Yang–Yang thermodynamics of one-dimensional Bose gases with anisotropic transversal confinement*

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By combining the thermodynamic Bethe ansatz and local density approximation, we investigate the Yang–Yang thermodynamics of interacting one-dimensional Bose gases with anisotropic transversal confinement. It is shown that with the increase of anisotropic parameter at low temperature, the Bose atoms are distributed over a wider region, while at high temperature the density distribution is not affected obviously. Both the temperature and transversal confinement can strengthen the local pressure of the Bose gases.

Keywords: Bose gas, thermodynamics, anisotropic transversal confinement

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1. Introduction

Since the experimental realization of one-dimensional (1D) quantum gas,[1−5] it has become a popular platform to study the many-body physics of strong correlated low-dimensional quantum systems. Low-dimensional quantum gas can be realized by confining the atom to the transversal ground state with a strong anisotropic magnetic trap or a two-dimensional optical lattice, which can be described with an effective 1D "toy" model theoretically.[6−8] The effective 1D interaction can be tuned in the full interacting regime with Feshbach resonance or confinement induced resonance by changing the magnetic field. As the Bose gases evolve from a weakly interacting regime to a strongly interacting Tonks–Girardeau (TG) regime, the ground state density profiles evolve from a Gauss-like density profile to a Fermi-like shell structure.[9−11] Besides the strongly interacting TG gases,[12−14] the bound states of atoms with repulsion[15] and super TG gas[16−21] have also been realized by a sudden quench of interaction, both of which have no analog in traditional condensed matter physics for strong dissipation.

It is the 1D ultracold atomic gases that enable us to investigate exotic quantum properties that are impossible in solid states. Theoretically, the strongly correlated 1D many-body system cannot be described by mean-field theory[22,23] and we have to adopt a non-perturbation method such as the Bethe ansatz,[8−11,24−26] the Bosonization method,[27] exact diagonalization,[28−30] and the Bose–Fermi mapping method (BFM).[31] In particular, the thermodynamic Bethe ansatz (TBA)[32−34] gives exact results for the thermodynamic properties of a 1D homogeneous many-body system at finite temperature. The exact theoretical result for the density profiles of interacting Bose gases in 1D has been compared with the experimental measurement and they match well quantitatively.[35] More recently, Haller *et al.* realized 1D quantum gas with tight anisotropic transversal confinement,[5] which was isotropic before. In this paper, we investigate the density profiles of 1D Bose gas trapped in a harmonic trap at finite temperature for different anisotropic transversal confinement by combining the TBA and the local density approximation (LDA).

2. Model and method

We consider Bose gas consisting of N atoms of mass m confined in an anisotropic harmonic trap $V_{\rm ext}(\boldsymbol{r}) = m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)/2$. We also introduce $\eta = \omega_y/\omega_z$ to denote the transversal anisotropic parameter. When the transversal confinement (y and z) is strong enough, the system can be described by

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the Hamiltonian in second quantized form as

$$\mathcal{H}_{1\mathrm{D}} = \int \mathrm{d}x \Bigg\{ \frac{\hbar^2}{2m} \frac{\partial \hat{\psi}^\dagger(x)}{\partial x} \frac{\partial \hat{\psi}(x)}{\partial x} + V_{\mathrm{ext}}(x)\, \hat{\psi}^\dagger(x)\hat{\psi}(x) + \frac{g_{1\mathrm{D}}}{2} \hat{\psi}^\dagger(x)\hat{\psi}^\dagger(x)\hat{\psi}(x)\hat{\psi}(x) \Bigg\}, \quad (1)$$

where the interacting constant $g_{1\mathrm{D}}$ depending on the transversal confinement and the atomic s-wave scattering length $a_{3\mathrm{D}}$ is[36]

$$g_{1\mathrm{D}} = \frac{2\hbar^2 a_{3\mathrm{D}}}{m d_\perp^2} \frac{\sqrt{\eta}}{1 - \mathcal{C}\,(a_{3\mathrm{D}}/d_\perp)}, \quad (2)$$

which can be tuned by magnetically Feshbach resonances from the strong attractive interaction limit to the strong repulsive interaction limit. The field operator $\Psi^\dagger(x)$ ($\Psi(x)$) creates (annihilates) an atom at position x. Here, the characteristic oscillator length in y direction is $d_\perp = \sqrt{2\hbar/m\omega_y}$ and the constant $\mathcal{C}$ is determined by the integral

$$\mathcal{C} = -\int_0^\infty \frac{\mathrm{d}t}{\sqrt{\pi t}} \left[\frac{\sqrt{\eta}}{\sqrt{(1 - \mathrm{e}^{-\eta t})(1 - \mathrm{e}^{-t})}} - \sqrt{\eta} - \frac{1}{t} \right].$$

For isotropic transversal confinement ($\eta = 1$), we have $\mathcal{C} \approx 1.4603$. It should be noticed that $\mathcal{C}$ is not monotonous with the increase of anisotropic parameter η.

