Relationships of exponents in multifractal detrended fluctuation analysis and conventional multifractal analysis*

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Multifractal detrended fluctuation analysis (MF-DFA) is a relatively new method of multifractal analysis. It is extended from detrended fluctuation analysis (DFA), which was developed for detecting the long-range correlation and the fractal properties in stationary and non-stationary time series. Although MF-DFA has become a widely used method, some relationships among the exponents established in the original paper seem to be incorrect under the general situation. In this paper, we theoretically and experimentally demonstrate the invalidity of the expression $\tau(q) = qh(q) - 1$ stipulating the relationship between the multifractal exponent $\tau(q)$ and the generalized Hurst exponent $h(q)$. As a replacement, a general relationship is established on the basis of the universal multifractal formalism for the stationary series as $\tau(q) = qh(q) - qH' - 1$, where H' is the nonconservation parameter in the universal multifractal formalism. The singular spectra, α and $f(\alpha)$, are also derived according to this new relationship.

Keywords: fractals, Hurst exponent, multifractal detrended fluctuation analysis, time series analysis

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1. Introduction

Fractals and multifractals commonly exist in the natural world.[1] Many methods have been developed to characterize these features over the years. Hurst[2] put forward a rescaled range analysis (R/S analysis) to study the Nile and the problems relating to water storage. More importantly, he gave an important exponent, generally known as the Hurst exponent H, to quantify the long-range correlation of the signal series. Halsey *et al.*[3] formulated the fixed-size box-counting algorithm, the most common classical multifractal analysis method, to calculate the multifractal exponents, such as the scaling exponent $\tau(q)$ and the generalized fractal dimension $D(q)$. This method has been used to analyze DNA and protein sequences.[4–7] However, the R/S analysis and the fixed-size box-counting algorithm can only handle stationary signals. To handle the fluctuation in non-stationary signals, Peng *et al.*[8] developed detrended fluctuation analysis (DFA) to determine the fractal scaling properties and the long-range correlation in both stationary and non-stationary time series. Taqqu *et al.*[9] subsequently gave the direct relationship between the scaling exponent of DFA and the Hurst exponent H for fractional Gaussian noise (fGn). In order to give a full description of the complicated scaling behaviours of the time series over multiple time scales, Kantelhart *et al.*[10] proposed multifractal detrended fluctuation analysis (MF-DFA) to analyze the long-range dependence with the crossover time scales. It is a generalized version of DFA. A concept of the generalized Hurst exponent, $h(q)$, was also proposed.[10] In particular, $h(2)$ is the scaling exponent of DFA. In addition, they established the relationship (expressed by Eq. (11)) between MF-DFA and the standard partition-function-based multifractal formalism for positive, stationary and normalized time series.

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Such a relationship has been employed to study the scaling behaviours of time series in different fields of research.[11−16] Movahed *et al.*[11] later gave the relationship (depicted in Eq.(6)) between H and $h(q=2)$. Furthermore, they proved such a relationship for fractional Brownian motion (fBm). Studies have also been conducted on some features of (MF-)DFA, such as the effects of non-stationarities,[17] trends[18] and extreme data loss[19] on DFA, the relation between DFA and power spectral density analysis,[20] and a comparative study of (MF-)DFA and some other correlation analysis methods.[21,22] Moreover, (MF-)DFA has been applied to solve problems in fields such as astronomy,[11,23−25] hydrology,[12,26,27] meteorology,[28] seismology[13,14,29−31] and electrics.[15] Another method, which can also handle the non-stationary series, is the wavelet transform modulus maxima (WTMM) proposed by Muzy *et al.*[32] A systematic discussion and comparison of the performance of the WTMM and MF-DFA on fractal and multifractal analyses were made by Oswiecimka *et al.*[33] Simulated numerical series and real-life examples were employed for comparison. The advantages and disadvantages of these two methods were pointed out. Then, Oswiecimka *et al.*[33] claimed that MF-DFA worked in a more automatic way, while the WTMM should be applied with care when applied to analyze real data.

Since the concept of the Hurst exponent, H, and the relationship between $h(q)$ and H are the critical issues of our discussion in the present paper, they are briefly reviewed here first. The scaling exponent β, which is determined by the power spectrum, is employed to describe many processes. Thus the Hurst exponent can also be defined using β. We can use a definition of the Hurst exponent H that requires different relations to connect the exponents according to the underlying processes, such as $H=(\beta-1)/2$ for fBm and $H=(\beta+1)/2$ for fGn.[34,35] Then H lies between 0 and 1 for both fBm and fGn. However, there is a different definition of the Hurst exponent H, giving $H=(\beta-1)/2$ for both fBms and fGns.[1,36] In this definition, H varies from -1 to 0 for fGn and from 0 to 1 for fBm. We denote the later definition of H as H_1 in the following discussion. The following statement gives the reasons why the former definition should be the one adopted in MF-DFA. As mentioned above, MF-DFA becomes DFA if $q=2$. Heneghan and McDarby[20] pointed out that $H=(\beta+1)/2$ for fGn and $H=(\beta-1)/2$ for fBm, and proved that the relationship between the DFA exponent $2h(2)$ and the power spectrum scaling exponent β is $\beta=2h(2)-1$. That is to say that $h(2)=H$ for fGn and $h(2)=H+1$ for fBm. The relationships between H and DFA scaling exponent $h(2)$ for fGn and fBm have been further confirmed by an analytical calculation. Taqqu *et al.*[9] analytically showed that the DFA scaling exponent h equals H for fGn. Furthermore, Movahed *et al.* derived the relationships $h(2)=H+1$ for fBm[11] and $h(2)=H$ for fGn.[12] These research results support that the scaling exponent $h(2)$ obtained by (MF-)DFA should be related to H rather than H_1. In addition, Kantelhart *et al.*[10] gave the relationship between H and β as $H=(\beta+1)/2$ for the stationary long-range correlated series. So in MF-DFA the former definition of the Hurst exponent is adopted.

