Modified scaling function projective synchronization of chaotic systems*

Xu Yu-Hua(徐玉华)[a)b)†], Zhou Wu-Neng(周武能)[c)], and Fang Jian-An(方建安)[c)]

[a)] *Department of Mathematics and Finance, Yunyang Teachers' College, Shiyan* 442000, *China*

[b)] *Computer School of Wuhan University, Wuhan* 430079, *China*

[c)] *College of Information Science and Technology, Donghua University, Shanghai* 201620, *China*

(Received 25 January 2011; revised manuscript received 6 April 2011)

This paper investigates a kind of modified scaling function projective synchronization of uncertain chaotic systems using an adaptive controller. The given scaling function in the new method can be an equilibrium point, a periodic orbit, or even a chaotic attractor in the phase space. Based on LaSalle's invariance set principle, the adaptive control law is derived to make the states of two chaotic systems function projective synchronized. Some numerical examples are also given to show the effectiveness of the proposed method.

Keywords: chaotic system, function projective synchronization, scaling function

PACS: 05.45.Gg, 05.45.Xt **DOI:** 10.1088/1674-1056/20/9/090509

1. Introduction

Chaos synchronization has been a hot topic since the pioneering work of Pecocra and Carroll.[1] Various methods and techniques have been proposed for the control and synchronization of chaotic systems. In the past, most synchronizing techniques, such as linear and nonlinear feedback control,[2−6] sliding mode control,[7] adaptive control[8,9] and the liner matrix inequality (LMI) technique, [10] have been designed for unidirectionally coupled systems.

In recent years, function projective synchronization has attracted a great deal of attention. Some results have been reported.[11−14] In function projective synchronization, the scaling factor is changed into a scaling function. So the function projective synchronization is a more general type of chaos synchronization. In function projective synchronization, setting the scaling function to a constant or unity will result in projective synchronization or identical synchronization, respectively. Compared with projective synchronization, function projective synchronization means that the master and slave systems can be synchronized up to a scaling function. This feature could be used to obtain more secure communication, because the unpredictability of the scaling function in the function projective synchronization can additionally enhance the security of the communication. So the study of function projective synchronization is well worth doing.[15,16] Most of the research efforts mentioned above have concentrated on studying a presetting scaling function with numerical examples. In this paper, we assume that the given scaling function can be an equilibrium point, a periodic orbit or even a chaotic attractor in the phase space. To the best of our knowledge, we find no similar studies in the existing literature. Motivated by the existing works, a kind of modified scaling function projective synchronization of uncertain chaotic systems using an adaptive controller is discussed in this paper.

2. Theoretical analyses

Let us consider a class of uncertain dynamical systems

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\alpha}). \tag{1}$$

Suppose the above systems can be rewritten as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{\alpha}, \tag{2}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 61075060), the Science and Technology Research Key Program for the Education Department of Hubei Province of China (Grant No. D20105001), the Open Project of State Key Laboratory of Industrial Control Technology, China (Grant No. ICT1007).

†Corresponding author. E-mail: yuhuaxu2004@163.com

© 2011 Chinese Physical Society and IOP Publishing Ltd

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$, $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{\alpha}) = (F_1(\boldsymbol{x}, \boldsymbol{\alpha}), \ldots, F_n(\boldsymbol{x}, \boldsymbol{\alpha}))^{\mathrm{T}} : \mathbf{R}^n \to \mathbf{R}^n$ is a nonlinear function, $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))^{\mathrm{T}} : \mathbf{R}^n \to \mathbf{R}^n$ is a vector-valued function, $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), \ldots, g_n(\boldsymbol{x}))^{\mathrm{T}} : \mathbf{R}^n \to \mathbf{R}^{n\times m}$ is a matrix function and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)^{\mathrm{T}} \in \mathbf{R}^m$ is the parameter vector. A number of well-known chaotic systems satisfy Eq. (2), such as Chua's circuit, the hyperchaotic Lü system and the generalized Lorenz system.

