Hopf bifurcation and uncontrolled stochastic traffic-induced chaos in an RED-AQM congestion control system*

Wang Jun-Song(王俊松)[a)†], Yuan Rui-Xi(袁睿翕)[b)],
Gao Zhi-Wei(高志伟)[c)], and Wang De-Jin(王德进)[d)]

[a)] *School of Biomedical Engineering, Tianjin Medical University, Tianjin* 300070, *China*
[b)] *School of Information Science, Tsinghua University, Beijing* 100084, *China*
[c)] *Department of Automation, Tianjin University, Tianjin* 300072, *China*
[d)] *Department of Automation, Tianjin University of Science and Technology, Tianjin* 300222, *China*

(Received 17 January 2011; revised manuscript received 7 April 2011)

We study the Hopf bifurcation and the chaos phenomena in a random early detection-based active queue management (RED-AQM) congestion control system with a communication delay. We prove that there is a critical value of the communication delay for the stability of the RED-AQM control system. Furthermore, we show that the system will lose its stability and Hopf bifurcations will occur when the delay exceeds the critical value. When the delay is close to its critical value, we demonstrate that typical chaos patterns may be induced by the uncontrolled stochastic traffic in the RED-AQM control system even if the system is still stable, which reveals a new route to the chaos besides the bifurcation in the network congestion control system. Numerical simulations are given to illustrate the theoretical results.

Keywords: stability, Hopf bifurcation, chaos, stochastic traffic

PACS: 05.45.–a, 47.52.+j **DOI:** 10.1088/1674-1056/20/9/090506

1. Introduction

Congestion control plays a crucial role in the success of the modern Internet. Since communication delay exists extensively in the network, the congestion control system is a typical time delay control system and bifurcation is often caused by the varying model parameters. Among the existing congestion control models, Misra's fluid-based model[1,2] and Kelly's congestion control model[3,4] are well-known models. Bifurcation problems in the two models have been extensively researched and great progress was made in Refs. [5]–[12]. Random early detection (RED), exponential RED and random early marking (REM), three well-known active queue management (AQM) schemes, have been a very active research area in the internet community.[1] The resulting stability and Hopf bifurcation analysis in the three kinds of AQM algorithms with communication delays are discussed in Refs. [13]–[17].

In this paper, we will first focus on the Hopf bifurcation analysis of a novel congestion model, namely the RED-AQM based rate congestion control, which was proposed in Refs. [18] and [19]. Secondly, to the best of our knowledge, until now it is still unknown what phenomena may be induced by the uncontrolled stochastic traffic when the delay is close to its critical value. Such an issue will be addressed to show that typical chaos patterns may be induced by the uncontrolled stochastic traffic in the RED-AQM control system even if the system is stable. Our analysis provides an insight into and a further understanding of the chaos mechanism of Internet traffic and some guidelines on how to design an efficient congestion controller to prevent the oscillatory behaviour and chaos patterns from occurring in the RED-AQM control system, which helps to achieve a high-level quality of service and improves the capability of the networks in general.

The rest of the paper is organized as follows. The transfer function model of the RED-AQM congestion control system is described in Section 2. The analysis of the stability and the Hopf bifurcation of an RED-AQM control system with communication delay

*Project supported by the National High Technology Research and Development Program of China (Grant No. 2007AA01Z480).
†Corresponding author. E-mail: wjsong2004@126.com
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is investigated in Section 3. In Section 4, an uncontrolled stochastic traffic-induced typical chaos pattern of the RED-AQM congestion control system is exhibited based on the recurrent plot approach. Finally, concluding remarks are given in Section 5.

2. A novel RED-AQM rate-based congestion control system

Recently, a novel model of the queue length in a router, namely the rate-based control, was been proposed in Refs. [18] and [19]. The model of the AQM router is illustrated as Fig. 1, where $v_i(t)$ is the controlled transmission rate of the source node, $i = 1, 2, \ldots, N$, N is the number of the controlled source nodes, $v_0(t)$ is the uncontrolled transmission rate of the guaranteed traffic flowing into the AQM router, which has a higher majority than others, $q(t)$ is the instantaneous queue length of the buffer in the AQM router and $\bar{u}$ is the service rate of the AQM router.

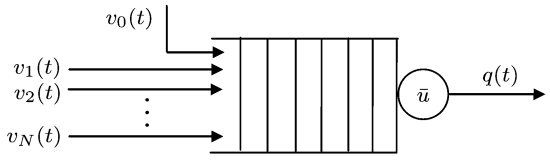


Fig. 1. System model of an AQM router.