Although the aforementioned methods have become popular over the years, there seem to be some problems in the relationships between the exponents given in these papers. The purpose of this paper is to reexamine these relationships and to propose the relationships valid under the general situation. We first state briefly the problems and our results.

In the development of MF-DFA, Kantelhardt *et al.*[10] established a relationship, which has been extensively applied ever since, between MF-DFA and the standard partition-function-based multifractal formalism $\tau(q)=qh(q)-1$ for the positive and stationary time series. This relationship has been employed as a basis for multifractal analysis in a good number of studies (see for example Ref. [11]–[15]). However, our investigation shows that such a relationship has a potential problem. It might be incorrect for some signals, such as the fGn. We investigate the problem from theoretical and experimental perspectives in this paper. Based on the universal multifractal formalism proposed and developed by Lovejoy *et al.*,[37−39] the relationship between $h(q)$ and $\tau(q)$ can be established as $h(q)=qh(q)-qH'-1$, where H' is the nonconservation parameter in the universal multifractal formalism. Such a relationship was first suggested, without detailed analysis, by Yu *et al.*[40] in their study of the daily rainfall in the Pearl River Basin of China.

In Section 2, we first describe the basic method and some important conclusions of MF-DFA. In Section 3, we point out a problem with MF-DFA, namely, the relationship between $\tau(q)$ and $h(q)$, via theoretical arguments and experimental substantiation. A new relationship is then given for the general situation. The singular spectra, α and $f(\alpha)$, are also proposed based on this new relationship. Numerical experiments are employed to substantiate our arguments

and support our postulation. We then conclude our discussion with a summary in Section 4.

2. Some existing results of MF-DFA

To facilitate our discussion, we first give a brief description of MF-DFA. Given a time series, $\{x_k\}_{k=1}^N$ of length N with compact support, the MF-DFA method consists of five main steps.[10]

Step 1 Determine the profile

$$Y(i) \equiv \sum_{k=1}^{i}(x_k - \langle x\rangle), \quad i = 1, \ldots, N, \tag{1}$$

where $\langle x\rangle$ means the average of $\{x_k\}$. It is noted in Ref. [10] that the subtraction of $\langle x\rangle$ is not compulsory, since it would be eliminated by the detrending procedure in step 3.

Step 2 For each given positive integer s, the profile can be divided into $N_s \equiv \mathrm{int}(N/s)$ non-overlapping local windows/segments with equal length s. In the case that N/s is not an integer, another N_s non-overlapping local windows/segments are constructed in the same way from the opposite end, which utilizes the remaining part of the profile in the construction of the first N_s local windows/segments. Then, $2N_s$ local windows are obtained altogether.

Step 3 For the v-th window of the $2N_s$ local ones, the variance can be determined as

$$F^2(v,s) \equiv \frac{1}{s}\sum_{i=1}^{s}\{Y[(v-1)s+i] - y_v(i)\}^2, \quad \text{for } v = 1, \ldots, N_s, \tag{2}$$

$$F^2(v,s) \equiv \frac{1}{s}\sum_{i=1}^{s}\{Y[N-(v-N_s)s+i] - y_v(i)\}^2, \quad \text{for } v = N_s+1, \ldots, 2N_s, \tag{3}$$

where y_v is the fitting polynomial representing the local trend in the v-th local window.

Step 4 Obtain the q-th order fluctuation function as

$$F_q(s) \equiv \left\{\frac{1}{2N_s}\sum_{v=1}^{2N_s}[F^2(v,s)]^{q/2}\right\}^{1/q}, \tag{4}$$

where q can be any real number.

Step 5 Determine the scaling behaviour of $F_q(s)$ for each q. If

$$F_q(s) \sim s^{h(q)}, \tag{5}$$

where $h(q)$ is called the generalized Hurst exponent and is related to the classical Hurst exponent H by

$$H = \begin{cases} h(2), & \text{for stationary case,} \\ h(2) - 1, & \text{for non-stationary case.} \end{cases} \tag{6}$$

According to the value of H, $\{x_k\}$ is considered as long-range anti-correlated if $0 < H < 0.5$; uncorrelated if $H = 0.5$; and long-range correlated if $H > 0.5$.

According to Kantelhardt *et al.*,[10] if the time series $\{x_k\}_{k=1}^N$ is stationary, positive and normalized, then it is not necessary to detrend in step 3. Thus DFA can be replaced by standard fluctuation analysis (FA) as

$$F^2_{\mathrm{FA}}(v,s) \equiv [Y(vs) - Y((v-1)s)]^2. \tag{7}$$

Then, combining Eq. (4) with Eq. (5), we obtain

$$\sum_{v=1}^{2N_s}|Y(vs) - Y((v-1)s)|^q \sim s^{qh(q)-1}. \tag{8}$$

To relate MF-DFA to the box-counting formalism, Kantlehardt *et al.*[10] gave the partition sum of the analyzed time series as

