Rapid Communication

Coherent state evolution in a Raman dispersion process*

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By using the technique of integration within an ordered product of operators, we derive the evolution of an initial coherent state in a Raman dispersion process.

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1. Introduction

In quantum optics, optical frequency conversion is a typical nonlinear process and is worth studying, for example, a second harmonic frequency generation will generate a squeezed state.[1,2] In this work, we tackle the evolution of an initial coherent state in a Raman dispersion process which is also a nonlinear process. The process involves the inelastic scattering of a photon when it is incident on a molecule. The photon loses some of its energy to the molecule or gains some from it, and so leaves the molecule with a lower or a higher frequency. The lower frequency components of the scattered radiation are called the Stokes lines and the higher frequency components are called the anti-Stokes lines. The Hamiltonian governing its dynamics is[3]

$$H = \omega a_1^\dagger a_1 + \omega a_2^\dagger a_2 + \omega a_3^\dagger a_3 + g_1(a_1 a_3^\dagger + a_1^\dagger a_3) + g_2(a_2 a_3^\dagger + a_2^\dagger a_3), \quad (1)$$

which involves the pairwise interactions of three separate field modes through the effect of nonlinear polarization. Here, g_1 and g_2 are the interaction constants, the photon operators obey the commutative relation $[a_i, a_j^\dagger] = \delta_{i,j}$. To our knowledge, the problem of how an initial coherent state evolves in this Raman dispersion process has not been exactly solved in the literature till now. The reason perhaps lies in the difficulty of theoretically disentangling the time evolution operator

$$\mathrm{e}^{-\mathrm{i}Ht} = \exp\{-\mathrm{i}t[\omega a_1^\dagger a_1 + \omega a_2^\dagger a_2 + \omega a_3^\dagger a_3 + g_1(a_1 a_3^\dagger + a_1^\dagger a_3) + g_2(a_2 a_3^\dagger + a_2^\dagger a_3)]\}. \quad (2)$$

In this work, we shall adopt the technique of integration within an ordered product (IWOP) of operators[4–7] to convert $\mathrm{e}^{-\mathrm{i}Ht}$ into its normally ordered form, then the problem can be solved analytically.

2. A useful multimode operator identity

To begin with, let us present a useful multimode operator identity,

$$\exp(a_i^\dagger \Lambda_{ij} a_j) =: \exp[a_i^\dagger (\mathrm{e}^{\boldsymbol{\Lambda}} - \boldsymbol{I})_{ij} a_j] :, \quad (3)$$

where the subscripts i and j can be from 1 to n, $\boldsymbol{\Lambda}$ is an invertible $n \times n$ matrix, the repeated index in a term implies summation from 1 to n, the symbol : : denotes normal ordering. To prove Eq. (3), we introduce the n-mode coherent state

$$|z_i\rangle = \exp\left[-\frac{1}{2}|z_i|^2 + a_i^\dagger z_i\right]|0_i\rangle, \quad (4)$$

with the completeness relation[8–10]

$$\int \prod_{i=1}^{n} \frac{\mathrm{d}^2 z_i}{\pi} |z_i\rangle\langle z_i|$$

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$$= \int \prod_{i=1}^{n} \frac{\mathrm{d}^2 z_i}{\pi} : \exp(-|z_i|^2 + z_i a_i^\dagger + z_i^* a_i - a_i^\dagger a_i) :$$
$$= 1, \tag{5}$$

where we have used

$$|0_i\rangle\langle 0_i| =: \exp(-a_i^\dagger a_i) : \tag{6}$$

and the integral formula

$$\int \frac{\mathrm{d}^2\alpha}{\pi} \exp(\zeta|\alpha|^2 + \xi\alpha + \eta\alpha^*) = -\frac{1}{\zeta} \exp\left(-\frac{\xi\eta}{\zeta}\right),$$
$$\mathrm{Re}\xi < 0. \tag{7}$$

Then we use the Baker–Hausdorff formula[11]

$$\mathrm{e}^A B \mathrm{e}^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots \tag{8}$$

to derive

$$\exp(a_i^\dagger \Lambda_{ij} a_j) a_k^\dagger \exp(-a_i^\dagger \Lambda_{ij} a_j) = a_i^\dagger (\mathrm{e}^{\boldsymbol{\Lambda}})_{il} a_l. \tag{9}$$