The corresponding first quantized Hamiltonian is

$$H = \sum_{j=1}^{N} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + V_{\mathrm{ext}}(x_j) \right] + g_{1\mathrm{D}} \sum_{j<l} \delta(x_j - x_l). \quad (3)$$

The model in the homogeneous trap is integrable and its exact solution can be obtained by the Bethe ansatz method.[37]

Subsequently, we review the Yang–Yang thermodynamics[32] for 1D Bose gas at finite temperature. The Bethe ansatz equations for the model with a periodical boundary condition in the absence of an external trap ($V_{\mathrm{ext}}(x_j) = 0$) take the formulation of

$$k_j L = 2\pi I_j - \sum_{l=1}^{N} \tan^{-1} \frac{k_j - k_l}{c}, \quad (4)$$

with $c = m g_{1\mathrm{D}}/\hbar^2$, where $\{k_j\}$ ($j = 1, 2, \ldots, N$) are quasi-momenta and $\{I_j\}$ are quantum numbers determining an eigenstate. In the thermodynamic limit ($N, L \to \infty$ with N/L finite), the Bethe ansatz Eq. (4) reduces to the form of integral equation

$$\rho(k) + \rho^h(k) = \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2c\rho(k')}{c^2 + (k - k')^2} \mathrm{d}k', \quad (5)$$

with the density of k roots $\rho(k)$ and the corresponding hole distribution $\rho^h(k)$. The energy density of the system can be expressed as

$$E/L = \int_{-\infty}^{\infty} \rho\,(k)\, k^2 \mathrm{d}k, \quad (6)$$

with particle density $N/L = \int_{-\infty}^{\infty} \rho\,(k)\,\mathrm{d}k$. The entropy of 1D Bose gas at finite temperature can be written as

$$S/L = \int_{-\infty}^{\infty} \mathrm{d}k \left[(\rho + \rho^h) \ln (\rho + \rho^h) - \rho \ln \rho - \rho^h \ln \rho^h \right]. \quad (7)$$

While the free energy is defined by

$$F = E - TS - \mu N,$$

where E and S are given by Eqs. (6) and (7), respectively, and the Lagrange multiplier μ is introduced to conserve the number of total atoms. At temperature T the thermal equilibrium can be arrived at by minimizing the free energy with the condition $\delta F = 0$ subjected to constraint Eq. (5), which leads to the following nonlinear integral equations, called TBA equations:

$$\epsilon\,(k) = -\mu + k^2 - \frac{Tc}{\pi} \int_{-\infty}^{\infty} \mathrm{d}k' \frac{\ln \{1 + \exp[-\epsilon\,(k')/T]\}}{c^2 + (k - k')^2}, \quad (8)$$

where $\rho^h(k) = \rho(k) \exp[\epsilon\,(k)/T]$ has been defined. Thus, Eq. (5) becomes

$$2\pi\rho(k) \{1 + \exp[\epsilon\,(k)/T]\} = 1 + 2c \int_{-\infty}^{\infty} \frac{\rho(k')\mathrm{d}k'}{c^2 + (k - k')^2}.$$

When $\epsilon\,(k)$ is determined, the thermodynamic quanta such as pressure can be obtained by

$$P = \frac{T}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k' \ln [1 + \exp(-\epsilon\,(k')/T)], \quad (9)$$

with which the free energy can be expressed as $F = -PL + \mu N$.