$$Z_q(s) \equiv \sum_{v=1}^{N/s}|p_s(v)|^q \sim s^{\tau(q)}, \tag{9}$$

where

$$p_s(v) \equiv \sum_{k=(v-1)s+1}^{vs} x_k = Y(vs) - Y((v-1)s). \tag{10}$$

Here, Eq (9) is employed with the aim of calculating the mass exponent $\tau(q)$ of the measured fractal.[3] From the above two equations, the relationship between $h(q)$ and $\tau(q)$ are established as

$$\tau(q) = qh(q) - 1. \tag{11}$$

Based on $\tau(q)$, the generalized fractal dimension $D(q)$ and the singular spectrum can be derived as

$$D(q) \equiv \frac{\tau(q)}{q-1}, \tag{12}$$

$$\alpha = \tau'(q), \quad f(\alpha) = q\alpha - \tau(q). \tag{13}$$

With Eq. (11), $D(q)$ and the singular spectrum can be expressed via $h(q)$ as

$$D(q) = \frac{qh(q) - 1}{q-1}, \tag{14}$$

$$\alpha = h(q) + qh'(q), \quad f(\alpha) = q[\alpha - h(q)] + 1. \tag{15}$$

3. Relationship between $\tau(q)$ and $h(q)$ in MF-DFA

As discussed in the introduction, a problem exists in the relationship between $\tau(q)$ and $h(q)$ established in Eq. (11). We would like to address this problem from theoretical and numerical points of view in this section.

3.1. Theoretical issue

It should be observed that for the positive, stationary and normalized time series $\{x_k\}_{k=1}^N$, if no trend has to be eliminated, then step 3 will not be performed. In such a case, step 1 of MF-DFA should be compulsory. If so,

$$Y(vs) - Y((v-1)s) = \sum_{k=(v-1)s+1}^{vs} (x_k - \langle x\rangle) \quad (16)$$

instead of Eq. (10). Even if $\{x_k\}_{k=1}^N$ is a positive series, $x_k - \langle x\rangle$ might still be negative. Without loss of generality, we assume

$$\sum_{k=(v-1)s+1}^{vs} (x_k - \langle x\rangle) > 0, \quad (17)$$

$$\sum_{k=vs+1}^{(v+1)s} (x_k - \langle x\rangle) < 0, \quad (18)$$

$$\left| \sum_{k=(v-1)s+1}^{vs} (x_k - \langle x\rangle) \right| > \left| \sum_{k=vs+1}^{(v+1)s} (x_k - \langle x\rangle) \right|. \quad (19)$$

Then, even if

$$\{x_k - \langle x\rangle\}_{k=(v-1)s+1}^{(v+1)s} \supseteq \{x_k - \langle x\rangle\}_{k=(v-1)s+1}^{vs}, \quad (20)$$

it is still possible to have

$$\left| \sum_{k=(v-1)s+1}^{(v+1)s} (x_k - \langle x\rangle) \right| < \left| \sum_{k=(v-1)s+1}^{vs} (x_k - \langle x\rangle) \right|. \quad (21)$$

Take the fGn as an example. Figure 1 depicts an enlarged plot of part of the fGn. We transform $\{x\}$ upwards by $|\min(x)| + 1$ to ensure that the new series, which could be denoted by $\{\tilde{x}\}$, is positive. Mathematically, $\{\tilde{x} - \langle\tilde{x}\rangle\} = \{x - \langle x\rangle\}$. Given $s = 3$, it is easy to see that Eqs. (18)–(21) are valid for $\{\tilde{x}\}$. However, as is well known, if μ is a measure defined on sets A and B, then

$$A \supseteq B \quad (22)$$

must be followed by

$$\mu(A) \geq \mu(B). \quad (23)$$

That is to say $|p_s(v)|$ in Eq. (9) cannot be treated as a measure defined on the time series $\{x_k\}_{k=1}^N$ since it does not possess the basic property of a measure. However, when the box-counting algorithm is used, it is necessary to have a measure defined on the set.[3] The partition-sum function based on $|p_s(v)|$ is not proper. Since Eq. (9) is incorrect, the relationship in Eq. (11), which is obtained by combining Eqs. (8), (9) and (10), might also be incorrect, at least for the transformed fGn. In the following subsection, we employ numerical experiments to support our arguments.

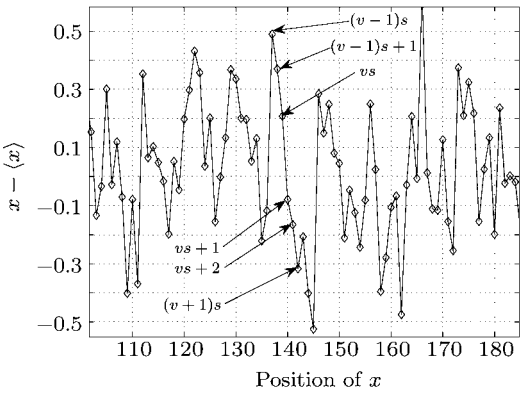


Fig. 1. An enlarged plot of part of the fGn $(x - \langle x\rangle)$ for $s = 3$.

3.2. Numerical experiments and discussion

We first substantiate our view on the problematic relationship between $\tau(q)$ and $h(q)$ established in Ref. [10] with a series of numerical experiments, and then give a general relationship between $\tau(q)$ and $h(q)$ in the next subsection. In order to numerically obtain the relationship between $\tau(q)$ and $h(q)$, we have to calculate $\tau(q)$ and $h(q)$ first. To be consistent, we consider the stationary, positive and normalized time series following the assumption in Ref. [10]. As is well known, the fGn is the stationary increment of the fBm. Thus the fGn is used to calculate $\tau(q)$ and $h(q)$. We construct our numerical examples by using the fixed-size box-counting algorithm given in Ref. [3] and the MF-DFA given in Ref. [10], respectively. The procedure to construct our examples is presented in detail as follows.

