymbol{x}_l, \bar{\boldsymbol{x}}_l) = \boldsymbol{f}_l(\boldsymbol{x}_l) - \boldsymbol{f}_l(\bar{\boldsymbol{x}}_l) - D\boldsymbol{f}_l(\bar{\boldsymbol{x}}_l)\boldsymbol{e}_l$. By using lemma 2 in Appendix A and $\boldsymbol{e}_l(t) = Z_l^{-1}(t)\boldsymbol{w}_l(t)$, we transform Eq. (16) into

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{w}_l}{\mathrm{d}t} &= \boldsymbol{B}_l\boldsymbol{w}_l + Z_l(t)\boldsymbol{F}_l(t, Z_l^{-1}(t)\boldsymbol{w}_l(t)) + Z_l(t) \\
&\quad \times \Big[- \boldsymbol{K}_l \sum_{j=1,j\neq l}^{p} c_{lj}(\bar{\boldsymbol{x}}_j + Z_j^{-1}\boldsymbol{w}_j) \\
&\quad - \boldsymbol{K}_l \sum_{j=p+1}^{N} c_{lj}(\bar{\boldsymbol{y}}_j + Z_j^{-1}\boldsymbol{w}_j)\Big]. \qquad (17)
\end{aligned}$$

Then the corresponding GS manifold $\boldsymbol{x}_l(t) = \bar{\boldsymbol{x}}_l(t) + \sigma_l(\boldsymbol{y}_{p+1}(t), \boldsymbol{y}_{p+2}(t), \ldots, \boldsymbol{y}_N(t))$ near $\bar{\boldsymbol{x}}_l(t)$ is transformed to

$$\boldsymbol{e}_l(t) = Z_l^{-1}(t)\boldsymbol{w}_l(t) = \sigma_l(\boldsymbol{y}_{p+1}(t), \boldsymbol{y}_{p+2}(t), \ldots, \boldsymbol{y}_N(t)),$$

where

$$\begin{aligned}
\boldsymbol{w}_l(t) &= Z_l(t)\sigma_l(t, \boldsymbol{y}_{p+1}(t), \boldsymbol{y}_{p+2}(t), \ldots, \boldsymbol{y}_N(t)) \\
&= \Sigma_l(\boldsymbol{y}_{p+1}(t), \boldsymbol{y}_{p+2}(t), \ldots, \boldsymbol{y}_N(t)).
\end{aligned}$$

Obviously, the existence of the first kind of Hölder GS manifold between systems (4) and (17) implies the existence of the second kind of Hölder GS manifold between systems (4) and (5). With lemmas 2 and 3 in Appendix A, we derive the following theorem, which is similar to theorem 1.

Theorem 2 In systems (4) and (17), if the following conditions are satisfied:

$$\Big| Z_l\boldsymbol{F}_l(Z_l^{-1}\boldsymbol{w}_l) - Z_l\boldsymbol{K}_l \sum_{j=1,j\neq l}^{p} c_{lj}(\bar{\boldsymbol{x}}_j + Z_j^{-1}\boldsymbol{w}_j)$$

$$
\begin{aligned}
&- Z_l\boldsymbol{K}_l \sum_{j=p+1}^{N} c_{lj}\boldsymbol{y}_j - Z_l\boldsymbol{F}_l(Z_l^{-1}\boldsymbol{w}_l') \\
&+ Z_l\boldsymbol{K}_l \sum_{j=1,j\neq l}^{p} c_{lj}(\bar{\boldsymbol{x}}_j + Z_j^{-1}\boldsymbol{w}_j') + Z_l\boldsymbol{K}_l \sum_{j=p+1}^{N} c_{lj}\boldsymbol{y}_j'\Big| \\
&\leq \lambda M_\alpha\Big[\sum_{j=1}^{p}|\boldsymbol{w}_j - \boldsymbol{w}_j'| + \sum_{j=p+1}^{N}|\boldsymbol{y}_j - \boldsymbol{y}_j'|^\alpha\Big], \\
&\quad 0<\alpha<1, \\
&\quad |Z_l\boldsymbol{F}_l(Z_l^{-1}\boldsymbol{w}_l) - Z_l\boldsymbol{K}_l \sum_{j=1,j\neq l}^{p} c_{lj}\boldsymbol{x}_j Z_j^{-1}\boldsymbol{w}_j \\
&\quad - Z_l\boldsymbol{K}_l \sum_{j=p+1}^{N} c_{lj}\boldsymbol{y}_j| \leq \tilde{N}, \\
&\quad |\mathrm{e}^{\boldsymbol{B}_l t}| \leq M\,\mathrm{e}^{-\beta t},
\end{aligned}
$$

