Maximal and total skew information for a two-qubit system using nonlinear interaction models*

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Maximal and total skew information is studied. For symmetric pure states of two-qubit, they are closely related to the linear entropy, the concurrence, and the spin squeezing parameter. For a two-qubit system implemented in three nonlinear interaction models with an external field, we give the exact state vectors and the expectation value $\langle S_z\rangle$ at any time t. Based on $\langle S_z\rangle^2$, we give the maximal and the total skew information and a condition in which the maximal and the total skew information can reach 1 and 2, respectively.

Keywords: maximal and total Wigner–Yanase skew information, nonlinear interaction models, two-qubit system

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1. Introduction

To generate the spin squeezing and the spin squeezed states, Kitagawa and Ueda first introduced the one-axis twisting and the two-axis countertwisting models.[1] Possible realizations of the one-axis twisting Hamiltonian have been studied in a two-component Bose–Einstein condensate[2−8] and in an atomic ensemble system in the dispersive regime. Sørensen *et al.*[2] also proposed that spin squeezing can be used as a measure of the many-particle quantum entanglement. In order to improve the spin squeezing and the entanglement, the one-axis twisting[9] and the two-axis countertwisting[10] models with external fields were proposed. Among the above models, only the one-axis twisting model can be solved analytically within the Heisenberg picture. However, for a few-particle case,[10−16] the exact solutions of the state vectors of the above models can be obtained at any time t.

On the other hand, the study of the quantum information[17,18] and the entanglement[19,20] is one of the most intriguing aspects of quantum theory. Recently, Pezzé and Smerzi introduced a different criterion for the multiparticle entanglement[21] in terms of the quantum Fisher information,[22−24] which plays an important role in the quantum statistical estimation[22] and can be used as a tool to study the quantum phase transition.[25] Moreover, it is shown that the Wigner–Yanase skew information[26] is essentially a particular kind of quantum Fisher information and is closely related to the Heisenberg uncertainty principle.[27,28] It can also be used to detect the entanglement.[29] In this work, we first present the maximal Wigner–Yanase skew information (MSI) and the total Wigner–Yanase skew information (TSI), and then assert that for symmetric pure states of two-qubit, they are intrinsically related to the spin squeezing and the concurrence, which is a measure of the entanglement of the two-qubit.[30−35] Further, we study the MSI and the TSI of states generated in the one-axis twisting and the two-axis countertwisting models with external fields and in the mix of the two models for the case of two-qubit. We present the states in terms of the eigenstates of the Hamiltonian at any time t and show that the expectation value $\langle S_z\rangle$ plays a key role and the external field can control the magnitude of the two kinds of information.

The rest of the present paper is organized as follows. In Section 2, we study the relations among the MSI, the TSI, and the linear entropy for sym-

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metric pure states. In Section 3, for the symmetric pure states of two-qubit, we give the relations among the MSI, the TSI, the linear entropy, the concurrence, and the spin squeezing parameter. In Section 4, we study the MSI and the TSI of the two-qubit states dynamically generated from $|11\rangle$ via three nonlinear interaction models. Finally, a conclusion is given in Section 5.

2. Maximal skew information, total skew information, and linear entropy

In order to further study the measurement theory from an information-theoretic point of view, Wigner and Yanase introduced the Wigner–Yanase skew information,[26] which is the amount of information about the values of the observables not commuting with operator A. In terms of density matrix ρ and operator A, the skew information (SI) is defined as[26]

$$I(\rho, A) = -\frac{1}{2}\mathrm{Tr}[\rho^{1/2}, A]^2. \tag{1}$$

From Eq. (1), we know that $I(\rho, A)$ may be interpreted as the degree of noncommutativity between ρ and A.[36] The SI can be rewritten as

$$I(\rho, A) = \mathrm{Tr}\rho A^2 - \mathrm{Tr}\rho^{1/2} A \rho^{1/2} A. \tag{2}$$

In particular, if a state is pure, then this state can be described by $|\psi\rangle$. The corresponding density matrix ρ has the property $\rho^k = \rho$. So the SI of the state has the form

$$I(\rho, A) = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2, \tag{3}$$

which is actually the fluctuation of operator A.

