Comments on "The feedback control of fractional order unified chaotic system"

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Our comments point out some mistakes in the main theorem given by Yang and Qi in Ref. [1] concerning the equivalent passivity method to design a nonlinear controller for the stabilizing fractional order unified chaotic system. The proof of this theorem is not reliable, since the mathematical basis of the fractional order calculus is not considered. Moreover, there are some algebraic mistakes in the inequalities used, thus making the proof invalid. We propose a proper Lyapunov function and the stability of Yang and Qi's Controller is investigated based on the fractional order Lyapunov theorem.

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1. Introduction

In this note, we present two comments on the proof of the main theorem given by Yang and Qi in Ref. [1]. The first comment points out some mathematical mistakes and suggests a new Lyapunov function, the second comment reveals some algebraic mistakes which invalidate the proof. In the last section, the inaccuracy of the proof procedure in Ref. [1] is shown and a new Lyapunov candidate is introduced. Using this new Lyapunov function, we investigate the stability of the proposed controller by Yang and Qi.

2. Comment 1

In Eq. (8) of Ref. [1], the derivative of the Lyapunov function with respect to time was calculated. The authors considered that $\dot{I}_a^\alpha x = D_a^\alpha x$ when $0 < \alpha < 1$. However, there are many references, such as Ref. [2], which show the common form of $\dot{I}_a^\alpha x$ to be

$$\dot{I}_a^\alpha x = D(I_a^\alpha x) = D_a^{1-\alpha} x. \tag{1}$$

Thus $\dot{I}_a^\alpha x = D_a^\alpha x$ is acceptable only for $\alpha = 0.5$.

By considering Eq. (1), the Lyapunov candidate in Ref. [1] is not suitable, since

$$\dot{V} = I_a^\alpha x D_a^{1-\alpha} x + I_a^\alpha y D_a^{1-\alpha} y + I_a^\alpha z D_a^{1-\alpha} z. \tag{2}$$

In the above derivative, αth-order integration and $(1-\alpha)$th-order derivative of the system dynamics appear, while only the αth-order derivative of the system dynamics is known to us. Thus, it is difficult to study. Therefore we propose changing the Lyapunov candidate to

$$V = \frac{1}{2}(I_a^{1-\alpha} x)^2 + \frac{1}{2}(I_a^{1-\alpha} y)^2 + \frac{1}{2}(I_a^{1-\alpha} z)^2, \tag{3}$$

whose derivative implies that

$$\dot{V} = I_a^{1-\alpha} x D_a^\alpha x + I_a^{1-\alpha} y D_a^\alpha y + I_a^{1-\alpha} z D_a^\alpha z, \tag{4}$$

in which only the αth-order derivative of the system dynamics appear, thus the investigation becomes easier.

3. Comment 2

The authors of Ref. [1] have used the direct Lyapunov theorem to show their controller stabilized fractional order unified chaotic system. The following inequalities are used in inequality (9) of Ref. [1] to show that the derivative of the Lyapunov function is negative definite:

$$x - y \leq \|x\| - \|y\|, \tag{5}$$

and

$$z \cdot w \leq z \cdot \|w\|. \tag{6}$$

Inequalities (5) and (6) are not acceptable in general. To show the incorrectness of these inequalities, x may

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be considered as a positive scalar and y, z and w may be regarded as negative scalars. The correct forms of inequalities (5) and (6) can be written, respectively, as[3]

$$x - y \leq \|x\| + \|y\|, \tag{7}$$

and

$$z \cdot w \leq \|z\| \cdot \|w\|. \tag{8}$$

4. Correction suggestion

In this part, we investigate the proof of theorem 2 in Yang and Qi's paper[1] by considering all of the above comments. Considering the Lyapunov candidate shown in Eq. (3) and using Eq. (4) leads to

$$\begin{aligned}\dot{V} &= I_a^{1-\alpha} x D_a^\alpha x + I_a^{1-\alpha} y D_a^\alpha y + I_a^{1-\alpha} z D_a^\alpha z \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau \cdot (25\theta + 10)(z - x) \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{y(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau \cdot \left(xz - \frac{8+\theta}{3} y \right) \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{z(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau \\ &\quad \times [-10(25\theta + 10)x - xy - kz]. \end{aligned} \tag{9}$$