We consider the drive and the response systems in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_{\mathrm{d}}(\boldsymbol{x}) + \boldsymbol{g}_{\mathrm{d}}(\boldsymbol{x})\boldsymbol{\beta}, \tag{3}$$

$$\dot{\boldsymbol{y}} = \boldsymbol{f}_{\mathrm{r}}(\boldsymbol{y}) + \boldsymbol{g}_{\mathrm{r}}(\boldsymbol{y})\boldsymbol{\alpha} - \boldsymbol{u}, \tag{4}$$

where $\boldsymbol{u}$ is the adaptive controller. Let $\tilde{\boldsymbol{\beta}}$, $\tilde{\boldsymbol{\alpha}}$ be the estimated values of parameters $\boldsymbol{\beta}$, $\boldsymbol{\alpha}$ for the drive and the response systems, respectively.

If there is a scaling function $\boldsymbol{H}(t)$, such that

$$\lim_{t\to\infty} \|\boldsymbol{e}(t)\| = \lim_{t\to\infty} \|\boldsymbol{y} - \boldsymbol{H}(t)\boldsymbol{x}\| = 0, \tag{5}$$

where $\boldsymbol{e}(t) = (e_1, e_2, \ldots, e_n)^{\mathrm{T}}$, $\boldsymbol{H}(t) = \mathrm{diag}(s_1(t), s_2(t), \ldots, s_n(t))$, and $(\dot{s}_1(t), \dot{s}_2(t), \ldots, \dot{s}_n(t))^{\mathrm{T}} = (G_1(s_1, s_2, \ldots, s_n, t), G_2(s_1, s_2, \ldots, s_n, t), \ldots, G_n(s_1, s_2, \ldots, s_n, t))^{\mathrm{T}}$. There exists a stable equilibrium point, a stable periodic orbit or a chaotic attractor in the phase space. Then we say that systems (3) and (4) achieve modified scaling function projective synchronization.

Remark 1 When $\boldsymbol{H}(t)$ is boundless, one problem should be taken into account if $\boldsymbol{e}(t) = \boldsymbol{y} - \boldsymbol{H}(t)\boldsymbol{x}$ is equal to infinite value. To achieve synchronization, that is, $\boldsymbol{e}(t) = \boldsymbol{y} - \boldsymbol{H}(t)\boldsymbol{x}$ is equal to 0 in the process of synchronization, we let $\boldsymbol{H}(t)$ have the boundary, thus, we assume that for system (5) there exist a stable equilibrium point, a stable periodic orbit or a chaotic attractor in the phase space, which satisfies the assumption.

It is easy to see that the modified scaling function projection synchronization encompasses the projective synchronization when the scaling function $\boldsymbol{H}(t)$ is taken as a constant.

Now we study the modified scaling function projective synchronization of uncertain chaotic systems using the adaptive controller.

The error system between drive system (3) and response system (4) can be expressed by

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{f}_{\mathrm{r}}(\boldsymbol{y}) - \boldsymbol{H}(t)\boldsymbol{f}_{\mathrm{d}}(\boldsymbol{x}) + \boldsymbol{g}_{\mathrm{r}}(\boldsymbol{y})\boldsymbol{\alpha} - \boldsymbol{H}(t)\boldsymbol{g}_{\mathrm{d}}(\boldsymbol{x})\boldsymbol{\beta} - \dot{\boldsymbol{H}}(t)\boldsymbol{x} - \boldsymbol{u}. \tag{6}$$

The problem of the modified scaling function projective synchronization of the original system transforms into the stability of the error system.