A collection of N spin-1/2 particles is represented by the collective operators

$$S_\mu = \sum_{i=1}^{2} S_{i,\mu},\ \ \mu \in \{x, y, z\}, \tag{4}$$

where $S_{i,\mu}$ are the spin operators for the i-th qubit and satisfy the cyclic commutation relations

$$[S_{i,\mu}, S_{i,\nu}] = \mathrm{i}\varepsilon_{\mu\nu\gamma} S_{i,\gamma}, \tag{5}$$

$$[S_\mu, S_\nu] = \mathrm{i}\varepsilon_{\mu\nu\gamma} S_\gamma, \tag{6}$$

$$\mu, \nu, \gamma \in \{x, y, z\},$$

with $\varepsilon_{\mu\nu\gamma}$ being the Levi-Civita symbol. For the pure states, as the SI is equal to the spin fluctuation, the maximal SI corresponds to the maximal spin fluctuation, which will be denoted as I_m. Based on the SI, we can define a quantity that is the sum of the SI of the x, y, and z directions, and call it the total skew information (TSI). For convenience, we denote it as

$$I_\mathrm{sum} = \sum_\mu \Delta S_\mu^2 = \langle\, S_\mu^2\rangle - \langle\, S_\mu\rangle^2. \tag{7}$$

On the other hand, the linear entropy is defined as[37,38]

$$E_\mathrm{L} = 1 - \mathrm{Tr}(\rho_i^2), \tag{8}$$

where ρ_i is the reduced matrix of the i-th particle. For the states belonging to the subspace with the maximal spin angular momentum $S = N/2$, there is a clear relation between the linear entropy and the expectation values of S_μ, expressed as

$$E_\mathrm{L} = \frac{1}{2}\left[1 - \frac{4}{N^2}(\langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2)\right]. \tag{9}$$

Then we can obtain the relation between the linear entropy and the TSI as

$$I_\mathrm{sum} = \frac{N}{2} + \frac{N^2}{2} E_\mathrm{L}. \tag{10}$$

If condition $\langle S_x\rangle = \langle S_y\rangle = 0$ can be satisfied in Eq. (9), for example, in the one-axis and the two-axis models,[39–41] then the linear entropy and I_sum depend only on $\langle S_z\rangle^2$,

$$E_\mathrm{L} = \frac{1}{2}\left[1 - \frac{4}{N^2}\langle S_z\rangle^2\right], \tag{11}$$

$$I_\mathrm{sum} = S(S+1) - \langle S_z\rangle^2. \tag{12}$$

Under the above restrictive condition, $\langle S_z\rangle^2$ is an essential ingredient of the linear entropy and the TSI. For the two-qubit system, the two equations become

$$E_\mathrm{L} = \frac{1 - \langle S_z\rangle^2}{2}, \tag{13}$$

$$I_\mathrm{sum} = 2 - \langle S_z\rangle^2. \tag{14}$$

Now we demonstrate that for arbitrary three mutually orthogonal directions, the sum of the SI equals I_sum given by Eq. (7). We construct a new coordinate frame as

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\sin\phi & \cos\phi & 0 \\ -\cos\theta\cos\phi & -\cos\theta\sin\phi & \sin\theta \end{pmatrix}$$

$$\times \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}, \tag{15}$$

where θ and ϕ are arbitrary polar and arbitrary azimuth angles, respectively. It is easy to demonstrate that

$$\langle S_1^2\rangle + \langle S_2^2\rangle + \langle S_3^2\rangle = \langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle, \tag{16}$$

$$\langle S_1\rangle^2 + \langle S_2\rangle^2 + \langle S_3\rangle^2 = \langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2. \tag{17}$$

Then we have the conclusion

$$I_{\rm sum} = \Delta S_x^2 + \Delta S_y^2 + \Delta S_z^2 = \Delta S_1^2 + \Delta S_2^2 + \Delta S_3^2. \tag{18}$$