By using the proof procedure in Ref. [1] and the above comments, Eq. (9) implies that

$$\begin{aligned}\dot{V} &\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \Big\{ \|x\| (25\theta + 10)(z - x) \\ &\quad + \|y\| \left(xz - \frac{8+\theta}{3} y \right) \\ &\quad + \|z\| [-(25\theta + 10)x - xy - kz] \Big\} \\ &\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \Big[(25\theta + 10)(\|z\| \cdot \|x\| + \|x\|^2) + \\ &\quad + \|x\| \cdot \|y\| \cdot \|z\| \\ &\quad + \frac{8+\theta}{3} \|y\|^2 + (25\theta + 10) \|x\| \cdot \|z\| \\ &\quad + \|x\| \cdot \|y\| \cdot \|z\| + k \|z\|^2 \Big]. \end{aligned} \tag{10}$$

It can be seen that inequality (10) cannot show $\dot{V} \leq 0$. So the stability proof of the mentioned controller is not valid.

In the following, we introduce the fractional order Lyapunov theorem.

Lemma 1[4] Let $x = 0$ be an equilibrium point of the non-autonomous fractional order system ${}_0D_t^\alpha x(t) = f(t, x)$. Assume that there exist a Lyapunov function $V(t, x(t))$ and class-k functions α_i $(i = 1, 2, 3)$ satisfying

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{11}$$

and

$${}_0D_t^\beta V(t, x) \leq -\alpha_3(\|x\|) \quad \beta \in (0, 1), \tag{12}$$

then $x = 0$ will be asymptotically stable.

We introduce a new Lyapunov candidate

$$V = x^2 + y^2 + z^2. \tag{13}$$

Referring to Ref. [5], the αth order derivative of $x^2(t)$ is introduced as

$${}_0D_t^\alpha x^2(t) = x(t) \cdot {}_0D_t^\alpha x(t) + p_x, \tag{14}$$

where $0 < \alpha < 1$,

$$p_x = \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} ({}_0D_t^k x)({}_0D_t^{\alpha-k} x)$$

and $|p_x| \leq \beta_1 |x|$.

Through the use of Eq. (13), the αth order derivative of the new Lyapunov candidate with respect to time is

$$\begin{aligned}{}_0D_t^\alpha V &= x \cdot {}_0D_t^\alpha x + p_x + y \cdot {}_0D_t^\alpha y + p_y \\ &\quad + z \cdot {}_0D_t^\alpha z + p_z, \end{aligned} \tag{15}$$

where $|p_x| \leq \gamma_1 |x|$, $|p_y| \leq \gamma_2 |y|$ and $|p_z| \leq \gamma_3 |z|$. Using the closed-loop system mentioned in Eq. (12) of Ref. [1] leads to

$$\begin{aligned}{}_0D_t^\alpha V &= x(25\theta + 10)(z - x) + p_x \\ &\quad + y\left(xz - \frac{8+\theta}{3} y \right) + p_y \\ &\quad + z[-10(25\theta + 10)x - xy - kz] + p_z \\ &\leq -(25\theta + 10)x^2 - \frac{8+\theta}{3} y^2 \\ &\quad - kz^2 + p_x + p_y + p_z \\ &\leq -(25\theta + 10)x^2 - \frac{8+\theta}{3} y^2 \\ &\quad - kz^2 + |p_x| + |p_y| + |p_z| \\ &\leq -(25\theta + 10)x^2 - \frac{8+\theta}{3} y^2 - kz^2 \\ &\quad + \gamma_1 |x| + \gamma_2 |y| + \gamma_3 |z| \\ &\leq |x| [-(25\theta + 10) |x| + \gamma_1] \\ &\quad + |y| \left(-\frac{8+\theta}{3} |y| + \gamma_2 \right) \\ &\quad + |z| (-k |z| + \gamma_3). \end{aligned} \tag{16}$$

Inequality (16) meets inequality (12) if the state variables satisfy the following conditions:

$$\frac{\gamma_1}{25\theta+10} \leq |x|\,, \quad \frac{3\gamma_2}{8+\theta} \leq |y|\,, \quad \frac{\gamma_3}{k} \leq |z|\,. \quad (17)$$

This demonstrates that by using control input $u = (1-29\theta-k)z-(38-10\theta)x$, in which $k>0$, the state variables enter into the following cubic regime and stay inside it,

$$|x| \leq \frac{\gamma_1}{25\theta+10}, \quad |y| \leq \frac{3\gamma_2}{8+\theta}, \quad |z| \leq \frac{\gamma_3}{k}\,. \quad (18)$$

In the other words, the considered passive control formulates all the states of the fractional order unified chaotic system norm bounded.

References

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