In this paper, we choose the controller, the scaling function and the parameters estimated adaptive law as

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{f}_{\mathrm{r}}(\boldsymbol{y}) + \boldsymbol{g}_{\mathrm{r}}(\boldsymbol{y})\tilde{\boldsymbol{\alpha}} - \boldsymbol{H}(t)\boldsymbol{f}_{\mathrm{d}}(\boldsymbol{x}) \\
&\quad - \boldsymbol{H}(t)\boldsymbol{g}_{\mathrm{d}}(\boldsymbol{x})\tilde{\boldsymbol{\beta}} - \dot{\boldsymbol{H}}(t)\boldsymbol{x}, \\
\dot{\tilde{\boldsymbol{\beta}}} &= -\boldsymbol{H}(t)(\boldsymbol{g}_{\mathrm{d}}(\boldsymbol{x}))^{\mathrm{T}}\boldsymbol{e} + \boldsymbol{e}_{\boldsymbol{\beta}}, \quad \dot{\tilde{\boldsymbol{\alpha}}} = (\boldsymbol{g}_{\mathrm{r}}(\boldsymbol{y}))^{\mathrm{T}}\boldsymbol{e} + \boldsymbol{e}_{\boldsymbol{\alpha}}, \\
\dot{\boldsymbol{H}}(t) &= \mathrm{diag}(\dot{s}_1(t), \dot{s}_2(t), \ldots, \dot{s}_n(t)),
\end{aligned}$$

where $\boldsymbol{e}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}$, and $\boldsymbol{e}_{\boldsymbol{\beta}} = \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}$.

A candidate Lyapunov function is chosen as

$$V = \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} + \frac{1}{2}\boldsymbol{e}_{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{e}_{\boldsymbol{\alpha}} + \frac{1}{2}\boldsymbol{e}_{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{e}_{\boldsymbol{\beta}}. \tag{7}$$

Then the differentiation of V along the trajectories of Eq. (6) is

$$\begin{aligned}
\dot{V} &= \boldsymbol{e}^{\mathrm{T}}\dot{\boldsymbol{e}} + \dot{\boldsymbol{e}}_{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{e}_{\boldsymbol{\alpha}} + \dot{\boldsymbol{e}}_{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{e}_{\boldsymbol{\beta}} \\
&= \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{g}_{\mathrm{r}}(\boldsymbol{y})\boldsymbol{\alpha} - \boldsymbol{g}_{\mathrm{r}}(\boldsymbol{y})\tilde{\boldsymbol{\alpha}} - \boldsymbol{H}(t)\boldsymbol{g}_{\mathrm{d}}(\boldsymbol{x})\boldsymbol{\beta} \\
&\quad + \boldsymbol{H}(t)\boldsymbol{g}_{\mathrm{d}}(\boldsymbol{x})\tilde{\boldsymbol{\beta}} - \dot{\boldsymbol{H}}(t)\boldsymbol{x} + \dot{\boldsymbol{H}}(t)\boldsymbol{x}) \\
&\quad - \boldsymbol{e}^{\mathrm{T}}\boldsymbol{g}_{\mathrm{r}}(\boldsymbol{y})(\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}) + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{H}(t)\boldsymbol{g}_{\mathrm{d}}(\boldsymbol{x})(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}) \\
&\quad - \boldsymbol{e}_{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{e}_{\boldsymbol{\alpha}} - \boldsymbol{e}_{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{e}_{\boldsymbol{\beta}} \\
&= -\boldsymbol{e}_{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{e}_{\boldsymbol{\alpha}} - \boldsymbol{e}_{\boldsymbol{\beta}}^{\mathrm{T}}\boldsymbol{e}_{\boldsymbol{\beta}}.
\end{aligned} \tag{8}$$

Thus $\dot{V} \le 0$. It is obvious that $\dot{V} = 0$ if, and only if, $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$ and $\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}$. So, set $E = \{e_i = 0, \tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}, \tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}, i = 1, 2, \ldots, n\}$ is the largest invariant set of set $E' = \{\boldsymbol{e} : \dot{V}(\boldsymbol{e})\big|_{(7)} = 0\}$ for error system (6). Then according to LaSalle's invariance set principle of differential equation, starting with arbitrary initial values of the augment system, we find that the orbits converge asymptotically to set E. It implies that the modified scaling function projective synchronization of systems (3) and (4) is achieved.