For the two-qubit case, equation (10) has a simple form

$$I_{\rm sum} = 1 + 2E_{\rm L}. \tag{19}$$

3. For two-qubit symmetric pure states

We begin with the fundamental symmetric state

$$|\psi\rangle = \sin\alpha|11\rangle + \cos\alpha|00\rangle \tag{20}$$

with concurrence $C = |\sin 2\alpha|$,[29,30,42] where $|0\rangle$ and $|1\rangle$ are the eigenstates of spin operator $S_{i,z}$ with eigenvalues $1/2$ and $-1/2$, respectively. First, we find the relation between the TSI and the concurrence. Note that condition $\langle S_x\rangle = \langle S_y\rangle = 0$ can be satisfied and $\langle S_z\rangle = \cos 2\alpha$, thereby, equations (13) and (14) can be simplified into

$$E_{\rm L} = \frac{C^2}{2}, \tag{21}$$

$$I_{\rm sum} = 1 + C^2. \tag{22}$$

The two equations above show that the TSI and the linear entropy are determined by the concurrence. Since $C \geq 0$, then the three quantities are mutually equivalent.

Next, we give a similar relation between the MSI and the concurrence. Considering that the mean spin direction $\boldsymbol{n}$ is along the z-axis and its collective angular momentum operators satisfy the condition $\langle [S_z, S_x]_+\rangle = \langle [S_z, S_y]_+\rangle = 0$, then the maximal spin fluctuation has the form[41]

$$\max(\Delta S_{\boldsymbol{n}})^2 = \max(\Delta S_z^2, \max \Delta S^2(x,y)),$$

where operator $S_{\boldsymbol{n}}$ is defined as $S_{\boldsymbol{n}} = \sin\theta\cos\phi S_x + \sin\theta\sin\phi S_y + \cos\theta S_z$, and $\max \Delta S^2(x,y)$ is the maximum fluctuation in plane (S_x, S_y).[41] Exactly, we can obtain the fluctuation in the z direction as

$$\Delta S_z^2 = \sin^2(2\alpha). \tag{23}$$

Furthermore, we can use the method in Refs. [43]–[45] to obtain

$$\begin{aligned} \max \Delta S^2(x,y) &= \frac{1}{2}[\langle S_x^2 + S_y^2\rangle + |\langle S_+^2\rangle|] \\ &= \frac{1 + |\sin 2\alpha|}{2}. \end{aligned} \tag{24}$$

By comparison, we can verify that $\max \Delta S^2(x,y)$ is just the maximal spin fluctuation in all directions. Finally, the MSI can be obtained by

$$I_{\rm m} = \frac{1}{2}(1 + |\sin 2\alpha|) = \frac{1}{2}(1 + C). \tag{25}$$

It is demonstrated that the spin squeezing parameter[43−45] ξ^2 is also closely related to the concurrence for the two-qubit symmetric states by[42,46,47]

$$\xi^2 = 1 - C. \tag{26}$$

By combining Eqs. (21) and (22), we can obtain a equality chain among these quantities,

$$\begin{aligned} I_{\rm sum} &= 1 + (2I_{\rm m} - 1)^2 \\ &= 1 + 2E_{\rm L} \\ &= 1 + C^2 = 1 + (1 - \xi^2)^2. \end{aligned} \tag{27}$$

In the following, we will demonstrate that for an arbitrary symmetric state of two-qubit

$$|\psi\rangle = a_1|00\rangle + a_2/\sqrt{2}(|01\rangle + |10\rangle) + a_3|11\rangle, \tag{28}$$

the equality chain is also valid. If we perform an arbitrary collective unitary transformation U on state $|\psi\rangle = \sin\alpha|11\rangle + \cos\alpha|00\rangle$, where $U = \mathrm{e}^{-\mathrm{i}\phi S_z}\,\mathrm{e}^{-\mathrm{i}\theta S_y}$ with an arbitrary polar angle θ and an arbitrary azimuth angle ϕ, then $|\psi\rangle$ becomes

$$|\tilde{\psi}\rangle = \sin\alpha|\tilde{1}\tilde{1}\rangle + \cos\alpha|\tilde{0}\tilde{0}\rangle, \tag{29}$$