In the experiment, we first used Matlab to synthesize 10 fBms with a length of 2^{15} for each assigned Hurst exponent H. The H was given in the range from 0.1 to 0.9 with a step size of 0.1. For each fBm, we obtained a corresponding fGn, $\{G_k\}$, with the same

H. As the fixed-size box-counting algorithm is based on the partition-sum function, it is necessary to keep all elements of the series non-negative. Thus, we replaced the original $\{G_k\}$ with $\{G_k - \min(G) + 1\}$, where $\min(G)$ denotes the minimum value of series $\{G_k\}$. In such a way, we transformed $\{G_k\}$ into a positive series, which was still stationary. Moreover, the detrending step in the MF-DFA ensured that the vertical translation held the same H as that of $\{G_k\}$. Without confusion, we still use $\{G_k\}$ to denote the vertically shifted fGns in the following description. To calculate $\tau(q)$ and $h(q)$, we let q increase from -5 to 5 with a step size of 1. Applying the box-counting algorithm,[3] we computed $\tau(q)$ for each series $\{G_k\}$. After normalizing the transformed series, we applied the fixed-size box-counting algorithm to get $\tau(q)$. The almost unvaried values of the averaged $D(q)$ shown in Table 1 indicate the monofractality of these vertically shifted fGns. Whereafter, we calculated $h(q)$ using MF-DFA. It should be noted that these normalized transformed fGns were stationary, positive and normalized series, totally satisfying the assumption in Ref. [10] of the derivation of the relationship between $\tau(q)$ and $h(q)$, i.e., Eq. (11). Since the Hurst exponents of the fGn here were all positive, as emphasized in the introduction, MF-DFA was directly performed without employing the double summation technique suggested by Kantelhardt *et al.*[10] Step 3, the detrending of the MF-DFA, was able to keep $h(q)$ of $\{G_k\}$ constant when a constant was added to each element G_k of the series. Now, we could obtain $\tau(q)$ and $h(q)$ for each H. Then the average values of ten $D(q)$ and $h(q)$ for each H were compared. The figures are shown in Fig. 2. Apparently, the actual curves of $\tau(q)$ versus $qh(q)$ do not follow the expression stipulated by Eq. (11). The deviations are actually rather drastic.

From Fig. 2, we can observe that the curves poorly fit the linear relationship when $H \leq 0.3$. As H increases, the curve of $\tau(q)$ versus $qh(q)$ is becomes increasingly like the straight line $\tau(q) = qh(q) - 1$. Table 2 gives the slopes and the intercepts of the linear fits of these actual curves. From the table, we can see that the linear fit is close to $\tau(q) = qh(q) - 1$ only when $H = 0.9$. There is also a significant difference between the actual curve and $\tau(q) = qh(q) - 1$ when $H \leq 0.8$, especially for small H. This numerical example shows that the relationship between $\tau(q)$ and $h(q)$ stipulated by Eq. (3) established by Kantelhardt *et al.*[10] appears to be incorrect.