For the Bose gas in a harmonic trap with $V_{\mathrm{ext}} = m\omega_x^2 x^2/2$, we can determine the density distribution of the 1D Bose gas with anisotropic transversal confinement by using the LDA.[8,38] According to the LDA, it is assumed that the system is in local thermodynamical equilibrium at each position x in the external trap. The density distribution of the quantum

gas can be obtained from the local equation of state $\mu(x) = \mu - m\omega_x^2 x^2/2$ under the Thomas–Fermi approximation. At specific temperature we firstly obtain $\epsilon(k)$ by solving Eq. (8) with an iteration method and then the quasimomentum distribution can be obtained with the same method. Combining it with the LDA, we can determine the local density at point x by the corresponding local chemical potential formulated as

$$n(x) = \int_{-\infty}^{\infty} \rho(\mu(x), x, k)\, \mathrm{d}k. \tag{10}$$

When the temperature is low enough that $k_{\mathrm{B}}T \ll \hbar\omega_{y,z}$, which is so small that the atoms cannot populate at the transversal excited states by thermal excitation, it is enough to consider only the transversal ground state such as the density distributions of Bose atoms at finite temperature determined completely by Eq. (10).

When the temperature is not very low, the transversal excited states might be occupied with an unnegligible fraction but the interaction only affects the density distribution of transversal ground state, and the atom population in transversal excited states can be described as ideal Bose gas, i.e.,

$$n_{\mathrm{ex}}^{n_{r_y,r_z}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\exp\left[\left(\dfrac{\hbar^2k^2}{2m} - \mu_{r_y,r_z}(x)\right)/k_{\mathrm{B}}T\right] - 1},$$

where the chemical potential of each excited state μ_{r_y,r_z} in thermal equilibrium is given by

$$\mu_{r_y,r_z}(x) = \mu(x) - (r_y\hbar\omega_y + r_z\hbar\omega_z).$$

The total density distribution of the 1D Bose gas with anisotropic transversal confinement can be given by summing over the transversal ground state and all transversal excited states as

$$n_{\mathrm{tot}}(x) = n(x) + \sum_{r_y+r_z=1}^{\infty} n_{\mathrm{ex}}^{r_y,r_z}(x).$$

3. Results and discussion

In the present paper, we investigate the quantum gas of ^{133}Cs confined in a harmonic trap with longitudinal trap frequency $\omega_x = 2\pi \times 16$ Hz and transversal trap frequency $\omega_z = 2\pi \times 13.2$ kHz in the z direction corresponding to the parameters in the experiment reported by Ref. [5]. The atomic s-wave scattering length in three dimensions is $a_{3\mathrm{D}} = 270a_0$ with a_0 being the Bohr radius. By changing the trap frequency ω_y in the y direction, we can tune the transversal anisotropic parameter η and thus the effective 1D interacting constant $U = g_{1\mathrm{D}}/l\hbar\omega_x$, where the characteristic oscillator length in x direction $l = \sqrt{m/\hbar\omega_x}$ equals the unit length. In Fig. 1, we plot the effective interaction regime to be investigated, where the interaction between atoms is repulsive and U becomes strong rapidly with the increase of anisotropic parameter η. In the following evaluation the unit of energy is taken as $\hbar\omega_x$, which is equal to the temperature 0.77 nK.

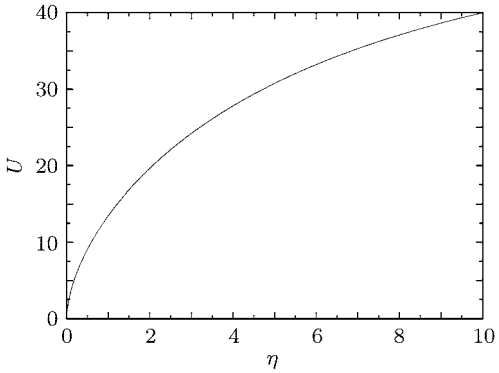


Fig. 1. Effective 1D interacting constant U versus anisotropic parameter η.

In Fig. 2, we display the total density distributions for the quantum gas of 200 Cs atoms for $\eta = 1.00$ ($U = 13.4536$) and 5 ($U = 30.743$) at different temperatures. In both situations the Bose gases are in a strongly interacting regime. As a comparison we plot the density profiles of Tonks gases at zero temperature. It is shown that at low temperature the density profiles behave as a Gaussian-like Bose distribution. As the temperature increases, the Bose atoms will be distributed over a wider region because more and more atoms are stimulated to the high excited states by the high temperature, including both the longitudinal excited states and the transversal excited states. Those atoms with high energy will be distributed with more probability in the region away from the centre of the harmonic trap. In addition, the increased anisotropic parameter will broaden the density profiles at low temperature, while those at high temperature are not affected obviously. That is to say, when the temperature is high enough the atom population at different energy levels will not be affected obviously by the strong interaction.

