Variational principles for two kinds of extended Korteweg–de Vries equations*

Cao Xiao-Qun(曹小群)†, Song Jun-Qiang(宋君强), Zhang Wei-Min(张卫民), and Zhao Jun(赵 军)

School of Computer Science, National University of Defense Technology, Changsha 410073, *China*

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Variational principles are constructed using the semi-inverse method for two kinds of extended Korteweg–de Vries (KdV) equations, which can be regarded as simple models of the nonlinear oceanic internal waves and atmospheric long waves, respectively. The obtained variational principles have also been proved to be correct.

Keywords: He's semi-inverse method, variational principles, oceanic internal wave, atmospheric long wave

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1. Introduction

The nonlinear Korteweg–de Vries (KdV) equation and its variants are very important nonlinear evolution equations (NLEEs) in the areas of theoretical physics, oceanography and meteorology.[1−10] The soliton phenomena were first observed by Russell[11] and the KdV equation was solved using the inverse scattering method by Gardner *et al.*[12] Finding exact solutions of NLEEs has become one of the most exciting and extremely active areas of research investigation. Many effective methods for obtaining exact solutions of various KdV equations and other NLEEs have been presented, such as the homotopy perturbation, the variational method, the asymptotic methods, the algebraic method, the homogeneous balance method, the tanh-function method, the Jacobi elliptic function method, the mapping method, the F-expansion method and the Exp-function method.[13−26]

Over the past decades, the combination of in situ and remote sensing observations has demonstrated that long nonlinear solitary-like waves are ubiquitous features of the atmosphere and the coastal oceans, examples include the oceanic internal wave[8−10] and the atmospheric Rossby solitary wave.[1−7] The nonlinear KdV-type theories and the nonlinear KdV-type equations have played a primary role in elucidating the essential features of the observations.[6−10] In recent years, variational principles in physics have resulted in a great amount of research.[27−35] Although variational principles for various differential equations have been studied for centuries,[27−35] the variational principles for the oceanic internal wave and the atmospheric long wave have been dealt with by only a few scientists. For example, Craig and Guyenne[36] derived a Hamiltonian formulation for one or several ideal fluids with a dynamic free surface, free interfaces, or both.[36] They also derived long wave models for internal waves and computed solitary wave solutions of those equations.[37] In the present paper, variational principles for two variant forms of the KdV equation, which can describe some main characteristics of the evolution of nonlinear internal waves and the motion of atmospheric long waves, are studied using the semi-inverse method.

2. Variational principle for oceanic internal waves

Huge internal waves in the ocean are often observed in various parts of the coastal oceans and their amplitudes can reach 50–100 meters. Usually, these large-amplitude solitary-like internal waves are modeled approximately with nonlinear wave equations of the KdV type. In this paper, the equation for the long nonlinear internal waves is

$$\frac{\partial \eta}{\partial t} + (C_0 + \alpha\eta + \alpha_1\eta^2)\frac{\partial \eta}{\partial x} + \beta\frac{\partial^3 \eta}{\partial x^3} = 0, \qquad (1)$$

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†Corresponding author. E-mail: caoxiaoqun@nudt.edu.cn

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where wave amplitude $\eta(x,t)$ is related to the isopycnal vertical displacement, t is time, x is the spatial variable in the direction of wave propagation. C_0 is the phase speed, α and α_1 are coefficients for the nonlinear and the cubic nonlinear terms, respectively, and β is the dispersive coefficient. Coefficients α, α_1 and β are determined by the background density stratification and the shear through the background horizontal current. Equation (1) is called the extended KdV (eKdV) equation, which includes the cubic nonlinearity and is a useful tool to study the long nonlinear internal waves in the ocean.[8−10] He[30] has obtained the variational principles for the general form KdV equation similar to Eq. (1) with variable coefficients.

We can rewrite Eq. (1) in a conservation form,

$$\eta_t + (C_0\eta + \alpha\eta^2/2 + \alpha_1\eta^3/3 + \beta\eta_{xx})_x = 0. \quad (2)$$

In this paper, we will apply the semi-inverse method[27−35] to construct the variational principle for oceanic internal waves. According to Eq. (2), we can introduce a potential function Φ defined as

$$\Phi_x = -\eta, \quad (3)$$

$$\Phi_t = C_0\eta + \alpha\eta^2/2 + \alpha_1\eta^3/3 + \beta\eta_{xx}, \quad (4)$$

so that Eq. (2) or (1) is automatically satisfied.