The queue length dynamics of the router can be expressed as[18,19]

$$G(s) = \frac{N\,\mathrm{e}^{-Rs}}{s}, \tag{1}$$

where s is the queue length and R is the average round-trip time of the network. Based on this model, an AQM congestion control scheme (namely the proportional-integral (PI) type rate-based AQM congestion control) was proposed in Refs. [18] and [19], which was proved to be suitable for network traffic control via the Internet.

Since the RED is another extensively used AQM scheme, we shall focus on the analysis of the RED-AQM type rate-based congestion control system in this paper. The block diagram of the RED-AQM rate-based congestion control system is shown in Fig. 2, where $q_{\mathrm{d}}(t)$ is the desired queue length in the router, $v(t) = \sum_{i=1}^{N} v_i(t)$ is the total controlled transmission rate of the source node, $C(s)$ is the transfer function of the RED-AQM controller located in the router and follows

$$C(s) = \frac{K_{\mathrm{RED}}}{Ts + 1}, \tag{2}$$

where K_{RED} and T are the control parameters.

The RED-AQM controller can clamp the steady queue length around the target buffer occupancy, thus providing the best-effort service traffic. The objective of this paper is to study the stability, the Hopf bifurcation and the chaos in the RED-AQM congestion control system.

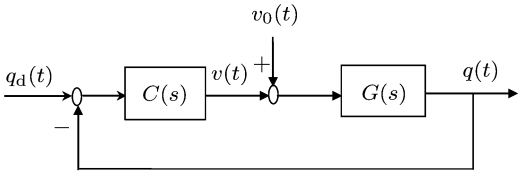


Fig. 2. Block diagram of the RED-AQM rate-based congestion control system.

3. Stability and Hopf-bifurcation analysis of the RED-AQM control system

3.1. Theoretical analysis

From Fig. 2, the feedback loop transfer function of the RED-AQM rate-based congestion control system can be obtained as

$$\phi(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}. \tag{3}$$

By substituting Eqs. (1) and (2) into Eq. (3), we can obtain the characteristic equation of the RED-AQM control system as

$$\Delta(s) = Ts^2 + s + NK_{\mathrm{RED}}\,\mathrm{e}^{-Rs} = 0. \tag{4}$$

Letting $s = \mathrm{j}\,\varpi$, where $\varpi > 0$, i.e., along the imaginary axis in the s-plane and partitioning the corresponding $\Delta(\mathrm{j}\,\varpi)$ into real and imaginary components yields

$$\begin{cases} -\varpi^2 T + NK_{\mathrm{RED}} \cos R\varpi = 0, \\ \varpi - NK_{\mathrm{RED}} \sin R\varpi = 0. \end{cases} \tag{5}$$

Therefore, we have the solution

$$\varpi_0 = \sqrt{\frac{-\dfrac{1}{T^2} + \sqrt{\dfrac{1}{T^4} + 4\dfrac{(NK_{\mathrm{RED}})^2}{T^2}}}{2}}, \tag{6}$$

$$R_0 = \frac{1}{\varpi_0} \arctan\left(\frac{1}{T\varpi_0}\right), \tag{7}$$

where R_0 is the critical value of the delay for the stability of the RED-AQM control system.

Lemma 1[20] For characteristic Eq. (4), we define

$$M(R) = \{s : \mathrm{Re}(s) \geq 0, \Delta(s, R) = 0\},$$

which denotes the number of roots with a nonnegative real part. Let $0 \leq R_1 < R_2$. Suppose that for $R \in [R_1, R_2]$, there are no roots for Eq. (4) on the imaginary axis, then $M(R_1) = M(R_2)$

Lemma 2 When $R < R_0$, all the roots for Eq. (4) have strictly negative real parts.

Proof When $R = 0$, the characteristic of Eq. (4) becomes

$$Ts^2 + s + NK_{\mathrm{RED}} = 0.$$

Since $T > 0$, $NK_{\mathrm{RED}} > 0$, there are no roots for the characteristics of Eq. (4) with nonnegative real parts, so we have $M(0) = 0$

From the above analysis, we know that there are no roots on the imaginary axis for Eq. (4) when $R < R_0$. By lemma 1, we have $M(R_0) = 0$, so $M(0) = M(R_0) = 0$. Therefore, when $R < R_0$, there are no roots for Eq. (4) with nonnegative real parts. This completes the proof.