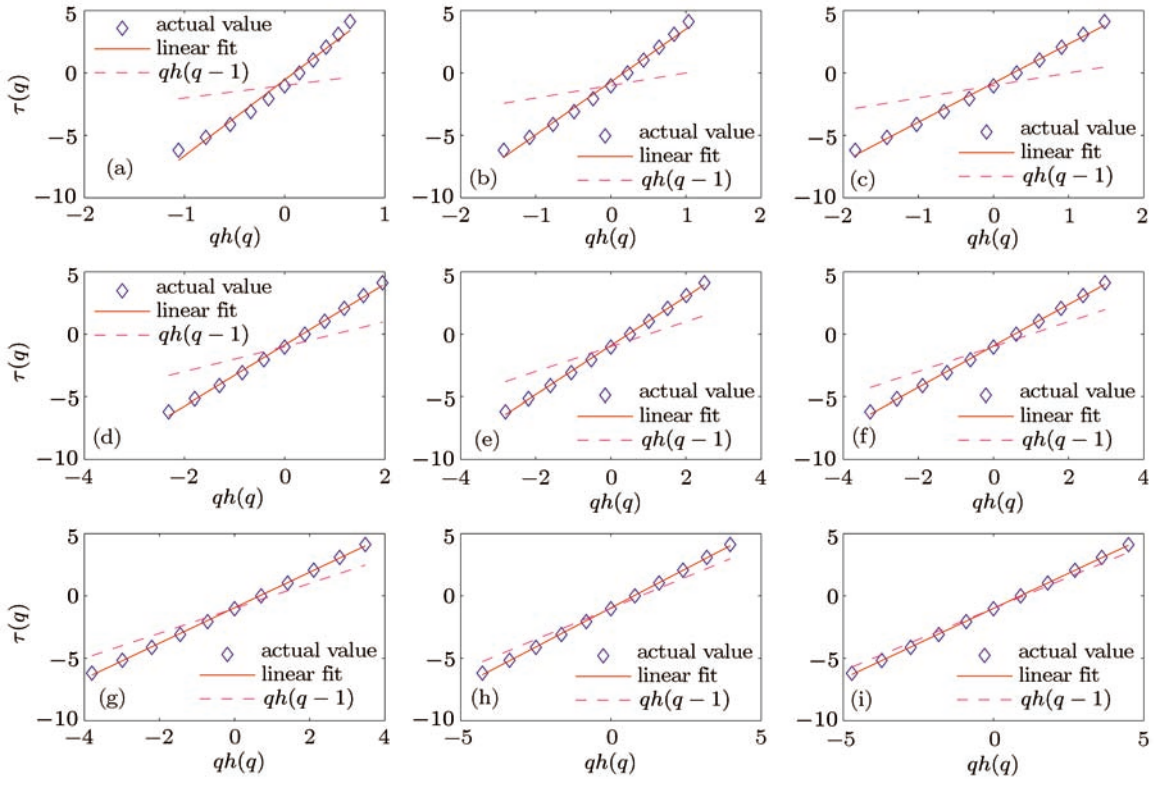


Fig. 2. (colour online) Relationships between $h(q)$ and $\tau(q)$ for numerical examples with $H = 0.1$ to 0.9. (a)–(i) correspond to $H = 0.1, 0.2, \ldots, 0.9$, respectively.

Table 1. Averaged $D(q)$ of vertical shifted fGns for different H and q.

	$q=-5$	$q=-4$	$q=-3$	$q=-2$	$q=-1$	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
$H=0.1$	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310
$H=0.2$	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310
$H=0.3$	1.0311	1.0311	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310
$H=0.4$	1.0311	1.0311	1.0311	1.0311	1.0311	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310
$H=0.5$	1.0311	1.0311	1.0311	1.0311	1.0311	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310
$H=0.6$	1.0311	1.0311	1.0311	1.0311	1.0311	1.0310	1.0310	1.0310	1.0310	1.0310	1.0310
$H=0.7$	1.0314	1.0314	1.0313	1.0312	1.0311	1.0310	1.0310	1.0309	1.0308	1.0307	1.0307
$H=0.8$	1.0321	1.0319	1.0317	1.0315	1.0313	1.0310	1.0308	1.0306	1.0304	1.0302	1.0300
$H=0.9$	1.0349	1.0341	1.0334	1.0326	1.0318	1.0310	1.0303	1.0295	1.0287	1.0280	1.0272

Table 2. Slopes and intercepts of the linear fits shown in Fig. 2.

H	Slope	Intercept
0.1	6.0839	–0.5593
0.2	4.2370	–0.7178
0.3	3.1375	–0.8149
0.4	2.4383	–0.8663
0.5	1.9604	–0.9184
0.6	1.6609	–0.9382
0.7	1.4240	–0.9504
0.8	1.2560	–0.9656
0.9	1.1264	–0.9995

Since the validity of the relationship has been given by Kantelhardt *et al.* [10] using the binomial cascade model, and our numerical experiments, on the other hand, are straight-forward and convincing, what then leads to the discrepancy between the theoretical and the experimental results? We attempt to provide an explanation and establish a general relationship in the following subsection.

3.3. A general relationship

In this subsection, we still focus on the stationary, positive and normalized time series. In the universal multifractal formalism, there are three parameters, α, C_1 and H', called the multifractal index, the codimension and the nonconservation parameter, respectively.[37] Comparing the codimension multifractal formalism based on scaling moment function $K(q)$ and codimension function $c(\gamma)$[38] with the dimension multifractal framework based on $\tau(q)$ and $f(\alpha)$,[3] we can observe that some relationships have already been established, particularly,

$$\tau(q) = (q-1)D - K(q), \tag{24}$$

where D is the dimension of the observing space (for time series, $D=1$),[3,37] and

$$K(q) = \frac{C_1}{\alpha - 1}(q^\alpha - q). \tag{25}$$

In addition, there is an interesting relationship for $q \geq 0$,[41]

$$\zeta(q) = qH' - \frac{C_1}{\alpha - 1}(q^\alpha - q), \tag{26}$$

where $\zeta(q)$ is the exponent obtained from the structure function $S_q(s) \equiv \langle |x_{k+s} - x_k|^q \rangle \sim s^{\zeta(q)}$ (see Ref. [42] for more details about the structure function and $\zeta(q)$). According to Koscielny-Bunde *et al.*, [43]

$$\zeta(q) = qh(q) - q, \tag{27}$$

and then

$$qh(q) = q + \zeta(q) = (H'+1)q - K(q). \tag{28}$$

It should be noted that if $q=1$, then according to Eq. (25), $K(q=1) \equiv 0$, Eq. (28) becomes

$$h(1) = H' + 1. \tag{29}$$

Combining Eq. (24) with Eq. (28), we can obtain for $q \geq 0$,

$$\tau(q) = qh(q) - qH' - 1. \tag{30}$$

Thus, the original relationship in Eq. (11), $\tau(q) = qh(q) - 1$, differs from the one we derived in Eq. (30) by $-qH'$.

If the two equations are compared using the vertically shifted fGn, which was employed to check the original relationship Eq. (11), with Eq. (12), we can then find that for $q=0$, both Eqs. (11) and (30) lead to $\tau(0) = -1$, so that $D(0) = \tau(0)/0 - 1 = -1/-1 = 1$. Since the experimental examples are monofractal, $D(q) \equiv$ const. and $h(q) \equiv H$ (H is the Hurst exponent

of the vertically shifted fGn here, which could be any value of 0.1, 0.2, ..., 0.9), then $D(q) = D(0) = 1$ and $\tau(q) = D(q)(q-1) = q-1$. However, the right-hand side of Eq. (11) is equal to $qH - 1$. Thus, Eq. (11) is valid only for $H = 1$. Based on Eq. (29), we obtain $H' = H - 1$. Then we can calculate the right-hand side of Eq. (30), and it indeed equals $\tau(q)$, since $qH - q(H-1) - 1 = qH - qH + q - 1 = q - 1$. Thus, Eq. (30) should be the correct relationship for the vertically shifted fGn.