It then follows from

$$\exp(-\mathrm{i} t a_i^\dagger \Lambda_{ij} a_j)|0_i\rangle = |0_i\rangle, \tag{10}$$

and by using the IWOP technique, we have

$$\exp(a_i^\dagger \Lambda_{ij} a_j)$$
$$= \int \prod_{i=1}^{n} \frac{\mathrm{d}^2 z_i}{\pi} \exp(a_i^\dagger \Lambda_{ij} a_j)|z_i\rangle\langle z_i|$$
$$= \int \prod_{i=1}^{n} \frac{\mathrm{d}^2 z_i}{\pi} \exp(a_i^\dagger \Lambda_{ij} a_j) \exp(a_i^\dagger z_i) \times \exp(-a_i^\dagger \Lambda_{ij} a_j)|0_i\rangle\langle z_i| \exp\left(-\frac{|z_i|^2}{2}\right)$$
$$= \int \prod_{i=1}^{n} \frac{\mathrm{d}^2 z_i}{\pi} \exp[a_i^\dagger (\mathrm{e}^{\boldsymbol{\Lambda}})_{ij} z_l]|0_i\rangle\langle z_i| \exp\left(-\frac{|z_i|^2}{2}\right)$$
$$= \int \prod_{i=1}^{n} \frac{\mathrm{d}^2 z_i}{\pi} : \exp[-|z_i|^2 + a_i^\dagger (\mathrm{e}^{\boldsymbol{\Lambda}})_{ij} z_l + z_i^* a_i - a_i^\dagger a_i] :$$
$$= : \exp[a_i^\dagger (\mathrm{e}^{\boldsymbol{\Lambda}} - \boldsymbol{I})_{ij} a_j] : . \tag{11}$$

Thus, Eq. (3) is proved. It then follows in the coherent state representation[12,13]

$$\exp(a_i^\dagger \Lambda_{ij} a_j)|z_i\rangle$$
$$= : \exp[a_i^\dagger (\mathrm{e}^{\boldsymbol{\Lambda}} - \boldsymbol{I})_{ij} a_j] : |z_i\rangle$$
$$= \exp\left[-\frac{1}{2}|z_i|^2 + a_i^\dagger (\mathrm{e}^{\boldsymbol{\Lambda}})_{ij} z_j\right]|0_i\rangle. \tag{12}$$

3. The evolution of three-mode coherent state in the Raman dispersion process

Now writing Eq. (1) as[14−16]

$$H = (\, a_1^\dagger \;\; a_2^\dagger \;\; a_3^\dagger \,) \Lambda \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_i^\dagger \Lambda_{ij} a_j,$$
$$\Lambda \equiv \begin{pmatrix} \omega & 0 & g_1 \\ 0 & \omega & g_2 \\ g_1 & g_2 & \omega \end{pmatrix}, \tag{13}$$

where the repeated index in a term implies summation from 1 to 3, and using Eq. (3), we have the normally ordered expansion of $\mathrm{e}^{-\mathrm{i}Ht}$,

$$\mathrm{e}^{-\mathrm{i}Ht} = \exp(-\mathrm{i} t a_i^\dagger \Lambda_{ij} a_j) =: \exp[a_i^\dagger (\mathrm{e}^{-\mathrm{i}t\boldsymbol{\Lambda}} - 1)_{ij} a_j] :$$
$$= : \exp\left\{ (\, a_1^\dagger \;\; a_2^\dagger \;\; a_3^\dagger \,) \left[\exp\left\{ -\mathrm{i}t \begin{pmatrix} \omega & 0 & g_1 \\ 0 & \omega & g_2 \\ g_1 & g_2 & \omega \end{pmatrix} \right\} - \boldsymbol{I} \right] \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right\} : . \tag{14}$$