In Refs. [11]–[16], research efforts have concentrated on studying a presetting scaling function with numerical examples. In this paper, the scaling function can be an equilibrium point, a periodic orbit or even a chaotic attractor in the phase space.

3. Applications

In this section, we give two examples to show the effectiveness of the above method.

Example 1 Two identical Lü chaotic systems.[17]

The drive system is

$$\begin{aligned}\dot{x}_1 &= a_1(x_2 - x_1),\\ \dot{x}_2 &= b_1 x_2 - x_1 x_3,\\ \dot{x}_3 &= -c_1 x_3 + x_1 x_2.\end{aligned}$$

The response system is

$$\begin{aligned}\dot{y}_1 &= a_2(y_2 - y_1) - u_1,\\ \dot{y}_2 &= b_2 y_2 - y_1 y_3 - u_2,\\ \dot{y}_3 &= -c_2 y_3 + y_1 y_2 - u_3.\end{aligned}$$

Let the adaptive controller be $\boldsymbol{u}^{\mathrm{T}} = (u_1, u_2, u_3)$, the scaling function and the parameters estimation adaptive law be

$$\begin{aligned}u_1 &= (y_2 - y_1)a_2^* - s_1(x_2 - x_1)a_1^* - x_1(s_2 s_3 - 2s_1),\\ u_2 &= s_2 x_1 x_3 - y_1 y_3 + y_2 b_2^* - s_2 x_2 b_1^*\\ &\quad -x_2((s_3 - 5)s_1 - 2s_2),\\ u_3 &= y_1 y_2 - s_3 x_1 x_2 - y_3 c_2^* + s_3 x_3 c_1^* - x_3(1 - s_1 s_2),\end{aligned}$$

$$\begin{aligned}\dot{a}_1^* &= -s_1(x_2 - x_1)e_1 + a_1 - a_1^*,\\ \dot{b}_1^* &= -s_2 x_2 e_2 + b_1 - b_1^*,\\ \dot{c}_1^* &= s_3 x_3 e_3 + c_1 - c_1^*,\end{aligned}$$

$$\begin{aligned}\dot{a}_2^* &= (y_2 - y_1)e_1 + a_2 - a_2^*,\\ \dot{b}_2^* &= y_2 e_2 + b_2 - b_2^*,\\ \dot{c}_2^* &= -y_3 e_3 + c_2 - c_2^*.\end{aligned}$$

The scaling function is the Rikitake chaotic system,[18]

$$\begin{aligned}\dot{s}_1 &= s_2 s_3 - 2s_1,\\ \dot{s}_2 &= (s_3 - 5)s_1 - 2s_2,\\ \dot{s}_3 &= 1 - s_1 s_2.\end{aligned}$$

In the simulations, the unknown parameters of the drive and the response systems are chosen as $(a_1, b_1, c_1) = (36, 20, 3)$ and $(a_2, b_2, c_2) = (35, 20, 4)$, respectively. The initial values of the drive and the response systems are taken as $(2, 3, 4)$ and $(6, 4, 5)$, respectively. The initial values estimated for unknown parameters are $(6, 7, 8)$ and $(5, 6, 7)$, and the scaling function initial values are $(9, 1.5, 9)$. The simulated results are shown in Figs. 1–4. In Fig. 1, three state errors versus time are shown. The state errors tend to zero asymptotically as time evolves. Figures 2 and 3 show that the arbitrary estimated initial values of parameters (a_1^*, b_1^*, c_1^*) and (a_2^*, b_2^*, c_2^*) finally evolve to the unknown parameters. The projections of the synchronized attractors in the plane are shown in Fig. 4.