Lemma 3 When $R = R_0$, Eq. (4) has a pair of purely imaginary roots $s = \pm \mathrm{j}\,\varpi_0$ and all the other roots have strictly negative real parts.

Proof Suppose that there exist a pair of roots for Eq. (4), $\lambda_{1,2} = \beta \pm \mathrm{j}\,\varpi_0$, where $\beta > 0$. Since the roots are continuous in parameter R for any sufficiently small positive number ε, there exists a positive number δ, which depends on ε such that $|\mathrm{Re}(\lambda_1) - \beta| < \varepsilon$ holds when $R \in (R_0 - \delta, R_0 + \delta)$. Let $\varepsilon = \beta/2$, we have $|\mathrm{Re}(\lambda_1)| > \beta/2$ when $R \in (R_0 - \delta, R_0)$. This contradicts the conclusion of lemma 2. Thus, lemma 3 must hold and this completes the proof.

Lemma 4 When $R = R_0$,

$$\left.\frac{\mathrm{d}\,\mathrm{Re}(s)}{\mathrm{d}R}\right|_{R=R_0} > 0.$$

Proof Differentiating Eq. (4) with respect to R and applying the implicit function theorem, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}R} = \frac{sNK_{\mathrm{RED}}\,\mathrm{e}^{-Rs}}{2Ts+1} = \frac{\varpi_0 NK_{\mathrm{RED}}(\sin R_0\varpi_0 + 2T\varpi_0 \cos R_0\varpi_0) + \mathrm{j}\,\varpi_0 NK_{\mathrm{RED}}(\cos R_0\varpi_0 - 2T\varpi_0 \sin R_0\varpi_0)}{1 + 4T^2\varpi_0^2}.$$

So

$$\left.\frac{\mathrm{d}\,\mathrm{Re}(s)}{\mathrm{d}R}\right|_{R=R_0} = \mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}R}\right)_{R=R_0} = \frac{\varpi_0 NK_{\mathrm{RED}}(\sin R_0\varpi_0 + 2T\varpi_0 \cos R_0\varpi_0)}{1 + 4T^2\varpi_0^2}.$$

Since

$$\sin R_0\varpi_0 = \frac{\varpi_0}{NK_{\mathrm{RED}}} > 0$$

and

$$\cos R_0\varpi_0 = \frac{\varpi_0^2 T}{NK_{\mathrm{RED}}} > 0,$$

it is obvious that

$$\left.\frac{\mathrm{d}\,\mathrm{Re}(s)}{\mathrm{d}R}\right|_{R=R_0} > 0.$$

This completes the proof.

Based on lemmas 3 and 4, and using the lemma given in Ref. [20], we can obtain the following lemma.

Lemma 5 When $R > R_0$, Eq. (4) has at least one root with a strictly positive real part.

Based on lemmas 2–5, we can obtain the following theorem about the local stability and the Hopf bifurcation of the RED-AQM control system by applying the stability theory and the Hopf bifurcation theorem for the delay differential equations.[21]

Theorem 1 The following conclusions about the stability and the Hopf bifurcation for the RED-AQM rate-based congestion control system hold.

(i) When $R < R_0$, the equilibrium point of the system is locally asymptotically stable.

(ii) When $R = R_0$, the system exhibits a Hopf bifurcation.

(iii) When $R > R_0$, the equilibrium point of the system is unstable.

Remark 1 The research focus of this section is the existence of the Hopf bifurcation in the RED-AQM control system, the analysis of the stability and the direction of the Hopf bifurcating periodic solutions of the system is not presented, the reader is referred to Refs. [5]–[12] for the details.

3.2. Simulation analysis

To illustrate the above analytic procedure for the RED-AQM control system, we consider an AQM router with 12 controlled source nodes. The average round-trip propagation delay is 0.2 s and the desired queue length $q_{\mathrm{d}}(t)$ is 150 packets. Therefore, the transfer function of the queue length model is given as $G_{\mathrm{p}}(s) = 12\,\mathrm{e}^{-0.2s}/s$, and the transfer

function of the designed RED-AQM rate controller is $C_{\rm RED}(s) = 0.09/(0.2s + 1)$.[22] We conduct the simulation using Matlab. The simulation result is shown in Fig. 3, which demonstrates that the RED-AQM control system is stable with good performance.