We exhibit both the density profiles of the transversal ground state and the transversal excited states with anisotropic parameter $\eta = 1$ and $\eta = 5$ in Fig. 3. As we expected, when the temperature is

low enough all Bose atoms populate at the transversal ground state even in the strongly interacting regime. In this case, the result from the TBA can describe the density plot of the 1D Bose gas. While because the temperature is in the same order of magnitude as the transversal confinement, we should consider the atoms excited to the high transversal energy states. With the increase of temperature more and more atoms will be stimulated to the transversal excited states and the atoms still populated at the transversal ground states are also distributed in the region away from the centre of the harmonic trap because the atoms are heated up. With the increase of anisotropic parameter η, stronger transversal confinement induces atoms in a smaller region in the transversal direction and, meanwhile, it makes the 1D interaction strength larger, which stimulates more atoms to populate at high longitudinal excited states and thus affects the density distribution in the longitudinal direction.

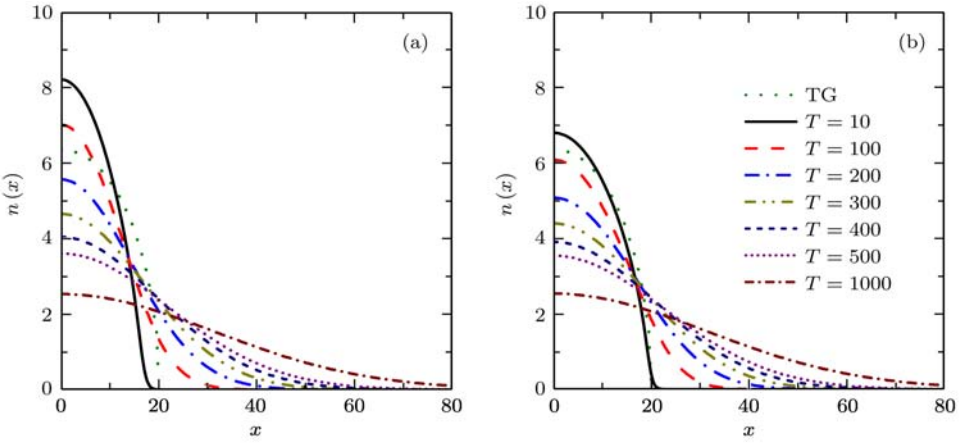


Fig. 2. (colour online) Density distribution for the 1D Bose gas with $N = 100$: (a) $\eta = 1$, (b) $\eta = 5$.