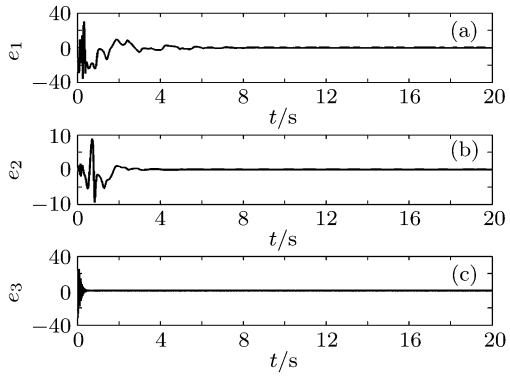


Fig. 1. Synchronization errors versus time: (a) e_1, (b) e_2, (c) e_3.

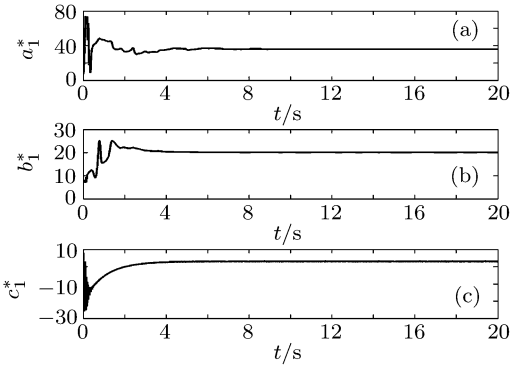


Fig. 2. Estimated parameters of the Lü drive system: (a) a_1^*, (b) b_1^*, (c) c_1^*.

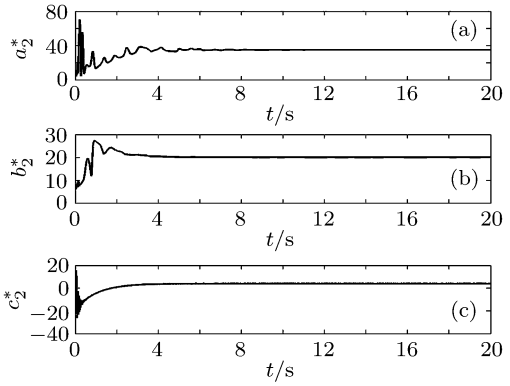


Fig. 3. Estimated parameters of the Lü response system: (a) a_2^*, (b) b_2^*, (c) c_2^*.

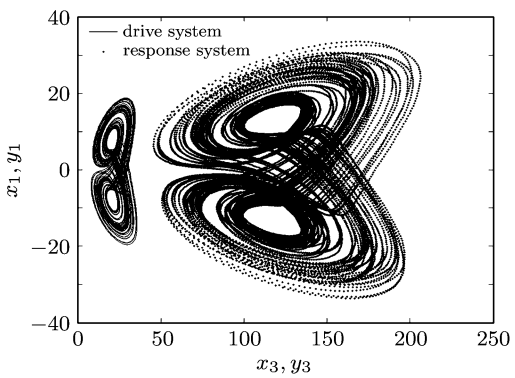


Fig. 4. Projection of the synchronized attractors in $x_3(y_3)$–$x_1(y_1)$ plane.

Example 2 Two different chaotic systems.

The drive system is given by the following Lü system

$$\begin{aligned}\dot{x}_1 &= a_1(x_2 - x_1),\\ \dot{x}_2 &= b_1 x_2 - x_1 x_3,\\ \dot{x}_3 &= -c_1 x_3 + x_1 x_2.\end{aligned}$$

The response system is given by the following controlled Chen system[19]

$$\begin{aligned}\dot{y}_1 &= a_2(y_2 - y_1) - u_1,\\ \dot{y}_2 &= (c_2 - a_2)y_1 + c_2 y_2 - y_1 y_3 - u_2,\\ \dot{y}_3 &= -b_2 y_3 + y_1 y_2 - u_3.\end{aligned}$$

