The effect of atomic spontaneous decay on the dynamics of the negativity of the Wigner function of radiation

Abdel-Baset A. Mohamed†

Faculty of Science, Assiut University, Assiut, Egypt

(Received 4 March 2011; revised manuscript received 14 April 2011)

The exact solution of the master equation for the case of a high-Q cavity with atomic decay is found. We use the negativity of the Wigner function (WF) as an indicator of nonclassicality. It is found that the negative values of the field WF are very sensitive to any change in the damping parameter. The atomic spontaneous decay leads to the simultaneous disappearance of both entanglement and nonclassicality of quantum states. Moreover, the purity of the field states is completely lost.

Keywords: non-classicality, entanglement, atomic decay

PACS: 03.65.Ud, 42.50.Lc **DOI:** 10.1088/1674-1056/20/9/090303

1. Introduction

One suitable representation for the quantum state of a system is the Wigner function (WF).[1,2] The WF formulism is fully equivalent to the density matrix representation, thus providing all the accessible information about the system for the observer. Several schemes for direct measuring values of the WF have been proposed[3,4] and performed experimentally.[5,6] The negative-valued part is related to the disappearance of the interference pattern in the mesoscopic quantum superpositions of coherent states due to the decoherence.[7−9] Physically, the Wigner distribution function describes a signal in space and frequency simultaneously, and it can be considered as the local frequency of the signal.[10] The applications of the Wigner distribution function in various areas of signal processing are presented in Ref. [11].

Recent experimental progress in quantum optics creates an increasing wealth of light fields with properties deviating further and further from coherent states (classical state also includes thermal states and even squeezed states).[12−14] In particular, since the invention of the laser, it has become clear that the statistical properties of light may be quite different from those of the classical sources.[15,16] Such nonclassical states can be distinguished from the classical ones by many features. Examples include the sub-Poissonian photon statistics,[17,18] the squeezing,[19,20] the photon number oscillations,[21] and the negative values of the Wigner function.[22,23] Most of these signatures can be quantified to define the degree of nonclassicality. Studying nonclassical effects of the quantum superposition of optical fields has recently attracted renewed attention, because they may play a key role in understanding the fundamentals of quantum physics and have many applications in quantum communication and quantum information technology,[24] such as quantum teleportation,[25] quantum cryptography,[26] and quantum computation.[27] The nonclassicality is a necessary input for the entangled states.

In Ref. [28] a quantitative measure of the nonclassicality based on negative regions of quasi-probability distributions was introduced. In Ref. [29] a measure of the nonclassicality of quantum states based on the volume of the negative part of the Wigner function was proposed. In our work, we use the negativity of the Wigner function as an indicator of the nonclassicality.

The decoherence takes place when well-defined phase relationships disappear due to both the dissipative interaction with the environment and the measurement process. The entanglement is a genuine quantum feature that gives rise to the most striking nonclassical effect.[30,31]

These important topics mentioned above and the experimental results on qubits coupled to the environment are the motivation to study the effect of the dissipation on entanglement, nonclassicality, and purity.

†Corresponding author. E-mail: abdelbastm@yahoo.com

© 2011 Chinese Physical Society and IOP Publishing Ltd

2. Physical model and its analytic solution

The Jaynes–Cummings (JC) model can describe a two-level system interacting with a quantum harmonic oscillator. The atomic states are denoted by $|e\rangle$ (for the exited state) and $|g\rangle$ (for the ground state). The creation and the annihilation operators for the harmonic oscillator are $\hat{a}^\dagger$ and $\hat{a}$, respectively. The cavity mode is assumed to be resonant with the atomic transition frequency, and the strength of the atom–photon coupling is denoted by λ. Under the rotating wave approximation, the Hamiltonian in the interaction picture is given by $\hat{H}_{\rm int} = \lambda(\hat{a}|e\rangle\langle g| + \hat{a}^\dagger|g\rangle\langle e|)$.