and solutions $\boldsymbol{y}_i(t) = \boldsymbol{y}_i(t;\tau,\boldsymbol{\eta}_i)$ $(i = p+1, p+2, \ldots, N)$ of Eq. (4) exist on $(-\infty,\tau)$, moreover, for any $\boldsymbol{\eta}_i$ and $\boldsymbol{\eta}_i'$, the following inequalities are satisfied:

$$
|\boldsymbol{x}_l| \leq D, \quad \frac{M\tilde{N}}{\beta} \leq D,
$$

$$
\begin{aligned}
&\frac{\lambda M M_\alpha\sqrt{2p}}{\beta} + \frac{\lambda M M_\alpha(1+\Delta p)\sqrt{p}\{M_1(N-p-1)\parallel \boldsymbol{K}c \parallel \sqrt{2}/\zeta\}^\alpha}{\beta - \alpha\zeta - \alpha M_1(L+\parallel \boldsymbol{K}c \parallel (N-p-1)(\Delta p+1))} < \Delta, \\
&\frac{\lambda M M_\alpha(1+\Delta p)\sqrt{p}M_1^\alpha}{\beta - \alpha\zeta - \alpha M_1(L+\parallel \boldsymbol{K}c \parallel (N-p-1)(\Delta p+1))} \leq \Delta,
\end{aligned}
$$

where λ, M, M_1, $\tilde{N}$, β, ζ, D, and Δ are non-negative constants, then a first kind of Hölder continuous GS manifold

$$
S = \{(\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_p, \boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N) | (\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_p) = \Sigma(\boldsymbol{y}_{p+1}, \boldsymbol{y}_{p+2}, \ldots, \boldsymbol{y}_N), -\infty < t < +\infty\}
$$

exists between systems (4) and (17), i.e., there exists a second kind of Hölder continuous GS manifold between systems (4) and (5): $\boldsymbol{x}_l(t) = \bar{\boldsymbol{x}}_l(t) + \sigma_l(\boldsymbol{y}_{p+1}(t), \boldsymbol{y}_{p+2}(t), \ldots, \boldsymbol{y}_N(t)) = \bar{\boldsymbol{x}}_l(t) + Z_l^{-1}(t)\Sigma_l(\boldsymbol{y}_{p+1}(t), \boldsymbol{y}_{p+2}(t), \ldots, \boldsymbol{y}_N(t))$.

3. Numerical simulations

In order to illustrate the aforementioned theoretical analysis clearly, we take a dynamical network with 10 nodes for example, the first 5 nodes are the Rössler systems, the other 5 nodes are the Lorenz systems. The network is descried by the following Eqs. (18) and (19).

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{y}_i}{\mathrm{d}t} &= \begin{bmatrix} -a_2 & a_2 & 0 \\ b_2 & -1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix}\boldsymbol{y}_i + \begin{bmatrix} 0 \\ -y_{i1}y_{i3} \\ y_{i1}y_{i2} \end{bmatrix} \\
&\quad - \boldsymbol{K}_i \sum_{j=6}^{10} c_{ij}\boldsymbol{y}_j - \boldsymbol{K}_i \sum_{j=1}^{5} c_{ij}\boldsymbol{x}_j, \qquad (18)
\end{aligned}
$$

where $i = 6, 7, \ldots, 10$, $\boldsymbol{y}_i = (y_{i1}, y_{i2}, y_{i3}) \in \mathbb{R}^3$, matrix $\boldsymbol{K}_i$ denotes the coupling strength, and c_{ij} is the linking parameter. We know that when $a_2 = 10, b_2 = 28, c_2 = 8/3$, the Lorenz system is chaotic.