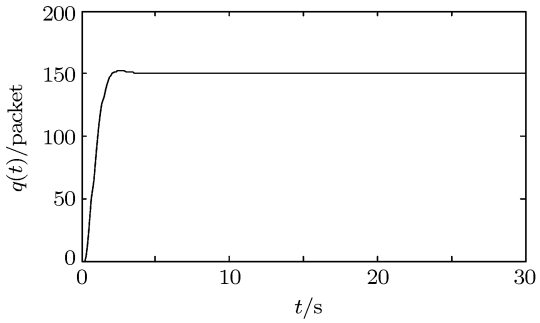


Fig. 3. Queue length $q(t)$ with $R = 0.2$.

In practice, delay R is a varying network parameter, so it is necessary to study what will be caused by the varying delay. According to Eqs. (6) and (7), we obtain $\varpi_0 = 1.0567$ and $R_0 = 1.2895$. So when

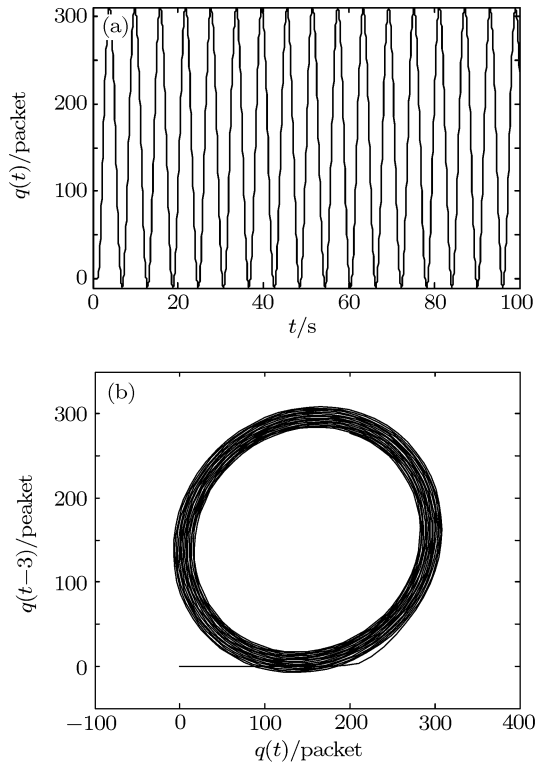


Fig. 4. (a) Queue length $q(t)$ and (b) phase plot of the queue length with $R = 1.2895$.

R passes through the critical value 1.2895, the RED–AQM control system loses its stability and a Hopf bifurcation occurs. Queue length $q(t)$ and its phase plot with $R = 1.2895$ are respectively shown in Figs. 4(a) and 4(b), which indicate that there exists a stable limit cycle.

4. Chaos induced by uncontrolled stochastic traffic in the RED-AQM control system

As shown in Figs. 1 and 2, there is some uncontrolled stochastic traffic input to the router. To the best of our knowledge, until now, it is still unknown what phenomena may be induced by uncontrolled stochastic traffic when the delay is close to its critical value. In this section, we shall address this problem, and some interesting and important results are obtained.

4.1. Recurrence plot approach

Natural processes may have distinct recurrent behaviours. The recurrence of states means that the system will return to an arbitrarily small neighborhood of the previous state after a period of time. The recurrence is not only a fundamental property of the deterministic dynamical systems, but is also typical for complex dynamical systems such as chaotic systems. For time series, the concept of recurrence was introduced by Eckmann by means of the recurrence plot (RP), a visual tool designed to display recurring patterns and to investigate nonstationary patterns.[23]

In this paper, the recurrence plot method is employed to reveal the physical implications of the queue length, i.e., a one-dimensional time series denoted as $q(i)$, $i = 1, 2, \ldots, n$, where n is the total number of $q(i)$. Following Takens' embedding theorem[24] and using an embedding dimension m and a time delay τ, we can reconstruct the phase space from the original time series $q(i)$ by

$$\begin{aligned} \boldsymbol{Q}_j &= (q_j, q_{j+\tau}, \ldots, q_{j+(m-1)\tau}), \\ & j = 1, 2, \ldots, K, \end{aligned} \tag{8}$$

where $\boldsymbol{Q}_j$ is the vector of the reconstructed states in the phase-space at the j-th sampling time; and $K = n - (m - 1)$ is the total number of $\boldsymbol{Q}_j$. Dimension m and time delay τ are estimated using C–C method.[25] In our study, without loss of generality, we set $m = 5$ and $\tau = 3$.