However, we investigated the two equations using the binomial cascade model. Lovejoy *et al.*[37] described the nonconservation parameter H' as $H' = 0$, which means that the set can be modeled as a pure multiplicative process, i.e., the set can be conserved from scale to scale. According to Lavallee *et. al.*,[39] the multiplicative processes were first developed as models of turbulent cascades. An example of such a multiplicative process in two-dimensional space is given in Ref. [39], which is actually the cascade model in two-dimensional space. Therefore, the binomial cascade model is the only one satisfying the $H' = 0$ condition. In fact, Eq. (19) and (20) in Ref. [10] ensure $h(1) = 1/q - \ln[a^q + (1-a)^q]/(q\ln(2))|_{q=1} = 1 - 0 = 1$, which indicates that $H' = h(1) - 1 = 0$. The a in the above formula is a parameter with a value between 0.5 and 1 indicating the probability of assigning the measure during the binomial cascade process in the binomial multifractal model. Under this situation, Eqs. (11) and (30) become identical.

Though our proposed relationship given in Eq. (30) is established on the basis of non-negative q, the analysis and the conclusion of the two examples above are actually independent of the sign of q. That means the new relationship is also valid for negative q. Thus we believe that the generalized relationship in Eq. (30) should be valid for all q. We would like to discuss this problem from empirical points of view.

Take the vertically shifted fGn with Hurst exponent H_0 (H_0 could be any value of 0.1, 0.2, ..., 0.9) as an example, it is known that the vertically shifted fGn is monofractal.[1] Then its generalized fractal dimension $D(q) = D \equiv 1$ and $h(q) = H_0$. As given by Halsey *et al.*,[3] $\tau(q) = D(q)\cdot(q-1) = q-1$. However, $qh(q) - 1 = qH_0 - 1$. Then Eq. (11) does not hold unless $H_0 = 1$. Therefore, the validity of the relationship given in Eq. (11) is questionable. Empirically, $\tau(q)$ should be 0 for $q = 1$ to ensure $D(1)$ finite, which comes from $D(1) = \tau(q)/(q-1)|_{q=1}$. Therefore, there should be one additional term $A(q)$ in Eq. (11) to ensure $\tau(1) = 0$. It is easy to obtain $A(1) = h(1) - 1$ for $q = 1$, if we set $\tau(1) = h(1) - A(1) - 1 = 0$. Then we can further derive the formula for $A(q)$ from the special situation (the fGn, mentioned above. It is already known that $\tau(q) = q-1$. Then if $qH_0 - A(q) - 1$ is expected to be equal to $q-1$, it can be obtained that $A(q) = q\cdot(H_0 - 1)$. Taking into account that $h(q) \equiv H_0$ for the fGn and $A(1) = h(1) - 1$, the new relationship can be established empirically as $\tau(q) = qh(q) - q\cdot(h(1)-1) - 1 = qh(q) - qH' - 1$. Therefore, it is reasonable to say that the original relationship, Eq. (11) proposed in Ref. [10], only holds for the special situation, i.e., $H' = 0$ or $h(1) = 1$ of the general relationship given in Eq. (30). Lovejoy *et al.*[37] pointed out that there are always cases in geophysical phenomena and the turbulence where the observables have $H' \neq 0$, e.g., $H' > 1$ for the temperature and the pressure in the verticals and $H' \approx -0.35$ for the rain rate in time on (climate) scales larger than one month.[44]

The numerical examples are employed again to check the new relationship we obtained in Eq. (30). The illustrative plots of $\tau(q)$ versus $qh(q) - qH'$ are shown in Fig. 3. The linear fits of the actual curves conform to the expected relationship $\tau(q) = qh(q) - qH' - 1$ very well. The slopes and the intercepts of the linear fits shown in Table 3 are very close to 1 and -1, respectively, which solidly support our new relationship.

Since H' seems to be similar to the Hurst exponent H, its meaning needs to be discussed here. What should be emphasized here is that although for some cases, H' and H are numerically equal, these two exponents have different physical meanings. It is already known that $\zeta(2) = \beta - 1$.[42] For a monofractal example, such as the fBm, we have $\tau(q) = D\cdot(q-1)$ and $h(q) \equiv H$, then from Eq. (24), we can obtain $K(q) \equiv 0$. Thus according to Eqs. (26) and (30), $\zeta(q) = qH'$. Specifically, for $q = 2$, we have $2H' = \beta - 1$, then $H' = \frac{\beta-1}{2}$, which is the Hurst exponent given by the definition in Refs. [1] and [36]. However, for the multifractal examples, $K(q) = D\cdot(q-1) - \tau(q)$ is not always equal to zero and H' should be $\frac{\beta-1}{2} + \frac{K(2)}{2} = H_1 + \frac{K(2)}{2}$. For the

case $K(q) \neq 0$, H' is different from H numerically. Actually, H' is the nonconservation parameter in the universal multifractal formalism and $H' = 0$ means that the set can be modeled as a pure multiplicative process, i.e., the set is conserved from scale to scale. While H is the Hurst exponent, which can be employed to measure the long-range correlation of the series, and $H = 0$ indicates the strong long-range anticorrelation. It is not uncommon to see that they have different values. Take the binomial multifractal model as an example, we have $H' = 0$ as mentioned above. However, $H = h(2) = 1/2 - \ln[a^2 + (1-a)^2]/(2\ln 2)$ according to Eq. (20) given in Ref. [10]. Then it obviously is not equal to 0.