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}_l}{\mathrm{d}t} &= \begin{bmatrix} 0 & -1 & -1 \\ 1 & a_1 & 0 \\ 0 & 0 & -c_1 \end{bmatrix}\boldsymbol{x}_l + \begin{bmatrix} 0 \\ 0 \\ x_{i1}x_{i3} + b_1 \end{bmatrix} \\
&\quad - \boldsymbol{K}_l \sum_{j=1}^{5} c_{lj}\boldsymbol{x}_j - \boldsymbol{K}_l \sum_{j=6}^{10} c_{lj}\boldsymbol{y}_j, \qquad (19)
\end{aligned}
$$

where $l = 1, 2 \ldots, 5$, $\boldsymbol{x}_l = (x_{l1}, x_{l2}, x_{l3}) \in \mathbb{R}^3$. When $a_1 = 0.2, b_1 = 0.2, c_1 = 5.7$, the Rössler system is also chaotic.

From Eqs. (18) and (19), we have two parts of the modified systems,

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{y}_i}{\mathrm{d}t} &= \begin{bmatrix} -a_2 & a_2 & 0 \\ b_2 & -1 & 0 \\ 0 & 0 & -c_2 \end{bmatrix}\boldsymbol{y}_i + \begin{bmatrix} 0 \\ -y_{i1}y_{i3} \\ y_{i1}y_{i2} \end{bmatrix} \\
&\quad - \boldsymbol{K}_i c_{ii}\boldsymbol{y}_i, \qquad (20) \\
\frac{\mathrm{d}\boldsymbol{x}_l}{\mathrm{d}t} &= \begin{bmatrix} 0 & -1 & -1 \\ 1 & a_1 & 0 \\ 0 & 0 & -c_1 \end{bmatrix}\boldsymbol{x}_l + \begin{bmatrix} 0 \\ 0 \\ x_{i1}x_{i3} + b_1 \end{bmatrix} \\
&\quad - \boldsymbol{K}_l c_{ll}\boldsymbol{x}_l. \qquad (21)
\end{aligned}
$$

We use the auxiliary system approach to verify the GS between systems (18) and (19). The auxiliary system

of Eq. (19) is introduced as

$$\frac{\mathrm{d}\boldsymbol{X}_l}{\mathrm{d}t} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a_1 & 0 \\ 0 & 0 & -c_1 \end{bmatrix} \boldsymbol{X}_l + \begin{bmatrix} 0 \\ 0 \\ X_{i1}X_{i3} + b_1 \end{bmatrix} - \boldsymbol{K}_l \sum_{j=1}^{5} c_{lj}\boldsymbol{X}_j - \boldsymbol{K}_l \sum_{j=6}^{10} c_{lj}\boldsymbol{y}_j. \quad (22)$$

The diagonal elements of matrix $\boldsymbol{C} = (c_{ij})_{10\times 10}$ are chosen to be (1, 2, 1, 1, 2, 2, 3, 4, 4, 4), the initial values of systems (19) and (21) are chosen as $\boldsymbol{x}(0) = (\boldsymbol{x}_1(0), \boldsymbol{x}_2(0), \ldots, \boldsymbol{x}_5(0)) = ((0.2, 0.4.0.6), (0.8, 1.0, 1.2), \ldots, (2.6, 2.8, 3.0))$, the initial values of systems (18) and (20) are $\boldsymbol{y}(0) = (\boldsymbol{y}_5(0), \boldsymbol{y}_6(0), \ldots, \boldsymbol{y}_{10}(0)) = ((3.2, 3.4, 3.6), (3.8, 4.0, 4.2), \ldots, (5.6, 5.8, 6.0))$, and for system (22), $\boldsymbol{X}(0) = (\boldsymbol{X}_1(0), \boldsymbol{X}_2(0), \ldots, \boldsymbol{X}_5(0)) = ((0.1, 0.2.0.3), (0.4, 0.5, 0.6), \ldots, (1.3, 1.4, 1.5))$.