One of the pioneering potential applications of the JC model in the context of quantum information is artificial atoms (qubits). For example, the coupled system of the Cooper pair box (artificial atom) and photons stored in a resonator (cavity mode) is described by the JC-model Hamiltonian.[32] Because of the size of the artificial atom, the quantum behaviour in a micromechanical system will be strongly influenced by the interaction with the environment. However, a real quantum system is inevitably influenced by its surrounding environment. The unavoidable mutual interaction often results in the dissipative evolution of the quantum coherence and the loss of the useful entanglement. Here, we consider only the source of decoherence acting on a two-level atom. Therefore, the two-level system and the harmonic oscillator interact with a reservoir of harmonic oscillators. The master equation that describes the time evolution of the system is then given by[33,34]

$$\begin{aligned}\frac{{\rm d}\hat{\rho}(t)}{{\rm d}t} =& -{\rm i}\lambda[\hat{a}|e\rangle\langle g| + \hat{a}^\dagger|g\rangle\langle e|,\ \hat{\rho}(t)] \\ &+\gamma(2|g\rangle\langle e|\hat{\rho}(t)\,|e\rangle\langle g| \\ &-|e\rangle\langle e|\hat{\rho}(t) - \hat{\rho}(t)|e\rangle\langle e|), \end{aligned} \tag{1}$$

where γ is the spontaneous emission rate from $|e\rangle$ to $|g\rangle$. The environment is assumed to be at zero temperature (no thermal excitation), i.e., only the spontaneous decay term exists, which leads to a purely dissipative process. The atomic spontaneous emission from the side of the cavity is similarly treated by coupling the atomic transition to another reservoir. However, in the optical regime, the spontaneous emission plays an important role in the atom–field dynamics. The influence of the atomic spontaneous emission decay on the collapse and the revival phenomena in the optical regime was first studied in Ref. [35]. It has been shown that revivals of the atomic inversion are more sensitive to the cavity damping than to the spontaneous emission. The analytic method to obtain the exact solution of Eq. (1) for the case of a high$-Q$ cavity (i.e., $\gamma \ll \lambda$) in the so-called dressed-states representation,[36–38] i.e., the representation consists of the complete set of Hamiltonian eigenstates, is used. The full set of dressed states in the case of exact resonance are given by $|\psi_n^\pm\rangle = \frac{1}{\sqrt{2}}(|e,n\rangle \pm |g,n+1\rangle)$. In an invariant subspace spanned by $|\varphi_n^+\rangle = |e,n\rangle$ and $|\varphi_n^-\rangle = |g,n+1\rangle$, the eigenvalues of $\hat{H}_{\rm int}$ are given by

$$\begin{aligned}\hat{H}_{\rm int}|g,0\rangle &= 0, \\ \hat{H}_{\rm int}|\psi_n^\pm\rangle &= \pm\mu_n|\psi_n^\pm\rangle = \pm\lambda\sqrt{n+1}|\psi_n^\pm\rangle.\end{aligned}$$

To solve the equation in the high-Q limit, we write the atomic operators appearing in Eq. (1) in terms of the dressed-states (see Refs. [37] and [38]). It is convenient to proceed in the interaction representation,

$$\dot{\hat{O}}(t) = {\rm e}^{{\rm i}\hat{H}_{\rm int}t}\frac{\partial\hat{\rho}(t)}{\partial t}\,{\rm e}^{-{\rm i}\hat{H}_{\rm int}t} + {\rm i}[\hat{H}_{\rm int}, \hat{O}(t)],$$

where $\hat{O}$ is an arbitrary operator of the atomic or the field subsystem.