The matrix $\boldsymbol{\Lambda}$ can be diagonalized as

$$\boldsymbol{\Lambda} = \boldsymbol{P} \boldsymbol{A} \boldsymbol{P}^{-1}, \tag{15}$$

where

$$\boldsymbol{P} = \begin{pmatrix} -\mu & \mu & -\nu/\mu \\ -\nu & \nu & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$
$$\boldsymbol{P}^{-1} = \begin{pmatrix} -\mu/2 & -\nu/2 & 1/2 \\ \mu/2 & \nu/2 & 1/2 \\ -\mu\nu & \mu^2 & 0 \end{pmatrix}, \tag{16}$$

where

$$\mu = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \quad \nu = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}, \tag{17}$$

with the eigenvalues

$$\lambda_1 = \omega - \sqrt{g_1^2 + g_2^2},$$
$$\lambda_2 = \omega + \sqrt{g_1^2 + g_2^2}, \quad \lambda_3 = \omega. \tag{18}$$

Using Eq. (15), we derive

$$\exp(-\mathrm{i}t\boldsymbol{\Lambda}) = \sum_{k=0}^{\infty}\frac{(-\mathrm{i}t\boldsymbol{\Lambda})^k}{k!} = P\begin{pmatrix} \sum\limits_{k=0}^{\infty}\frac{(-\mathrm{i}t\lambda_1)^k}{k!} & 0 & 0 \\ 0 & \sum\limits_{k=0}^{\infty}\frac{(-\mathrm{i}t\lambda_2)^k}{k!} & 0 \\ 0 & 0 & \sum\limits_{k=0}^{\infty}\frac{(-\mathrm{i}t\omega)^k}{k!} \end{pmatrix}\boldsymbol{P}^{-1}$$

$$= \begin{pmatrix} -\mu & \mu & -\nu/\mu \\ -\nu & \nu & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} \exp(-\mathrm{i}t\lambda_1) & 0 & 0 \\ 0 & \exp(-\mathrm{i}t\lambda_2) & 0 \\ 0 & 0 & \exp(-\mathrm{i}t\omega) \end{pmatrix}\begin{pmatrix} \mu/2 & -\nu/2 & 1/2 \\ \mu/2 & \nu/2 & 1/2 \\ -\mu\nu & \mu^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\mu^2\xi + \nu^2\mathrm{e}^{-\mathrm{i}t\omega} & \frac{1}{2}\mu\nu\xi - \mu\nu\mathrm{e}^{-\mathrm{i}t\omega} & \frac{1}{2}\mu\eta \\ \frac{1}{2}\mu\nu\xi - \mu\nu\mathrm{e}^{-\mathrm{i}t\omega} & \frac{1}{2}\nu^2\xi + \mu^2\mathrm{e}^{-\mathrm{i}t\omega} & \frac{1}{2}\nu\eta \\ \frac{1}{2}\mu\eta & \frac{1}{2}\nu\eta & \frac{1}{2}\xi \end{pmatrix} \equiv \boldsymbol{W}, \tag{19}$$

where

$$\xi = \mathrm{e}^{-\mathrm{i}t\lambda_1} + \mathrm{e}^{-\mathrm{i}t\lambda_2}, \quad \eta = \mathrm{e}^{-\mathrm{i}t\lambda_2} - \mathrm{e}^{-\mathrm{i}t\lambda_1}. \tag{20}$$

When the initial state is a 3-mode coherent state $|z_1, z_2, z_3\rangle$, then at time t it evolves into

$$\mathrm{e}^{-\mathrm{i}Ht}|z_1, z_2, z_3\rangle = \exp\left\{(\, a_1^\dagger\ a_2^\dagger\ a_3^\dagger\,)[\boldsymbol{W} - \boldsymbol{I}]\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right\}|z_1, z_2, z_3\rangle$$

$$= \exp\left[-\sum_{i=1}^{3}\frac{|z_i|^2}{2}\right]\exp\left[(\, a_1^\dagger\ a_2^\dagger\ a_3^\dagger\,)\boldsymbol{W}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right]|000\rangle, \tag{21}$$

the final state keeps a coherent state.