In order to establish a variational formulation, we use the semi-inverse method and construct a trial function in the form

$$J(\eta,\Phi) = \iint L_1 \,\mathrm{d}x\,\mathrm{d}t, \quad (5)$$

where

$$L_1 = \eta\Phi_t + (C_0\eta + \alpha\eta^2/2 + \alpha_1\eta^3/3 + \beta\eta_{xx})\Phi_x + F_1(\eta), \quad (6)$$

with F_1 being an unknown function of η and/or its derivatives. There exist alternative approaches to the construction of the trial function, see Refs. [27]–[35]. The advantage of the above trial Lagrange lies on the fact that the stationary condition with respect to Φ results in the Euler–Lagrange equation

$$\frac{\partial L_1}{\partial \Phi} - \frac{\partial}{\partial x}\left(\frac{\partial L_1}{\partial \Phi_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L_1}{\partial \Phi_t}\right) = 0, \quad (7)$$

or

$$-\frac{\partial}{\partial x}(C_0\eta + \alpha\eta^2/2 + \alpha_1\eta^3/3 + \beta\eta_{xx}) - \frac{\partial \eta}{\partial t} = 0,$$

which is equivalent to Eq. (2).

Now calculating the stationary condition of the above trial Lagrange L_1 with respect to η, we obtain the following Euler equation

$$\frac{\partial L_1}{\partial \eta} + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L_1}{\partial \eta_{xx}}\right) + \frac{\delta F_1}{\delta \eta} = 0, \quad (8)$$

where $\delta F_1/\delta\eta$ is called the He's variational derivative[32] with respect to η and is defined as

$$\frac{\delta F_1}{\delta \eta} = \frac{\partial F_1}{\partial \eta} - \frac{\partial}{\partial x}\frac{\partial F_1}{\partial \eta_x} - \frac{\partial}{\partial t}\frac{\partial F_1}{\partial \eta_t} + \frac{\partial^2}{\partial x^2}\frac{\partial F_1}{\partial \eta_{xx}} + \cdots.$$

In view of Eq. (6), Eq. (8) can be re-written as

$$\Phi_t + (C_0 + \alpha\eta + \alpha_1\eta^2)\Phi_x + \beta\Phi_{xxx} + \frac{\delta F_1}{\delta \eta} = 0. \quad (9)$$

The above equation should be consistent with field Eqs. (3) and (4). Substituting Eqs. (3) and (4) into Eq. (9) results in

$$\frac{\delta F_1}{\delta \eta} = \alpha\eta^2/2 + 2\alpha_1\eta^3/3. \quad (10)$$

From the above equation, F_1 can be identified as

$$F_1 = (\alpha\eta^3 + \alpha_1\eta^4)/6. \quad (11)$$

We therefore obtain the variational formulation needed for the extended KdV equation describing the oceanic internal wave, which reads

$$J(\eta,\Phi) = \iint [\eta\Phi_t + (C_0\eta + \alpha\eta^2/2 + \alpha_1\eta^3/3 + \beta\eta_{xx})\Phi_x + (\alpha\eta^3 + \alpha_1\eta^4)/6]\,\mathrm{d}x\,\mathrm{d}t, \quad (12)$$

and is subject to the constraint of field Eq. (3).

Proof Making Eq. (12) stationary with respect to Φ and η, respectively, we obtain the Euler–Lagrange equations

$$\delta\Phi: \ -\eta_t - (C_0\eta + \alpha\eta^2/2 + \alpha_1\eta^3/3 + \beta\eta_{xx})_x = 0, \quad (13)$$

$$\delta\eta: \ \Phi_t + (C_0 + \alpha\eta + \alpha_1\eta^2)\Phi_x + \beta\Phi_{xxx} + \alpha\eta^2/2 + 2\alpha_1\eta^3/3 = 0, \quad (14)$$

where $\delta\Phi$ and $\delta\eta$ are the first order variations of Φ and η, respectively. Obviously, Eq. (13) is equivalent to Eq. (2) or (1). Using constraint $\Phi_x = -\eta$, Eq. (14) turns out to Eq. (4).

3. Variational principle for atmospheric long waves

When the long-wave approximation is made, the barotropic nonlinear vorticity equation is equivalent to the generalized KdV equation,[1−3] which controls the motion of the long waves in the atmosphere, such

as the Rossby solitary wave.[4−7] The generalized KdV equation to be studied in this part is

$$\alpha\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\gamma\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^3}-\varepsilon\frac{\partial^5 u}{\partial x^5}=0, \quad (15)$$

where $\alpha = L^2\Phi/(\beta D^5)$, $\gamma = (U_0-\beta D^2)L^2/(\beta D^4)$, $\varepsilon = D^2/L^2$, u is the disturbance stream function, β is the Rossby parameter, U_0 is the typical velocity of the basic flow, Φ is the scale of the stream function, D and L are the characteristic vertical and horizontal length scales for the flow variation, respectively. Because all large-scale motions of geophysical fluid is thin, we have a small aspect ratio, $D/L \ll 1$.

We can rewrite the generalized KdV Eq. (15)[1−3] in a conservation form

$$(\alpha u)_t+(\alpha u^2/2+\gamma u-u_{xx}-\varepsilon u_{xxxx})_x=0. \quad (16)$$

In this part, we will apply the He's semi-inverse method[27−35] to construct the variational principle for the atmospheric long wave. According to Eq. (16), we can introduce a special function Π defined as

$$\Pi_x=-\alpha u, \quad (17)$$

$$\Pi_t=\alpha u^2/2+\gamma u-u_{xx}-\varepsilon u_{xxxx}, \quad (18)$$

so that Eq. (15) or (16) is automatically satisfied.