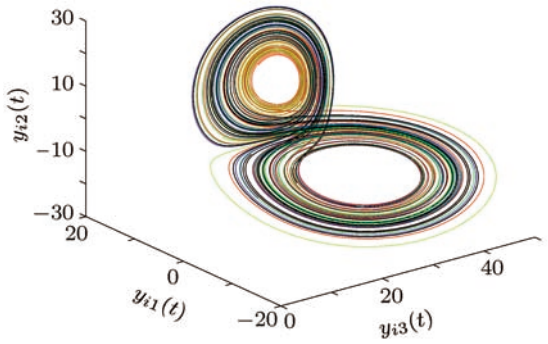


Fig. 1. (colour online) Graphical representations of the modified systems (20) ($i = 6, 7, \ldots, 10$, and five chaotic attractors with different colors coincide with each other).

The first kind of GS In this case, the solutions of Eq. (21) have an asymptotically stable equilibrium and the systems in (20) are chaotic. We select $\boldsymbol{K}_l = \mathrm{diag}\{0.5, 0.5, 0.5\}$, $(l = 1, 2, \ldots, 5)$ and $\boldsymbol{K}_i = \mathrm{diag}\{0.05, 0.05, 0.05\}$, $(i = 6, 7, \ldots, 10)$. Then we find that the systems in Eq. (21) approach asymptotically stable equilibrium, which are $(-0.0084, -0.0280, 0.0322)$, $(-0.0132, -0.0166, 0.0298)$, $(-0.0084, -0.0280, 0.0322)$, $(-0.0084, -0.0280, 0.0322)$ and $(-0.0132, -0.0166, 0.0298)$, respectively, while the systems in Eq. (20) are still chaotic (see Fig. 1). The systems in Eq. (19) and the auxiliary systems in Eq. (22) carry out the CS, which is shown in Fig. 2. Based on the auxiliary system method, we show that systems (18) and (19) realize the GS.

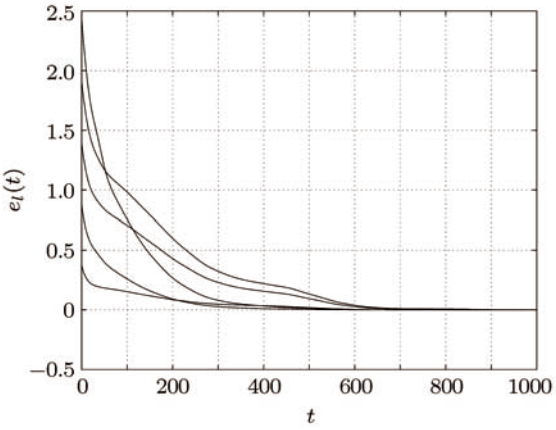


Fig. 2. Graphical representations of errors between systems (19) and (22). $e_l(t) = \sqrt{\sum_{i=1}^{3}(x_{li}(t) - X_{li}(t))^2}$, $l = 1, 2, \ldots, 5$.

The second kind of GS In this case, the systems in Eq. (21) collapse to orbital asymptotically stable periodic solutions and the systems in Eq. (20) are still chaotic. We choose $\boldsymbol{K}_l = \mathrm{diag}\{0.06, 0.06, 0.6\}$, $(l = 1, 3, 4)$, $\boldsymbol{K}_l = \mathrm{diag}\{0.03, 0.03, 0.3\}$, $(l = 2, 5)$, and $\boldsymbol{K}_i = \mathrm{diag}\{0.05, 0.05, 0.05\}$, $(i = 6, 7, \ldots, 10)$. The results are shown in Figs. 3–5. Especially, in Fig. 3, we can see a colourful curve. In fact, it is coincided with five curves, which have different colours and represent the periodic solutions of the modified systems (21).

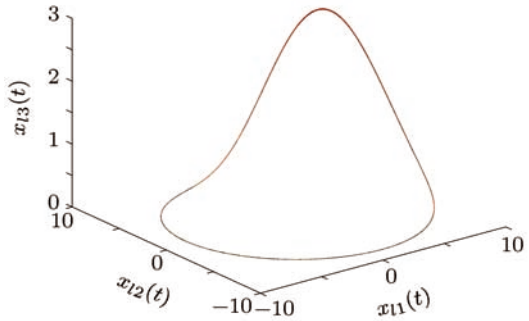


Fig. 3. (colour online) Graphical representations of the modified systems (21) ($l = 1, 2, \ldots, 5$, and five curves with different colours coincide with each other).