4. The partition function for Raman dispersion process

From Eq. (19), it will be very convenient for us to derive the partition function $\mathrm{Tr}\exp(-\beta H)$ of the system described by the Hamiltonian in Eq. (1). According to Eq. (19), let $\mathrm{i}t \to \beta$, we have

$$\exp(-\beta\boldsymbol{\Lambda}) = \begin{pmatrix} \frac{1}{2}\mu^2\xi' + \nu^2\mathrm{e}^{-\beta\omega} & \frac{1}{2}\mu\nu\xi' - \mu\nu\mathrm{e}^{-\beta\omega} & \frac{1}{2}\mu\eta' \\ \frac{1}{2}\mu\nu\xi' - \mu\nu\mathrm{e}^{-\beta\omega} & \frac{1}{2}\nu^2\xi' + \mu^2\mathrm{e}^{-\beta\omega} & \frac{1}{2}\nu\eta' \\ \frac{1}{2}\mu\eta' & \frac{1}{2}\nu\eta' & \frac{1}{2}\xi' \end{pmatrix} \equiv \boldsymbol{W}', \tag{22}$$

where

$$\xi' = \mathrm{e}^{-\beta\lambda_1} + \mathrm{e}^{-\beta\lambda_2}, \quad \eta' = \mathrm{e}^{-\beta\lambda_2} - \mathrm{e}^{-\beta\lambda_1}. \tag{23}$$

From Eqs. (21) and (23), we can obtain

$$Z = \mathrm{Tr}\,\mathrm{e}^{-\beta\boldsymbol{H}} = \int\prod_{i=1}^{3}\left[\frac{\mathrm{d}^2 z_i}{\pi}\right]\langle z_1, z_2, z_3|\mathrm{e}^{-\beta H}|z_1, z_2, z_3\rangle$$

$$= \int\prod_{i=1}^{3}\left[\frac{\mathrm{d}^2 z_i}{\pi}\right]\langle z_1, z_2, z_3| : \exp\left\{(\, a_1^\dagger\ a_2^\dagger\ a_3^\dagger\,)[\exp\{-\beta\boldsymbol{\Lambda}\} - \boldsymbol{I}]\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right\} : |z_1, z_2, z_3\rangle$$

$$= \int \prod_{i=1}^{3} \left[\frac{\mathrm{d}^2 z_i}{\pi}\right] \exp\left\{-\sum_{i=1}^{3} |z_i|^2 + (\, z_1^* \;\; z_2^* \;\; z_3^* \,)\boldsymbol{W}' \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right\}$$

$$= \int \prod_{i=1}^{3} \left[\frac{\mathrm{d}^2 z_i}{\pi}\right] \exp\left\{-(\, z_1^* \;\; z_2^* \;\; z_3^* \,)(\boldsymbol{I} - \boldsymbol{W}') \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right\}$$

$$= [\det(\boldsymbol{I} - \boldsymbol{W}')]^{-1} = 4[(1 - \mathrm{e}^{-\beta\omega})(\xi'^2 - 4\xi' - \eta'^2 + 4)]^{-1}. \tag{24}$$

Substituting Eqs. (18) and (23) into Eq. (24), we have

$$Z = \{(1 - \mathrm{e}^{-\beta\omega})[1 + \mathrm{e}^{-\beta\omega}(\mathrm{e}^{-\beta\omega} - \mathrm{e}^{\beta\sqrt{g_1^2+g_2^2}} - \mathrm{e}^{-\beta\sqrt{g_1^2+g_2^2}})]\}^{-1}. \tag{25}$$

In particular, when $g_1 = g_2 = 0$, Z reduces to

$$Z \to (1 - \mathrm{e}^{-\beta\omega})^{-3}, \tag{26}$$

which is just the partition function of three independent harmonic oscillators. Using

$$U = -\frac{\partial}{\partial\beta} \ln Z, \tag{27}$$

we derive the internal energy of the system

$$U = Z\{\omega \mathrm{e}^{-\beta\omega}[1 + \mathrm{e}^{-\beta\omega}(\mathrm{e}^{-\beta\omega} - \mathrm{e}^{\beta k} - \mathrm{e}^{-\beta k})] + \mathrm{e}^{-\beta\omega}(1 - \mathrm{e}^{-\beta\omega})[(\omega - k)\mathrm{e}^{\beta k} + (\omega + k)\mathrm{e}^{-\beta k} - 2\omega \mathrm{e}^{-\beta\omega}]\}. \tag{28}$$

where $k = \sqrt{g_1^2 + g_2^2}$.

5. Conclusion

In summary, we have derived the normally ordered time evolution operator and the partition function for the Raman process, this again exhibits the merit of using the IWOP technique.

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