The recurrence plot is a two-dimensional binary diagram representing the recurrences that occur in an m-dimensional phase space within an arbitrarily defined threshold at different time i, j. The main step of the RP is the calculation of a $K \times K$ matrix $\boldsymbol{A}$, which can be mathematically expressed as

$$A_{i,j} = \Theta(\|\boldsymbol{Q}_i - \boldsymbol{Q}_j\| - \varepsilon), \quad i, j = 1, \ldots, K, \tag{9}$$

where $\|\cdot\|$ is a norm (e.g. Euclidean norm) and $\Theta(\cdot)$ is the Heaviside step function described by

$$\Theta(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases} \tag{10}$$

The ε is a threshold distance defined as $\varepsilon = a \cdot \mathrm{std}(x_i)$, with $\mathrm{std}(x_i)$ being the standard deviation function and a a threshold coefficient. Although there is no general rule for the estimation of a, it is usually set to about 0.3. We take the default value ($a = 0.3$) in this study.

4.2. Analysis of the chaos induced by uncontrolled stochastic traffic based on the recurrence plot

Supposing that the power P of the uncontrolled stochastic traffics input to the router in Figs. 1 and 2 is equal to 1.0, as shown in Fig. 5. Since delay R is a varying parameter, we consider the following two cases.

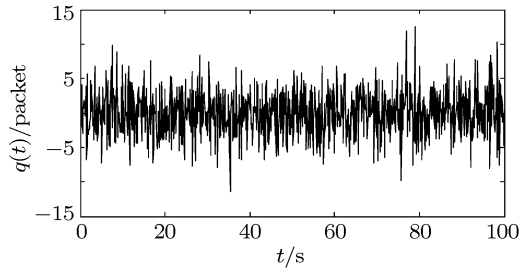


Fig. 5. Uncontrolled stochastic traffic with power $P = 1.0$.

Case 1 $R = 0.2$. In this case, $R \ll R_0$, according to theorem 1 and Fig. 3, the RED-AQM congestion control system is very stable. When there is some uncontrolled stochastic traffic input to the router, the queue length in the router is illustrated in Fig. 6. As predicted, the RED-AQM congestion control system is still stable with satisfying performance.

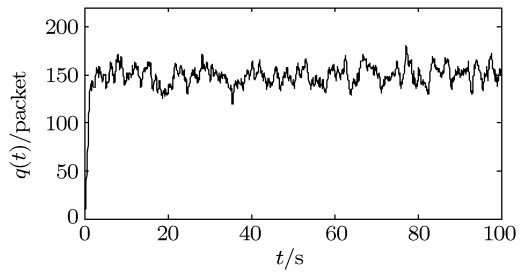


Fig. 6. Queue length $q(t)$ with $R = 0.2$ and uncontrolled stochastic traffic.

Case 2 $R = 1.2$. In this case, the delay is close to its critical value, i.e., $R_0 = 1.2895$. Obviously, the RED-AQM congestion control system is still stable, however, with a small degree if stability. When there is no uncontrolled stochastic traffic input to the router, the queue length is shown in Fig. 7(a), which illustrates

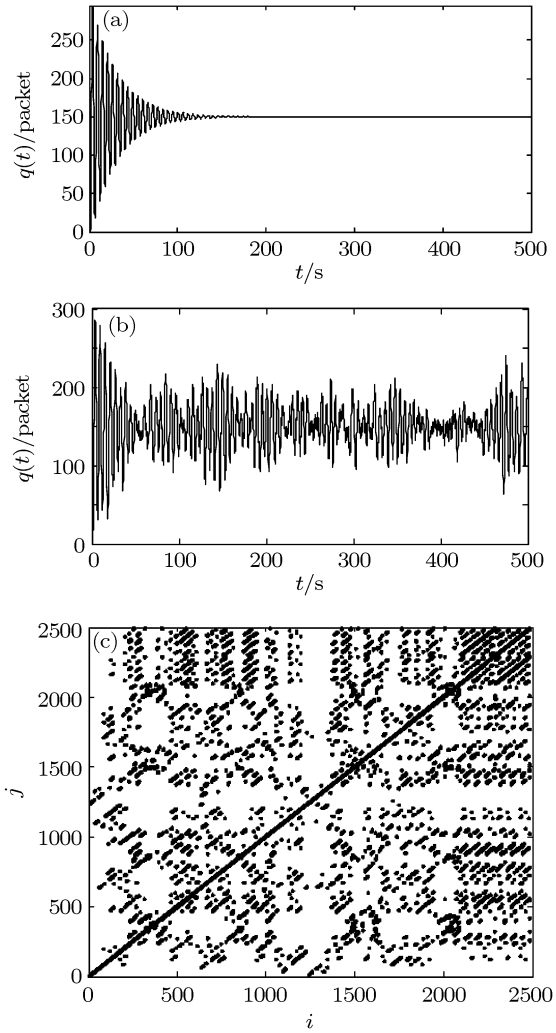


Fig. 7. Queue length $q(t)$ without (a) and with (b) uncontrolled stochastic traffic. (c) Recurrent plot of the queue length with the uncontrolled stochastic traffic. The $R = 1.2$.