Let the adaptive controller be $u^{\mathrm{T}} = (u_1, u_2, u_3)$, the scaling function and the parameter estimation adaptive law be

$$\begin{aligned}u_1 &= (y_2 - y_1)a_2^* - s_1(x_2 - x_1)a_1^* - x_1(s_2 s_3 - 2s_1),\\ u_2 &= s_2 x_1 x_3 - y_1 y_3 - y_1 a_2^* + (y_1 + y_2)c_2^* - s_2 x_2 b_1^*\\ &\quad -x_2((s_3 - 5)s_1 - 2s_2),\\ u_3 &= y_1 y_2 - s_3 x_1 x_2 - y_3 b_2^* + s_3 x_3 c_1^* - x_3(1 - s_1 s_2),\end{aligned}$$

$$\begin{aligned}\dot{a}_1^* &= -s_1(x_2 - x_1)e_1 + a_1 - a_1^*,\\ \dot{b}_1^* &= -s_2 x_2 e_2 + b_1 - b_1^*,\\ \dot{c}_1^* &= s_3 x_3 e_3 + c_1 - c_1^*,\end{aligned}$$

$$\begin{aligned}\dot{a}_2^* &= (y_2 - y_1)e_1 - y_1 e_2 + a_2 - a_2^*,\\ \dot{c}_2^* &= (y_1 + y_2)e_2 + c_2 - c_2^*,\\ \dot{b}_2^* &= -y_3 e_3 + b_2 - b_2^*.\end{aligned}$$

The scaling function is the Rikitake chaotic system

$$\begin{aligned}\dot{s}_1 &= s_2 s_3 - 2s_1,\\ \dot{s}_2 &= (s_3 - 5)s_1 - 2s_2,\\ \dot{s}_3 &= 1 - s_1 s_2.\end{aligned}$$

In the simulations, the unknown parameters of the drive and the response systems are chosen as $(a_1, b_1, c_1) = (36, 20, 3)$ and $(a_2, b_2, c_2) = (35, 3, 28)$ respectively. The initial values of the drive and the response systems are taken as $(2, 3, 4)$ and $(6, 4, 5)$, respectively. The initial values estimated for the unknown parameters are $(6, 7, 8)$ and $(5, 6, 7)$, and the scaling function initial values are $(9, 1.5, 9)$. The simulated results are shown in Figs. 5–8. In Fig. 5, three state errors versus time are shown. The state errors tend to zero asymptotically as time evolves. Figures 6 and 7 show that the arbitrary estimated initial values of parameters (a_1^*, b_1^*, c_1^*) and (a_2^*, b_2^*, c_2^*) finally evolve to the unknown parameters. The projections of the synchronized attractors in the plane are shown in Fig. 8.

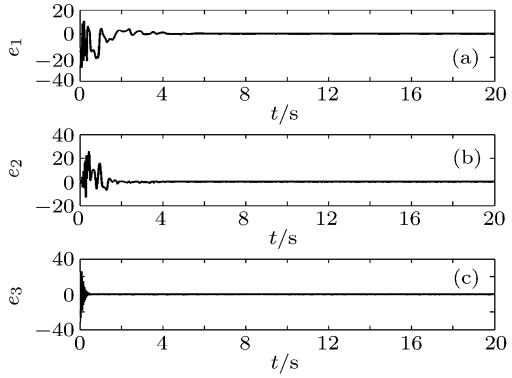


Fig. 5. Synchronization errors versus time: (a) e_1, (b) e_2, (c) e_3.

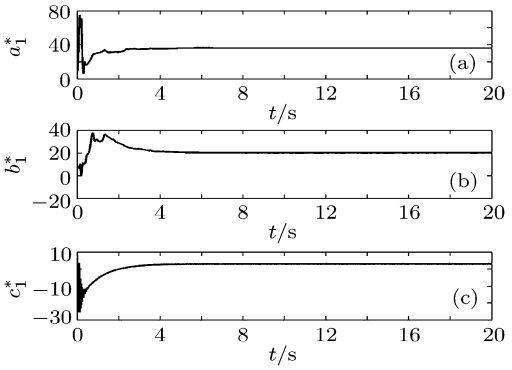


Fig. 6. Estimated parameters of the Lü drive systems: (a) a_1^*, (b) b_1^*, (c) c_1^*.