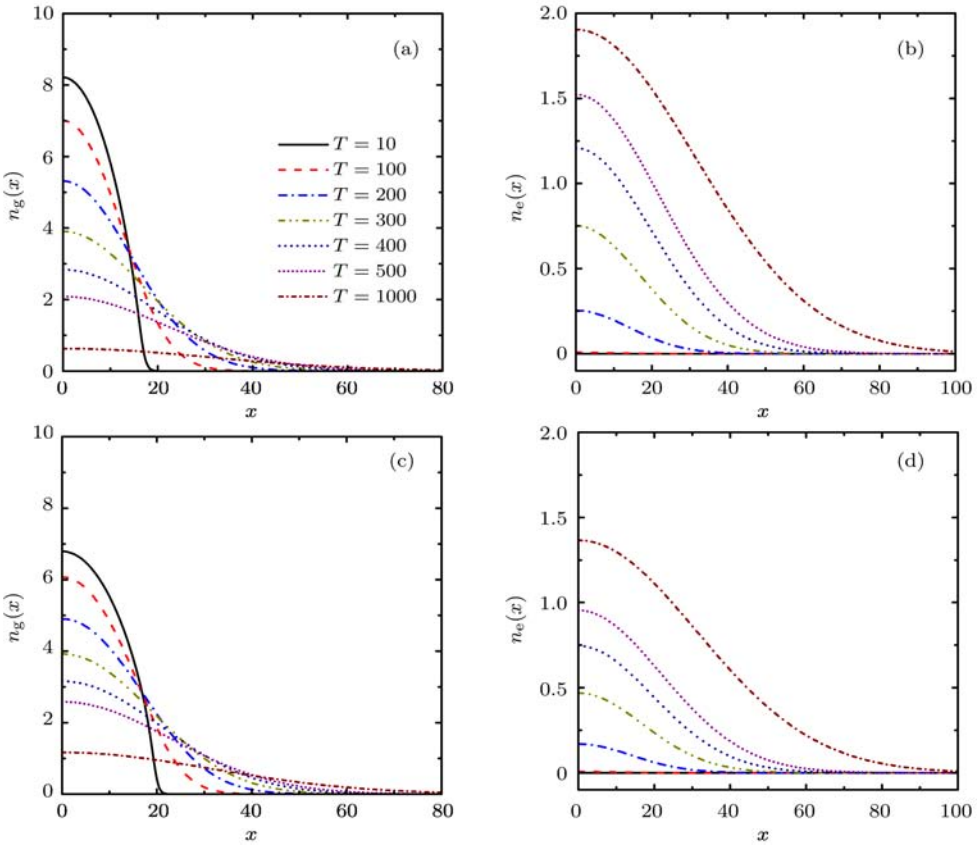


Fig. 3. (colour online) Density distribution of the 1D Bose gas with $N = 200$ for anisotropic parameter $\eta = 1$ ((a) and (b)) and $\eta = 5$ ((c) and (d)). Left column: transversal ground state; right column: transversal excited states.

In Fig. 4, we display the atom number in the transversal ground state and transversal excited states with the change of temperature. It is shown that when the transversal confinement is strong enough and the temperature is not so high ($T < 200\hbar\omega_x$), considering only the transversal ground state is enough and the TBA can describe the density distributions of 1D Bose gases very well. Similar to the evaluation of density profiles, the local pressure can be obtained by combining Eq. (9) with the LDA. In Fig. 5, we display the local pressure of the 1D Bose gases trapped in the harmonic trap. The atoms located at the centre of the trap exhibit a stronger pressure and both the increased anisotropic parameter and temperature strengthen the pressure of the Bose gases obviously. Although with the increase of temperature the atoms at the transversal ground state become small, the system exhibits stronger pressure. As the transversal confinement becomes strong, more atoms will stay at the transversal ground state and exhibit stronger pressure.

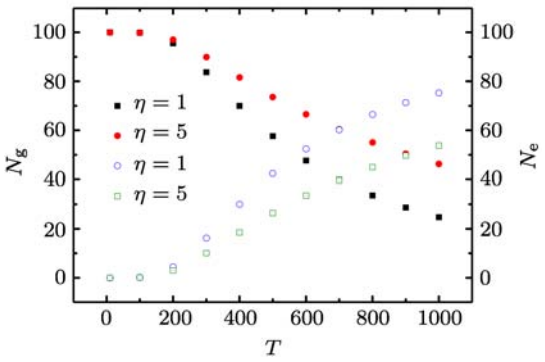


Fig. 4. (colour online) The atom number versus temperature for the 1D Bose gas with $N = 200$ for $\eta = 1$ (a) and $\eta = 5$ (b).

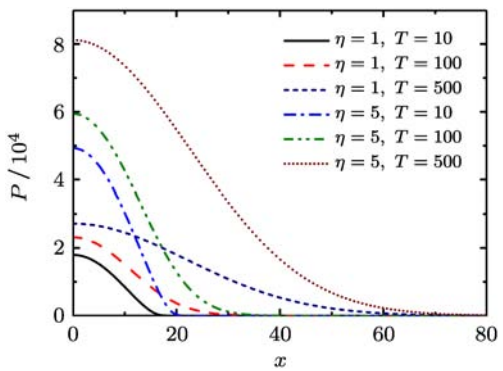


Fig. 5. (colour online) The local pressure for the 1D Bose gas with $N = 200$ for $\eta = 1$ and $\eta = 5$.

4. Conclusion

We have investigated 1D Bose gases in a harmonic trap with anisotropic transversal confinement at finite temperature. The density profiles and pressure are evaluated by combining the TBA and the LDA. In the case of low temperature, as the transversal anisotropic parameter increases, the Bose atoms are redistributed in the trap. More atoms populate in the region away from the centre of the harmonic trap, where higher potential energy can be obtained. The density profiles become broader and decrease to zero rapidly at the boundary, which is similar to polarized Fermi distributions. The system can be described well by the TBA. In the case of higher temperature, more atoms populate in the longitudinal excited states and we have to consider the atoms stimulated to the transversal excited states. The density distribution will not be affected dramatically by the transversal anisotropic parameter. The investigation on the pressure of transversal ground state shows that although with the increase of temperature the number of atoms populated at the transversal ground state becomes small, its pressure becomes strong. The increased transversal confinement also makes the pressure strong.