that the queue length converges to the desired one finally, although some oscillatory behaviour is present in the transient process. When there is uncontrolled stochastic traffic input to the router, the queue length is shown in Fig. 7(b), which illustrates that there is violently oscillatory behaviour in the queue length from the beginning to the end. The recurrent plot of the queue length is shown in Fig. 7(c), in which a large number of homogeneous discrete points indicate the existence of a random property and an obvious line

style structure along the principal diagonal indicates the existence of deterministic behaviour. The above features indicate that the queue length is of a typical chaos pattern.[23]

5. Conclusion

In this paper, we have studied the bifurcation and the chaos of a kind of RED-AQM rate-based congestion control system. By choosing the communication delay as the bifurcation parameter, we have shown that there is a critical value of the time delay for the stability of the RED-AQM congestion control system. When the time delay exceeds the critical value, the system will lose its local stability and the Hopf bifurcation will occur at the equilibrium point. Furthermore, our study has demonstrated that typical chaos patterns of the queue length may be induced by uncontrolled stochastic traffic if the stability degree of the RED-AQM congestion control system is low, which reveals a new route to chaos besides bifurcation in the congestion control system of the Internet. Our analysis results provide an insight into and an understanding of the chaos mechanism of Internet traffic, and imply that it is necessary to develop a congestion controller with a high enough degree of stability so that uncontrolled stochastic traffic-induced chaos can be prevented in the network, which helps to achieve a good quality of service and improves the capability of the network.

References

[1] Misra V, Gong W B and Towsley D 2000 *Proc. ACM/SIGCOMM* **30**

[2] Hollot C V, Misra V, Towsley D and Gong W B 2002 *IEEE Trans. Automatic Control* **47** 945

[3] Kelly F P, Maulloo A and Tan D K H 1998 *J. Oper. Res. Soc.* **49** 237

[4] Kelly F P 2000 *Philos Trans. Roy. Soc.* A **358** 2335

[5] Li C G, Chen G R, Liao X F and Yu J 2004 *Chaos Soliton. Fract.* **19** 853

[6] Wang Z and Chu T 2006 *Chaos Soliton. Fract.* **28** 161

[7] Raina G 2005 *IEEE Trans. Automatic Control* **50** 1135

[8] Zheng Y G and Wang Z H 2010 *Nonlinear Anal. Real.* **11** 1552

[9] Ding D, Zhu J, Luo X and Liu Y 2009 *Nonlinear Anal. Real.* **10** 2873

[10] Liu Y L, Zhu J and Luo X 2009 *Chin. Phys.* B **18** 3772

[11] Ding D, Zhu J and Luo X 2009 *Nonlinear Anal. Real.* **10** 824

[12] Raina G and Heckmann O 2007 *Performance Evaluation* **64** 266

[13] Hu H and Huang L 2010 *Nonlinear Dyn.* **59** 463

[14] Guo S, Liao X, Liu Q and Wu H 2009 *Nonlinear Anal. Real.* **10** 2225

[15] Guo S, Liao X and Li C 2008 *Nonlinear Anal. Real.* **9** 1292

[16] Guo S, Liao X and Li C 2008 *Nonlinear Anal. Real.* **9** 1768

[17] Yang H and Tian Y 2005 *Chaos Soliton. Fract.* **25** 1093

[18] Yang H and Oliver W W Y 2006 *IEE Proc. Commun.* **153** 5

[19] Yang H and Oliver W W Y 2007 *IEEE T. Parall. Distr.* **18** 550

[20] Cooke K and Grossman Z 1982 *J. Math. Anal. Appl.* **86** 592

[21] Hale J 1977 *Theory of Functional Differential Equations* (Berlin: Spring-Verlag)

[22] Wang J S 2010 *Postdoctor Research Report* (Tsinghua University)

[23] Eckmann J P, Kamphorst S O and Ruelle D 1987 *Europhys. Lett.* **15** 973

[24] Zbilut J P and Webber J 1992 *Phys. Lett.* A **171** 199

[25] Kim H S, Eykholt R and Salas J D 1999 *Physica* D **127** 48