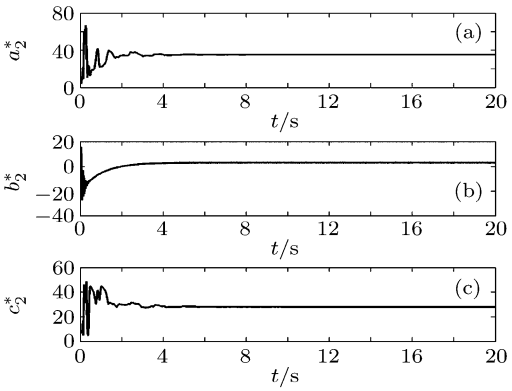


Fig. 7. Estimated parameters of the Chen response systems: (a) a_2^*, (b) b_2^*, (c) c_2^*.

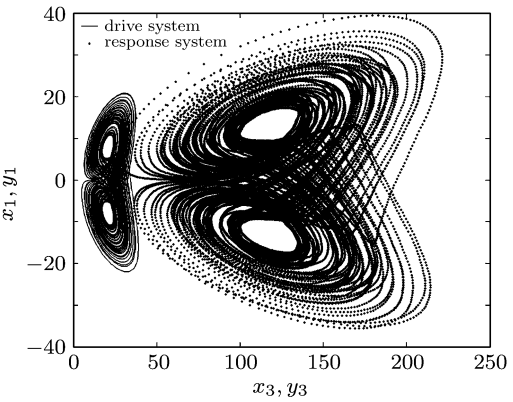


Fig. 8. Projection of the synchronized attractors in $x_3(y_3) - x_1(y_1)$ plane.

4. Conclusion

In this paper, the problem of modified scaling function projective synchronization has been investigated. Based on LaSalle's invariance set principle we have provided the theoretical analysis for the occurrence of the synchronization. The unknown parameters have been identified. The method proposed has been proved to be valid to realize the function synchronization in coupled systems. Finally, some numerical simulations have been given to show the effectiveness of the proposed method.

References

[1] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821

[2] Zhang H B, Yu Y B and Zhang J 2010 *Chin. Phys.* B **19** 080509-1

[3] Cao J, Wang Z and Sun Y 2007 *Physica* A **385** 718

[4] Cao J and Li L 2009 *Neural Networks* **22** 335

[5] He W and Cao J 2009 *Chaos* **19** 013118

[6] Lu J, Cao J and Daniel W C 2008 *IEEE Trans. Circuits Syst.* I **55** 1347

[7] Li W L and Song Y Z 2008 *Phys. Rev. Lett.* **57** 51

[8] Hu J B, Han Y and Zhao L D 2009 *Acta Phys. Sin.* **58** 1441 (in Chinese)

[9] Zhang R X, Yang Y and Yang S P 2009 *Acta Phys. Sin.* **58** 6039 (in Chinese)

[10] Kuntanapreeda S 2009 *Phys. Lett.* A **373** 2837

[11] Zou Y L and Zhu J 2006 *Acta Phys. Sin.* **55** 1965 (in Chinese)

[12] Wu D and Li J 2010 *Chin. Phys.* B **19** 120505

[13] Zhang R and Xu Z 2010 *Chin. Phys.* B **19** 120511

[14] Lü L, Cai Y and Luan L 2010 *Chin. Phys.* B **19** 080506

[15] Zhang R X and Yang S P 2008 *Acta Phys. Sin.* **57** 4073 (in Chinese)

[16] Zhou P and Cao Y X 2010 *Chin. Phys.* B **19** 100507

[17] Lü J H and Chen G R 2002 *Int. J. Bifurcat. Chaos* **12** 659

[18] Ito K 1980 *Earth Planet. Sci. Lett.* **51** 451

[19] Chen G R and Ueta T 1999 *Int. J. Bifurcat. Chaos* **9** 1465