In this work, we suppose that the atom is initially in an excited state, i.e., $\hat{\rho}_{\rm a}(0) = |e\rangle\langle e|$, and the field is initially in an even coherent state, i.e.,

$$\begin{aligned}|{\rm ECS}\rangle &= \sum_{n=0}^{\infty} p_n|n\rangle, \\ &= \frac{1}{\sqrt{\cosh(|\nu|^2)}}\sum_{n=0}^{\infty}\frac{\nu^{2n}}{\sqrt{(2n)!}}|2n\rangle. \end{aligned} \tag{2}$$

The solution of the master equation in the high-Q limit with the previous initial states is given by

$$\hat{\rho}(t) = \sum_{m,n=0}\left(\sum_{i,j=+,-}\omega_{m,n}^{ij}|\varphi_m^i\rangle\langle\varphi_n^j|\right), \tag{3}$$

where

$$\omega_{m,n}^{\pm\pm} = \begin{cases} p_m p_n^*\,{\rm e}^{-\gamma t}[\cos(\mu_m-\mu_n)t \\ \quad \pm\cos(\mu_m+\mu_n)t], \quad \forall\ m\neq n, \\ \dfrac{{\rm e}^{-\gamma t}}{2}\left[\left(\dfrac{\bar{n}}{\gamma t}\right)^{n/2}{\rm I}_n(2\sqrt{\bar{n}\gamma t})\,{\rm e}^{-\bar{n}}\right. \\ \quad \left.\pm|p_n|^2\cos(2\mu_n t)\right], \quad \forall\ m=n, \end{cases} \tag{4}$$

$$\begin{aligned}\omega_{m,n}^{+-} &= (\omega_{m,n}^{-+})^\dagger \\ &= {\rm i}\,p_m p_n^*\,{\rm e}^{-\gamma t}\ \sin(\mu_m t)\cos(\mu_n t) \quad \forall\ m,n, \end{aligned} \tag{5}$$

${\rm I_n}$ is the modified Bessel function, and $\bar{n} = |\nu|^2$. The coherence properties, the entanglement, and the dynamics of the nonclassicality of quantum states will be

studied using the eigenvalues of the previous density matrix $\rho(t)$ and the marginal density matrices of the field and the atom $\rho_{\mathrm{f(a)}}(t)$.

3. Nonclassicality, purity, and entanglement

The potential applications of the WF, which are related to our study, are as follows: the negativity of the WF used as an indicator for the nonclassicality and the decoherence of the created superposition states can be further determined by measuring the WF of the cavity field or the qubit state.[39] Here, we are interested in the negativity of the WF serving as an indicator for the nonclassicality and the effect of the atomic damping on it. The approaches that quantify the nonclassicality are based on negative regions of the quasi-probability distributions.[28] Another measurement of the nonclassicality of quantum states based on the volume of the negative part of the WF is proposed in Ref. [29].

If a two-level particle interacts with a single-mode field via the single-photon transition process with atomic damping, the time evolution of the reduced density matrix of the field is given by $\hat{\rho}_{\mathrm{f}}(t) = \sum_{n,m=0}^{\infty} \omega_{m,n}^{++}|m\rangle\langle n| + \omega_{m,n}^{--}|m+1\rangle\langle n+1|$. The WF is usually expressed in an integral form, which is not always easy to compute. An expression for the WF in terms of displaced number states to avoid a tedious phase space integration is given in Refs. [40] and [41]. By using the methods in those references, the time evolution of the quasi-probability Wigner function is presented as

$$W(\alpha,t) = \sum_{l,m,n=0}^{\infty} \sum_{s=+,-} \frac{2(-1)^l}{\pi} \times \omega_{m,n}^{ss}(t)\ell_{m+\delta_{-,s}}^{l}\ell_{n+\delta_{-,s}}^{*l}, \tag{6}$$

where $\delta = 1$ if $i = j$, otherwise is equal to zero, and

$$\ell_j^n = \begin{cases} \alpha^{n-j}\sqrt{\frac{j!}{n!}}\mathrm{L}_j^{n-j}(|\alpha|^2), \ n > j, \\ \alpha^{j-n}\sqrt{\frac{n!}{j!}}\mathrm{L}_n^{j-n}(|\alpha|^2), \ n < j, \end{cases} \tag{7}$$

where L_n^m is the associated Laguerre polynomial. Because there are several reconstruction methods that have been proposed and experimentally demonstrated to obtain the WF,[43] we simultaneously study the temporal evolution of the negative values of the WF, the purity of field states, and the field–atom entanglement under the atomic damping. The entanglement of final state (3) is quantified by the negativity,[43] which is defined by $N(\rho) = \frac{\|\rho^{\mathrm{T}}\|-1}{2}$, where ρ^{T} is the matrix obtained by partially transposing the density matrix ρ. When $N(\rho) = 0$, the states are separable. The purity loss of the field is measured by the cavity entropy, $S_{\mathrm{f}} = -\mathrm{Tr}(\hat{\rho}_{\mathrm{f}} \ln \hat{\rho}_{\mathrm{f}})$.