where states $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$ are eigenstates of operator $\boldsymbol{\sigma}_i \cdot \boldsymbol{n}$ with the arbitrary direction $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and are given by

$$|\tilde{0}\rangle = \mathrm{e}^{-\mathrm{i}\phi/2}\cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{\mathrm{i}\phi/2}\sin\frac{\theta}{2}|1\rangle, \tag{30}$$

$$|\tilde{1}\rangle = \mathrm{e}^{-\mathrm{i}\phi/2}\sin\frac{\theta}{2}|0\rangle - \mathrm{e}^{\mathrm{i}\phi/2}\cos\frac{\theta}{2}|1\rangle \tag{31}$$

with eigenvalues 1 and -1, respectively. Now we turn to the S_z representation, state $|\tilde{\psi}\rangle$ can be rewritten as

$$|\psi\rangle = A_1|00\rangle + A_2/\sqrt{2}(|01\rangle + |10\rangle) + A_3|11\rangle \quad (32)$$

with coefficients

$$\begin{aligned} A_1 &= \mathrm{e}^{-\mathrm{i}\phi}\left(\cos\alpha\cos^2\frac{\theta}{2} + \sin\alpha\sin^2\frac{\theta}{2}\right), \\ A_2 &= \sqrt{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\cos\alpha - \sin\alpha), \\ A_3 &= \mathrm{e}^{\mathrm{i}\phi}\left(\cos\alpha\sin^2\frac{\theta}{2} + \sin\alpha\cos^2\frac{\theta}{2}\right). \end{aligned} \quad (33)$$

With an appropriate choice of α, θ, and ϕ, we can achieve that A_1, A_2, and A_3 are equal to a_1, a_2, and a_3, respectively. We do corresponding transformation on collective operator $\hat{F}$, which becomes $U\hat{F}U^{-1}$. It is obvious that the expectation values leading to Eq. (27) do not change. In other words, we only let $\langle\psi|\hat{F}|\psi\rangle$ referred to in Eq. (27) become $\langle\psi U^{-1}U|\hat{F}|U^{-1}U\psi\rangle$. However, this simple operation suggests that the equality chain given by Eq. (27) is always valid for the arbitrary symmetric pure state of two-qubit.

4. Maximal and total skew information generated by three nonlinear interaction models with two-qubit

Having established an equality chain among the MSI, the TSI, the linear entropy, the concurrence, and the spin squeezing parameter, now we study the MSI and the TSI of two-qubit states dynamically generated from $|11\rangle$ via three nonlinear interaction Hamiltonian models. As these models do not change the parity of the initial state, the expectation value $\langle S_z\rangle$ plays a pivotal role in these models.

Quantum mechanics shows that if the initial state of the system can be expressed as a linear combination of the eigenstates of a Hamiltonian that is independent of time, i.e., $\psi(0) = \sum_n c_n\phi_n$, then the final state can be given by $\psi(t) = \sum_n \mathrm{e}^{-\mathrm{i}E_n t}c_n\phi_n$, where E_n is the eigenvalue corresponding to eigenstate ϕ_n (assume $\hbar = 1$). So, we can study the information included in the final state $\psi(t)$. For our problem, the first thing we need to do is to find the eigenvalues and the eigenstates of these nonlinear interaction Hamiltonian models. In particular, for the case with small particle number, this work is easy. Here we consider only the two-qubit case. Our central task is to calculate the expectation value $\langle S_z\rangle$ at any time t.