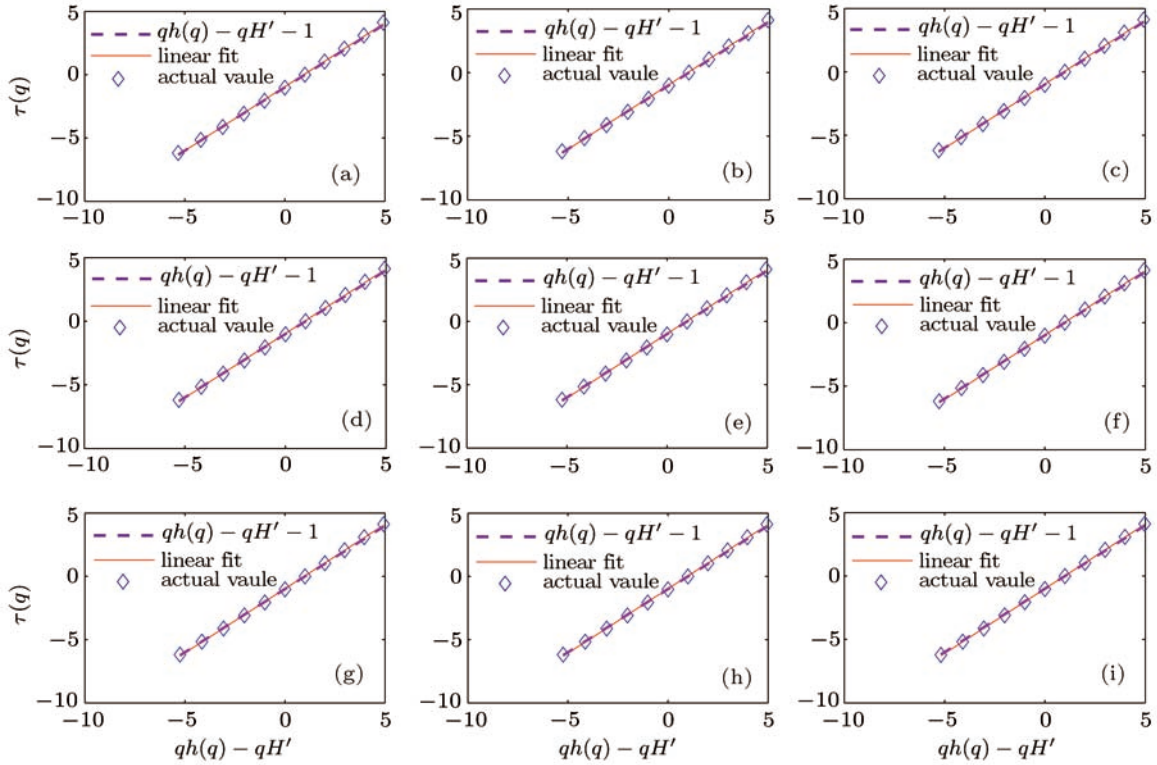


Fig. 3. (colour online) Relationships between $h(q)$ and $\tau(q)$ for numerical examples with $H = 0.1$ to 0.9 (a)–(i) correspond to $H = 0.1, 0.2, \ldots, 0.9$, respectively.

Table 3. Calculated coefficients of $qh(q) - qH'$ and intercepts of the linear fits shown in Fig. 3.

H	Coefficient	Intercept
0.1	1.0114	–0.9526
0.2	1.0126	–0.9562
0.3	1.0134	–0.9612
0.4	1.0129	–0.9626
0.5	1.0156	–0.9727
0.6	1.0157	–0.9743
0.7	1.0166	–0.9737
0.8	1.0172	–0.9785
0.9	1.0203	–1.0032

Based on this new relationship (Eq. (30)), we can further obtain the singular spectra, α and $f(\alpha)$, as

$$\alpha = \frac{\mathrm{d}}{\mathrm{d}q}\tau(q) = h(q) + q\frac{\mathrm{d}}{\mathrm{d}q}h(q) - H', \tag{31}$$

$$f(\alpha) = q\alpha - \tau(q) = q(\alpha - h(q) + H') + 1. \tag{32}$$

Therefore, H' should be estimated before we calculate the multifractal spectrum using $h(q)$ from the MF-DFA.

4. Conclusion

MF-DFA is a generalized version of DFA developed for characterizing the fractal and the multifractal properties of stationary and non-stationary signals. Kantelhardt *et al.*[10] established the relationship for the stationary positive series, which connects $h(q)$ to $\tau(q)$ for MF-DFA. This relationship has been used in a large number of research studies. We have shown in this paper that the relationship is in fact valid only

under the special situation $H' = 0$. Our argument has also been substantiated by numerical experiments and the theoretic derivation, which suggest that the generalized relationship between $h(q)$ and $\tau(q)$ should be $\tau(q) = qh(q) - qH' - 1$ instead of the one stipulated in Eq. (11) for the stationary positive time series. The singular spectra, α and $f(\alpha)$, based on $h(q)$ have also been proposed in Eqs. (31) and (32), respectively, to replace the existing ones stipulated in Eq. (15).

To recapitulate, we have conceptually argued and experimentally demonstrated one problem in MF-DFA, which is the relationship between $h(q)$ and $\tau(q)$. We have suggested a plausible form of the relationship.