In Figs. 1 and 2, the time evolution of the Wigner function at a particular point $\alpha = (\mathrm{Re}, \mathrm{Im}) = (2, 0)$ in the phase space, cavity entropy S_{f} and negativity $N(\rho)$ are plotted as functions of scaled time λt for $\bar{n} = 10$ with different values of atomic damping parameter γ. From these curves, we can perceive that the amplitudes of the local maxima and the local minima of the negativity decrease with the increase of parameter γ. The negativity vanishes, state (3) finally goes into the disentanglement state, and its coherence is lost completely (see Fig. 2). This means that after a particular time, the atomic damping destroys the entanglement of the global atom–field state.

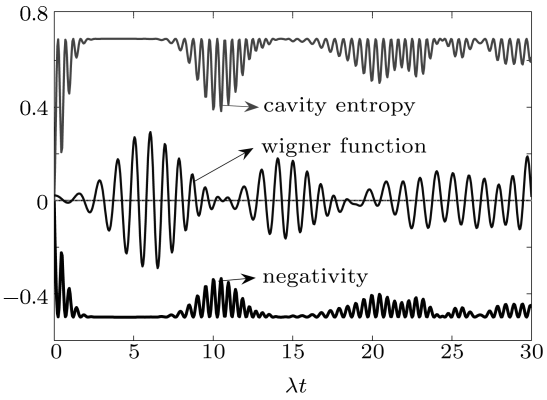


Fig. 1. Time evolution of the Wigner function, cavity entropy, and negativity $N(\rho)$.

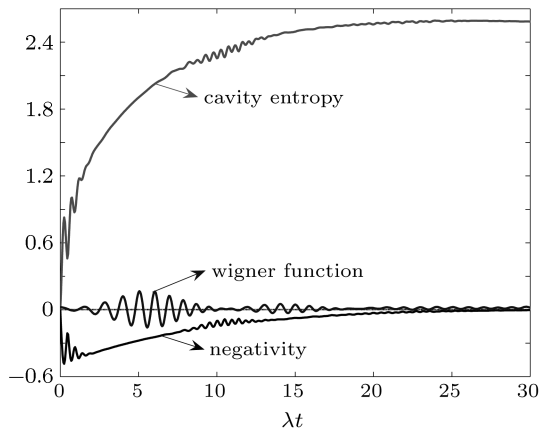


Fig. 2. Time evolution of the Wigner function, cavity entropy, and negativity $(N\rho)$ with $\gamma/\lambda = 0.1$.

Physically, the considered interaction of the atom subsystem with the environment leads to the spontaneous decay of the excited state, in which the atom makes a transition from its internal excited state to its ground state and loses its excess energy. That leads to the incoherent decay of the atom into the external reservoir of modes. The photon emitted by the atom irreversibly escapes into the environment. This process of occupying state $|g\rangle$ via the spontaneous emission from state $|e\rangle$ is, according to the term $\mathrm{e}^{-\gamma t}$, the reason for the exponential decay of both the states of atom and field. Finally, the atom is traced out, and therefore the entanglement between the atom and the cavity mode disappears.

If we look at the time evolution of the purity of the field state, we note that the atomic damping causes the amplitude suppression of the oscillations of S_{f}. At the maximum, the field loses its coherence, and it partially gains its coherence at the minimum. However, the field loses more coherence than it gains due to the atomic damping, and finally completely loses its coherence and falls into a mixed state.