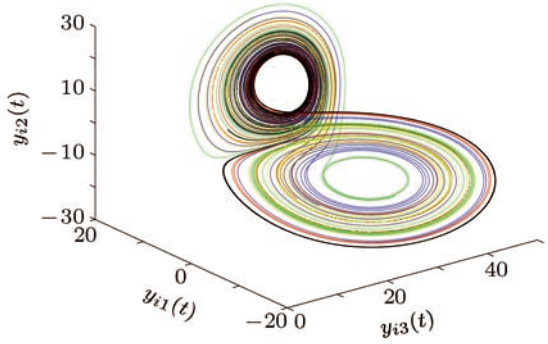


Fig. 4. (colour online) Graphical representations of the modified systems (20) ($i = 6, 7, \ldots, 10$, and five chaotic attractors with different colours coincide with each other).

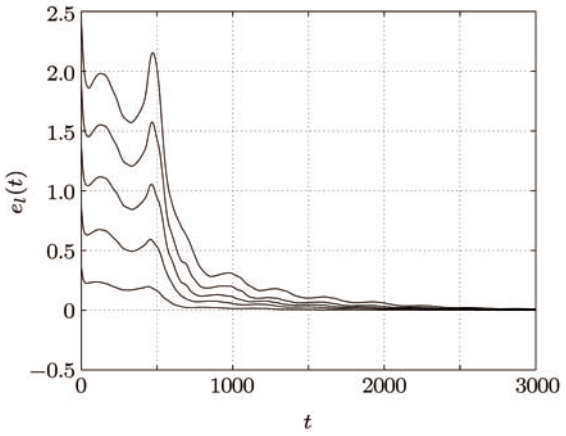


Fig. 5. Graphical representations of errors between the systems (19) and (22). $e_l(t) = \sqrt{\sum_{i=1}^{3}(x_{li}(t) - X_{li}(t))^2}$, $l = 1, 2, \ldots, 5$.

On the whole, from the results of the above simulations, it can be seen that the chaotic systems in network (1) achieve the two kinds of GS.

4. Conclusion and discussion

Although the GS phenomenon exists in nature universally, the mathematical mechanism of the GS of chaotic systems has been unclear over the years due to its complexity. Theoretical analysis of the GS in complex dynamical networks, which are being studied in many fields of science and engineering today, is still insufficient. The problem is solved partly in this paper. However, theorems derived here give the sufficient but not the necessary conditions. It is valuable to point out that some other kinds of GS can be analyzed in the same way, for example, a network in which some modified systems collapse to orbital asymptotically stable quasi-period solutions, while the others are chaotic. There are still many problems to be solved, such as the problem where all modified systems are chaotic. We will focus on such topics in the future.

Appendix A

Proof of inequality (6) Due to

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{y}_i}{\mathrm{d}t} &= \boldsymbol{A}_i\boldsymbol{y}_i + \boldsymbol{f}_i(\boldsymbol{y}_i) - \boldsymbol{K}_i\sum_{j=p+1}^{N} c_{ij}\boldsymbol{y}_j - \boldsymbol{K}_i\sum_{j=1}^{p} c_{ij}\boldsymbol{x}_j,\\ & i = p+1, p+2, \ldots, N,\\ \frac{\mathrm{d}\boldsymbol{y}'_i}{\mathrm{d}t} &= \boldsymbol{A}_i\boldsymbol{y}'_i + \boldsymbol{f}_i(\boldsymbol{y}'_i) - \boldsymbol{K}_i\sum_{j=p+1}^{N} c_{ij}\boldsymbol{y}'_j - \boldsymbol{K}_i\sum_{j=1}^{p} c_{ij}\boldsymbol{x}'_j,\\ & i = p+1, p+2, \ldots, N,\end{aligned}$$

then

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{y}_i - \mathrm{d}\boldsymbol{y}'_i}{\mathrm{d}t} &= \boldsymbol{A}_i(\boldsymbol{y}_i - \boldsymbol{y}'_i) + \boldsymbol{f}_i(\boldsymbol{y}_i) - \boldsymbol{f}_i(\boldsymbol{y}'_i)\\ &\quad - \boldsymbol{K}_i\sum_{j=p+1}^{N} c_{ij}(\boldsymbol{y}_j - \boldsymbol{y}'_j)\\ &\quad - \boldsymbol{K}_i\sum_{j=1}^{p} c_{ij}(\boldsymbol{x}_j - \boldsymbol{x}'_j).\end{aligned}$$