In order to establish the variational formulation, we use the semi-inverse method[27−35] and construct a trial function in the form

$$J(u,\Pi)=\iint L_2\,\mathrm{d}x\,\mathrm{d}t, \quad (19)$$

where

$$L_2 = \alpha u\Pi_t+(\alpha u^2/2+\gamma u-u_{xx}-\varepsilon u_{xxxx})\Pi_x + F_2(u), \quad (20)$$

with F_2 being an unknown function of u and/or its derivatives. There exist alternative approaches to the construction of the trial function, see Refs. [27]–[35]. The merit of the trial Lagrangian is that the stationary condition with respect to Π results in the following Euler–Lagrange equation

$$\frac{\partial L_2}{\partial\Pi}-\frac{\partial}{\partial x}\left(\frac{\partial L_2}{\partial\Pi_x}\right)-\frac{\partial}{\partial t}\left(\frac{\partial L_2}{\partial\Pi_t}\right)=0, \quad (21)$$

or

$$-\frac{\partial}{\partial x}(\alpha u^2/2+\gamma u-u_{xx}-\varepsilon u_{xxxx})-\frac{\partial(\alpha u)}{\partial t}=0,$$

which is equivalent to Eq. (16).

Now calculating the stationary condition of the above trial Lagrange L_2 with respect to u, we obtain the following Euler equation

$$\frac{\partial L_2}{\partial u}+\frac{\partial^2}{\partial x^2}\left(\frac{\partial L_2}{\partial u_{xx}}\right)+\frac{\partial^4}{\partial x^4}\left(\frac{\partial L_2}{\partial u_{xxxx}}\right)+\frac{\delta F_2}{\delta u}=0, \quad (22)$$

where $\delta F_2/\delta u$ is the He's variational derivative[32] with respect to u and is defined as

$$\frac{\delta F_2}{\delta u}=\frac{\partial F_2}{\partial u}-\frac{\partial}{\partial x}\frac{\partial F_2}{\partial u_x}-\frac{\partial}{\partial t}\frac{\partial F_2}{\partial u_t}+\frac{\partial^2}{\partial x^2}\frac{\partial F_2}{\partial u_{xx}}+\cdots.$$

In view of Eq. (20), Eq. (22) can be re-written as

$$\alpha\Pi_t+(\alpha u+\gamma)\Pi_x-\Pi_{xxx}-\varepsilon\Pi_{xxxxx}+\frac{\delta F_2}{\delta u}=0. \quad (23)$$

The above equation should be equivalent to field Eqs. (17) and (18). Substituting Eqs. (17) and (18) into Eq. (23) yields

$$\frac{\delta F_2}{\delta u}=\alpha^2u^2/2. \quad (24)$$

From the above equation, unknown function F_2 can be determined as

$$F_2=\alpha^2u^3/6. \quad (25)$$

Finally, we obtain the variational formulation needed for the generalized KdV equation describing the atmospheric long wave, which reads

$$J(u,\Pi) = \iint[\alpha u\Pi_t+(\alpha u^2/2+\gamma u-u_{xx} -\varepsilon u_{xxxx})\Pi_x+\alpha^2u^3/6]\,\mathrm{d}x\,\mathrm{d}t, \quad (26)$$

and is subject to the constraint of Eq. (17).

Proof Making Eq. (26) stationary with respect to u and Π, respectively, we obtain the following Euler–Lagrange equations

$$\delta\Pi:\ -(\alpha u)_t-(\alpha u^2/2+\gamma u-u_{xx}-\varepsilon u_{xxxx})_x=0, \quad (27)$$

$$\delta u:\ \alpha\Pi_t+(\alpha u+\gamma)\Pi_x-\Pi_{xxx}-\varepsilon\Pi_{xxxxx} +\alpha^2u^2/2=0, \quad (28)$$

where $\delta\Pi$ and δu are the first order variations for Π and u, respectively. It is obvious that Eq. (27) is equivalent to Eq. (16) or (15). Using constraint $\Pi_x=-\alpha u$, Eq. (28) turns out to be Eq. (18).

4. Conclusion

The semi-inverse method was proposed by He to establish generalized variational principles for physical problems, which can eliminate the variational crisis brought by the Lagrange multiplier method. It has been successfully applied to many nonlinear equations arising in mathematical physics and nonlinear

sciences.[27−35] Via the semi-inverse method, the generalized variational principles were constructed for two variants of the KdV equation. The obtained variational principles have also been proved to be correct. According to the obtained variational principles, we can study the laws of motion for related waves in geophysical fluid and the problems discussed can also be solved numerically by using direct variational methods, such as the finite element method and Ritz's method

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