There are two sources for the coherence loss in the present system. One of them is due to the unitary atom–field interaction. This process is usually called the entanglement. And from the point of view of one of the subsystems, the coherence loss will take place. Also, this process explains why the coherence of the field oscillates.

The aim of our work is to observe the effect of the atomic damping on the temporal evolution of the negative value of the WF and compare that with its counterparts of the purity and the entanglement. In the absence of damping ($\gamma/\lambda = 0$), we see that (Fig. 1) the WF has the local maximum of positive value, which means that there is no quantum interference between the states of the field. At $\lambda t > 0$, the minimum of the Wigner function oscillates between positive and negative values. This is due to the unitary cavity–atom interaction. From the point of view of $N(\rho)$ or S_{f}, this process is usually called the entanglement or the purity loss. This means that in the absence of the damping, there are entanglement, purity loss, no quantum interference (positive values of the WF), and quantum interference (negative values of the WF).

For larger values of γ, say $\gamma/\lambda = 0.1$, negative values of both the negativity (measure the entanglement) and the Wigner function (measure the nonclassicality of quantum states) disappear completely (see Fig. 2). On the other hand, the death of the nonclassicality of quantum states is completely consistent with the purity loss and the falling into a mixed state. The statistically mixed state has no negative value of the WF.

To confirm the phenomenon of the death of the nonclassicality of quantum states, we display in the inset of Fig. 3 the Wigner function $W(X, Y)$ in the phase space at time $\lambda t = 0.0$, where $X = \mathrm{Re}(\alpha)$ and $Y = \mathrm{Im}(\alpha)$. The time evolution of the minimum negative value of the WF with different damping parameters $\gamma/\lambda = 0.0,\ 0.05,\ 0.1$ is displayed in Fig. 3. From Eq. (4), we note that the phenomenon of the death of the nonclassicality is due to the factor of atomic damping $\mathrm{e}^{-\gamma t}$, which decays to zero at $t \to \infty$.

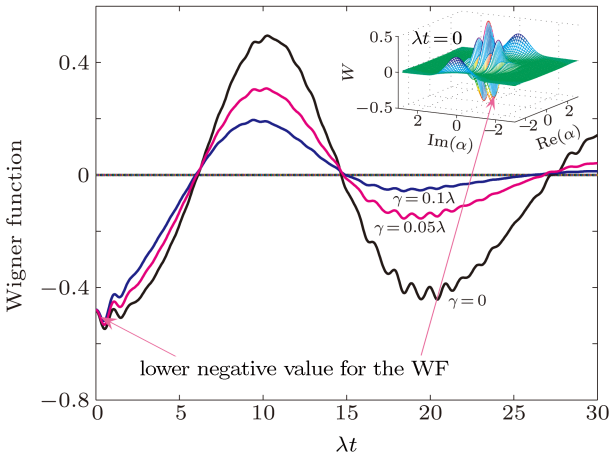


Fig. 3. (colour online) Time evolution of minimum negative value for the WF with different damping parameter $\gamma/\lambda = 0.0,\ 0.05,\ 0.1$. The inset displays the Wigner function $W(X, Y)$.

4. Conclusion

The dynamics of the nonclassicality of quantum states is investigated by using the negative value of the WF. It is found that the negative value of the field WF is very sensitive to any change in the damping parameter. The dissipation simultaneously leads to the long-time death for both the entanglement and the nonclassicality of even the coherent state, and the purity of the field state is lost completely.

References

[1] Royer A 1989 *Found. Phys.* **19** 3
[2] Lvovsky A I and Raymer M G 2009 *Rev. Mod. Phys.* **81** 299
[3] Lutterbach L G and Davidovich L 1997 *Phys. Rev. Lett.* **78** 2547
[4] Kim M S, Antesberger G, Bodendorf C T and Walther H 1998 *Phys. Rev.* A **58** R65