The solution of the equation can be described as:

$$\begin{aligned}\boldsymbol{y}_i - \boldsymbol{y}'_i &= \mathrm{e}^{(\boldsymbol{A}_i - \boldsymbol{K}_i c_{ii})(\tau - t)}(\boldsymbol{\eta}_i - \boldsymbol{\eta}'_i)\\ &\quad + \int_t^{\tau} \mathrm{e}^{(\boldsymbol{A}_i - \boldsymbol{K}_i c_{ii})(s-t)}[(\boldsymbol{f}_i(\boldsymbol{y}_i) - \boldsymbol{f}_i(\boldsymbol{y}'_i))\\ &\quad - \boldsymbol{K}_i\sum_{j=p+1, j\neq i}^{N} c_{ij}(\boldsymbol{y}_j - \boldsymbol{y}'_j)\\ &\quad - \boldsymbol{K}_i\sum_{j=1}^{p} c_{ij}(\boldsymbol{x}_j - \boldsymbol{x}'_j)]\mathrm{d}s,\\ & i = p+1, p+2, \ldots, N.\end{aligned}$$

Consequently, the inequality (6) holds.

Lemma 1[31] Suppose $\phi(t)$, $\psi(t)$, and $\omega(t)$ are continuous functions defined on $[a, b]$, $\omega(t) > 0$, $\psi(t)$ is a monotonous non-negative and non-decreasing function, if $\phi(t) \leq \psi(t) + \int_a^t \omega(s)\phi(s)\mathrm{d}s$, then $\phi(t) \leq \psi(t)\,\mathrm{e}^{\int_a^t \omega(s)\mathrm{d}s}$.

Schauder fixed point theorem[31] Let $\hat{M}$ be a non-empty, closed and convex bounded subset of a Banach space B, and suppose $F : \hat{M} \to \hat{M}$ is a compact operator, then F has a fixed point.

Arzela theorem[31] Suppose Ω is a bounded region in $\mathbb{R}^n$, $\tilde{M}$ is the subset of $C_0(\bar{\Omega})$. If (i) for any $f \in \tilde{M}$, there exists a constant K such that $\mathrm{Sup}_{\boldsymbol{x}\in\Omega}|f(\boldsymbol{x})| \leq K$; (ii) for any fixed $\varepsilon > 0$, there exists $\delta > 0$, $|\boldsymbol{x} - \boldsymbol{y}| < \delta$, $\boldsymbol{x}$, $\boldsymbol{y} \in \Omega$ such that $|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \varepsilon$, $\forall f \in \tilde{M}$, then there exists the function sequence $\{f_n\}$ in $\tilde{M} \subset C_0(\bar{\Omega})$ converges to f.

Lemma 2[32] For a linear periodic differential equation described as:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x, \tag{A1}$$

where $A(t)$ is a continuous function, $A(t+T) = A(t)$, $T > 0$. Then there exists a continuous function $Z(t)$ with period T, and the equation (A1) can be transformed into (A2) using $w(t) = Z(t)x$,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \boldsymbol{B}w, \tag{A2}$$

where $\boldsymbol{B}$ is a constant matrix.

Lemma 3[32] Suppose the real parts of the eigenvalues of the matrix $\boldsymbol{B}$ in the equality (A2) are $\beta_1, \beta_2, \ldots, \beta_n$, the eigenvalues of $\frac{1}{2}[A(t) + A^*(t)]$ are $\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)$. Then there exists a unitary matrix $\boldsymbol{S}(t) = [S_{i,j}(t)]$ such that $\beta_i = \frac{1}{T}\int_0^T \sum_{j=1}^n |S_{i,j}(t)|^2 \alpha_{i,j}(t)\mathrm{d}t, i = 1, 2, \ldots, n$.

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