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References

[1] Mandelbrot B B 1983 *The Fractal Geometry of Nature* (New York: W.H. Freeman & Co Ltd)
[2] Hurst H E 1951 *Trans. Amer. Soc. Civ. Eng.* **116** 770
[3] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986 *Phys. Rev.* A **33** 1141
[4] Yu Z G, Anh V V, Gong Z M and Long S C 2002 *Chin. Phys.* **11** 1313
[5] Yu Z G, Xiao Q J, Shi L, Yu J W and Anh V 2010 *Chin. Phys.* B **19** 068701
[6] Zhu S M, Yu Z G and Anh V 2011 *Chin. Phys.* B **20** 010505
[7] Han J J and Fu W J 2010 *Chin. Phys.* B **19** 010205
[8] Peng C K, Buldyrev S V, Havlin S, Simons M, Stanley H E and Goldberger A L 1994 *Phys. Rev.* E **49** 1685
[9] Taqqu M S, Teverovsky V and Willinger W 1995 *Fractals* **3** 785
[10] Kantelhardt J W, Zschiegner S A, Koscielny-Bunde E, Havlin S, Bunde A and Stanley H E 2002 *Physica A* **316** 87
[11] Movahed M S, Jafari G R, Ghasemi F, Rahvar S and Tabar M R R 2006 *J. Stat. Mech.* P02003
[12] Movahed M S and Hermanis E 2008 *Physica* A **387** 915
[13] Telesca L, Lapenna V and Macchiato M 2004 *Chaos Solition. Fract.* **19** 1
[14] Telesca L, Lapenna V and Macchiato M 2005 *New J. Phys.* **7** 214
[15] Kimiagar S, Movahed M S, Khorram S, Sobhanian S and Tabar M R R 2009 *J. Stat. Mech.* P03020
[16] Luo S H and Zeng J S 2009 *Acta Phys. Sin.* **58** 150 (in Chinese)
[17] Chen Z, Ivanov P Ch, Hu K and Stanley H E 2002 *Phys. Rev.* E **65** 041107
[18] Hu K, Ivanov P Ch, Chen Z, Carpena P and Stanley H E 2001 *Phys. Rev.* E **64** 011114
[19] Ma Q D Y, Bartsch R P, Bernaola-Galvan P, Yoneyama M and Ivanov P Ch 2010 *Phys. Rev.* E **81** 031101
[20] Heneghan C and McDarby G 2000 *Phys. Rev.* E **62** 6103
[21] Xu L, Ivanov P Ch, Hu K, Chen Z, Carbone A and Stanley H E 2005 *Phys. Rev.* E **71** 051101
[22] Bashan A, Bartsch R, Kantelhardt J W and Havlin S 2008 *Physica* A **387** 5080
[23] Anh V, Yu Z G and Wanliss J 2007 *Nonlinear. Proc. Geophys.* **14** 701
[24] Anh V V, Yong J M and Yu Z G 2008 *J. Geophys. Res.* **113** A10215
[25] Yu Z G, Anh V and Eastes R 2009 *J. Geophys. Res.* **114** A05214
[26] Zhang Q, Xu C Y, Chen Y D and Yu Z 2008 *Hydrol. Proc.* **22** 4997
[27] Zhang Q, Xu C Y, Yu Z, Liu C and Chen Y D 2009 *Physica* A **388** 927
[28] Talkner P and Weber R O 2000 *Phys. Rev.* E **62** 150
[29] Varotsos P A, Sarlis N V and Skordas E S 2002 *Phys. Rev.* E **66** 011902
[30] Varotsos P A, Sarlis N V and Skordas E S 2003 *Phys. Rev.* E **67** 021109
[31] Telesca L, Colangelo G, Lapenna V and Macchiato M 2004 *Phys. Lett.* A **332** 398
[32] Muzy J F, Bacry E and Arneodo A 1994 *Int. J. Bifur. Chaos* **4** 245
[33] Oswiecimka P, Kwapien J and Drozdz S 2006 *Phys. Rev.* E **74** 016103
[34] Mandelbrot B B and Van Ness J W 1968 *SIAM Rev.* **10** 422
[35] Barton R J and Poor H V 1988 *IEEE Trans. Inf. Theory* **34** 943
[36] Flandrin P 1992 *IEEE Trans. Inf. Theory* **38** 910
[37] Lovejoy S, Schertzer D and Allaire V C 2008 *Atmos. Res.* **90** 10
[38] Schertzer D and Lovejoy S 1987 *J. Geophys. Res.* **92** 9693
[39] Lavallee D, Lovejoy S, Schertzer D and Ladoy P 1993 *Fractals in Geography* (Englewood Cliffs: PTR Prentice Hall) p. 158
[40] Yu Z G, Leung Y, Chen Y D, Zhang Q and Anh V 2011 *Multifractal Analyses of Daily Rainfall in the Pearl River Basin of China* (submitted to J. Hydrol.)
[41] Seuront L, Schmitt F, Lagadeuc Y, Schertzer D and Lovejoy S 1999 *J. Plankton Res.* **21** 877
[42] Davis A, Marshak A, Wiscombe W and Cahalan R 1994 *J. Geophys. Res.* **99** 8055
[43] Koscielny-Bunde E, Kantelhardt J W, Braun P, Bunde A and Havlin S 2006 *J. Hydrol.* **322** 120
[44] Tessier Y, Lovejoy S, Hubert P, Schertzer D and Pecknold S 1996 *J. Geophys. Res.* **101** 26427