[5] Leibfried D, Meekhof D M, King B E, Monroe C, Itano W M and Wineland D J 1996 *Phys. Rev. Lett.* **77** 4281
[6] Nogues G, Rauschenbeutel A, Osnaghi S, Bertet P, Brune M, Raimond J M, Haroche S, Lutterbach L G and Davidovich L 2000 *Phys. Rev.* A **62** 054101
[7] Buzek V and Knight P L 1995 *Prog. Opt.* **34** 1
[8] Zurek W H 1981 *Phys. Rev.* D **24** 1516
[9] Zeh H D 1970 *Found. Phys.* **1** 69
[10] Zayed E M E, Daoud A S, AL-Laithy M A and Naseem E N 2005 *Chaos Soliton. Fract.* **24** 967
[11] Dragoman D 2005 *EURASIP J. Appl. Sig. P.* **10** 1520
[12] Turchette Q A, Georgiades N Ph, Hood C J and Kimble H J 1998 *Phys. Rev.* **58** 4056
[13] Lvovsky A I and Mlynek J 2002 *Phys. Rev. Lett.* **88** 250401
[14] Zavatta A, Viciani S and Bellini M 2004 *Science* **306** 660
[15] Arecchi F T 1965 *Phys. Rev. Lett.* **15** 912
[16] Scully M O and Lamb Jr W E 1967 *Phys. Rev.* **159** 208
[17] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press) pp. 736–738
[18] Dalton B J 1986 *Phys. Scr.* **T12** 43
[19] Yuen H P 1976 *Phys. Rev.* A **13** 2226
[20] Giacobino E and Febre C 1992 *Appl. Phys.* B: *Photophys. Laser. Chem.* **55** 189
[21] Schleich W and Wheeler J A 1987 *Nature* **326** 574
[22] Lütkenhaus N and Barnett S M 1995 *Phys. Rev.* A **51** 3340
[23] Lvovsky A I, Hansen H, Aichele T, Benson O, Mlynek J and Schiller S 2001 *Phys. Rev. Lett.* **87** 050402
[24] Bouwmeester D, Ekert A and Zeilinger A 2000 *The Physics of Quantum Information* (NewYork: Springer)
[25] Braunstein S L, D'Ariano G M, Milburn G J and Sacchi M F 2000 *Phys. Rev. Lett.* **84** 3486
[26] Tittel W, Brendel J, Zbinden H and Gisin N 1998 *Phys. Rev. Lett.* **81** 3563
[27] Chuang I L and Nielsen M A 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[28] Lütkeńhaus N and Barnett S M 1995 *Phys. Rev.* A **51** 3340
[29] Kenfack A and Życzkowski K 2004 *J. Opt.* B: *Quantum Semiclass. Opt.* **6** 396
[30] Wheeler J A and Zurek W H 1984 *Quantum Theory and Measurement* (Princeton: Princeton University Press)
[31] Bell J S 1964 *Physics* **1** 195
[32] Blais A, Huang R S, Wallraff A, Girvin S and Schoelkopf R 2004 *Phys. Rev.* A **69** 062320
[33] Diedrich F, Bergquist J C, Itano W M and Wineland D J 1989 *Phys. Rev. Lett.* **62** 403
[34] Marzoli I, Cirac J I, Blatt R and Zoller P 1994 *Phys. Rev.* A **49** 2771
[35] Quang T, Knight P L and Buzek V 1991 *Phys. Rev.* A **44** 6092
[36] Puri R R and Agarwal G S 1986 *Phys. Rev.* A **33** 3610
[37] Hessian H A and Mohamed A B A 2008 *Chin. Phys. Lett.* **25** 2492
[38] Hessian H A 2008 *Int. J. Mod. Phys.* B **22** 4017
[39] Liu Y X, Wei L F and Nori F 2005 *Phys. Rev.* A **72** 033818
[40] Moya-Cessa H and Knight P L 1993 *Phys. Rev.* A **48** 2479
[41] Hessian H A and Mohamed A B A 2008 *Laser Physics* **18** 1217
[42] Bertet P, Auffeves A, Maioli P, Osnaghi S, Meunier T, Brune M, Raimond J M and Haroche S 2002 *Phys. Rev. Lett.* **89** 200402
[43] Vidal G and Werner R F 2002 *Phys. Rev.* A **65** 032314