References

[1] Stöferle T, Moritz H, Schori C, Köhl M and Esslinger T 2004 *Phys. Rev. Lett.* **92** 130403
[2] Paredes B, Widera A, Murg V, Mandel Q, Fölling S, Cirac I, Shlyapnikov G V, Hänsch T W and Bloch I 2004 *Nature* **429** 277
[3] Kinoshita T, Wenger T and Weiss D S 2004 *Science* **305** 1125
[4] Cheng F, Ke M, Li X L, Tang J Y and Wang Y Z 2009 *Chin. Phys.* B **18** 4259
[5] Haller E, Mark M J, Hart R, Danzl J G, Röllner L, Melezhik V, Schmelcher P and Nägerl H 2010 *Phys. Rev. Lett.* **104** 153203
[6] Olshanii M 1998 *Phys. Rev. Lett.* **81** 938
[7] Petrov D S, Shlyapnikov G V and Walraven J T M 2000 *Phys. Rev. Lett.* **85** 3745
[8] Dunjko V, Lorent V and Olshanii M 2001 *Phys. Rev. Lett.* **86** 5413
[9] Hao Y, Zhang Y, Liang J Q and Chen S 2006 *Phys. Rev.* A **73** 063617
[10] Hao Y, Zhang Y and Chen S 2007 *Phys. Rev.* A **76** 063601
[11] Han J R, Liu X F, Wang Y J, Wang Y Z, Xu G Y and Yan L 2008 *Chin. Phys.* B **17** 4158
[12] Girardeau M D 1960 *J. Math. Phys.* **1** 516
[13] Girardeau M D 1965 *Phys. Rev.* **139** B500

[14] Tonks L 1936 *Phys. Rev.* **50** 955
[15] Winkler K, Thalhammer G, Lang F, Grimm R, Denschlag J H, Daley A J, Kantian A, Büchler H P and Zoller P 2006 *Nature* **441** 853
[16] Astrakharchik G E, Boronat J, Casulleras J and Giorgini S 2005 *Phys. Rev. Lett.* **95** 190407
[17] Batchelor M T, Bortz M, Guan X W and Oelkers N 2005 *J. Stat. Mech.*: *Theory Exp.* L10001
[18] Chen S, Guan L, Yin X, Hao Y and Guan X W 2010 *Phys. Rev.* A **81** 031609(R)
[19] Chen S, Guan X W, Yin X, Guan L and Batchelor M T 2010 *Phys. Rev.* A **81** 031608(R)
[20] Guan L and Chen S 2010 *Phys. Rev. Lett.* **105** 175301
[21] Wang L, Hao Y and Chen S 2010 *Phys. Rev.* A **81** 063637
[22] Zhou Q, Lu J F and Yin J P 2010 *Chin. Phys.* B **19** 093202
[23] Wang Z X, Ni Z G, Cong F Z, Liu X S and Chen L 2010 *Chin. Phys.* B **19** 113205
[24] Fuchs J N, Gangardt D M, Keilmann T and Shlyapnikov G V 2005 *Phys. Rev. Lett.* **95** 150402
[25] Guan L, Chen S, Wang Y and Ma Z Q 2009 *Phys. Rev. Lett.* **102** 160402
[26] Batchelor M T, Bortz M, Guan X W and Oelkers N 2006 *J. Stat. Mech.* P03016
[27] Cazalilla M A and Ho A F 2003 *Phys. Rev. Lett.* **91** 150403
[28] Deuretzbacher F, Bongs K, Sengstock K and Pfannkuche D 2007 *Phys. Rev.* A **75** 013614
[29] Hao Y and Chen S 2009 *Eur. Phys. J.* D **51** 261
[30] Hao Y, Zhang Y, Guan X W and Chen S 2009 *Phys. Rev.* A **79** 033607
[31] Girardeau M D and Minguzzi A 2007 *Phys. Rev. Lett.* **99** 230402
[32] Yang C N and Yang C P 1969 *J. Math. Phys.* **10** 1115
[33] Yin X, Chen S and Zhang Y 2008 *Phys. Rev.* A **79** 053604
[34] Zhao E, Guan X W, Liu W V, Batchelor M T and Oshikawa M 2009 *Phys. Rev. Lett.* **103** 140404
[35] Amerongen A H V, Es J J P V, Wicke P, Kheruntsyan K V and Druten N J V 2008 *Phys. Rev. Lett.* **100** 090402
[36] Peng S G, Bohloul S S, Liu X J, Hu H and Drummond P D 2010 *Phys. Rev.* A **82** 063633
[37] Lieb E H and Liniger W 1963 *Phys. Rev.* **130** 1605
[38] Menotti C and Stringari S 2002 *Phys. Rev.* A **66** 043610