4.1. One-axis twisting model with an external field

For the one-axis twisting model with an external field, the Hamiltonian can be described as

$$H_1 = 2\chi_1 S_x^2 + \Omega S_z, \quad (34)$$

where $\Omega > 0$ is the strength of the external field. We find that only two eigenstates $\phi_{1\pm}$ are useful for our problem by solving the eigenvalue equation of the Hamiltonian. The ϕ_{1+} corresponds to the maximal energy $E_{1+} = \chi_1 + \sqrt{\chi_1^2 + \Omega^2}$, and ϕ_{1-} corresponds to the minimum energy $E_{1-} = \chi_1 - \sqrt{\chi_1^2 + \Omega^2}$. Their exact expressions are

$$\phi_{1\pm} = \frac{1}{A_\pm}\begin{pmatrix} (\Omega \pm \sqrt{\chi_1^2 + \Omega^2})/\chi_1 \\ 0 \\ 1 \end{pmatrix}, \quad (35)$$

where $A_{1\pm}$ are normalization factors,

$$A_{1\pm} = \chi_1 / \left[2\left(\chi_1^2 + \Omega^2 \pm \Omega\sqrt{\chi_1^2 + \Omega^2}\right)\right]^{1/2}. \quad (36)$$

Under the initial condition $c_{1+}\phi_{1+} + c_{1-}\phi_{1-} \equiv |11\rangle$, we can obtain coefficients $c_{1\pm}$ and present the final state vector as

$$c_{1\pm} = \frac{\sqrt{\chi_1^2 + \Omega^2} \mp \Omega}{2\sqrt{\chi_1^2 + \Omega^2}}, \quad (37)$$

$$\psi_1(t) = (a_1(t) \;\; 0 \;\; b_1(t))^{\mathrm{T}}, \quad (38)$$

$$a_1(t) = \chi_1 \frac{\mathrm{e}^{-\mathrm{i}E_{1+}t} - \mathrm{e}^{-\mathrm{i}E_{1-}t}}{2\sqrt{\chi_1^2 + \Omega^2}}, \quad (39)$$

$$\begin{aligned} b_1(t) = {} & \frac{(\mathrm{e}^{-\mathrm{i}E_{1+}t} + \mathrm{e}^{-\mathrm{i}E_{1-}t})}{2} \\ & - \frac{(\mathrm{e}^{-\mathrm{i}E_{1+}t} - \mathrm{e}^{-\mathrm{i}E_{1-}t})}{2\Omega^{-1}\sqrt{\chi_1^2 + \Omega^2}}. \end{aligned} \quad (40)$$

By a straightforward calculation, we can obtain the expectation value as

$$\langle S_z\rangle = \frac{1}{\chi_1^2 + \Omega^2}\left[-\Omega^2 - \chi_1^2\cos\left(2\sqrt{\chi_1^2 + \Omega^2}t\right)\right]. \quad (41)$$

In Fig. 1(a), we present the time evolution of $\langle S_z\rangle$. We emphasize $|\langle S_z\rangle|$. From Eq. (41), we can see that it is a periodical function with an elegant form. In the absence of the external field,

the mean spin-flip occurs at $t_{\rm c} = (2n+1)\pi/\Delta E$ as shown by the dashed dotted line in Fig. 1(a), where $\Delta E = E_{1+} - E_{1-} = 2\sqrt{\chi_1^2 + \Omega^2}$ is the energy difference between the second excited state and the ground state. In the presence of the external field, if $\Omega \leq \chi_1$, then its minimum value can be equal to zero, which occurs at $t_{\rm m} = \arccos(-\Omega^2/\chi_1^2)/\Delta E$. We can find that the system possesses the maximal entanglement and spin squeezing at $t_{\rm m}$. The $\Omega = \chi_1$ is a critical point. For the case of $\Omega > \chi_1$, its minimum value $|\chi_1^2 - \Omega^2|/(\chi_1^2 + \Omega^2)$ occurs at $t_{\rm c}$, which is larger than zero, and the concurrence cannot reach one. Under the condition $\Omega \gg \chi_1$, it will be close to the initial value -1, as shown by the solid line in Fig. 1(a). In fact, for the $\Omega \gg \chi_1$ case, the nonlinear term $2\chi_1 S_x^2$ can be treated as a perturbation, then states of the system and the expectation value $\langle \hat{F} \rangle$ do not deviate too far away from those at the initial time. In Figs. 1(b) and 1(c), we present the time evolution of the MSI and the TSI. From Eqs. (14) and (27), it is known that the two quantities together with $|\langle S_z \rangle|$ can reach the extreme values synchronously. So, we do not discuss them further.

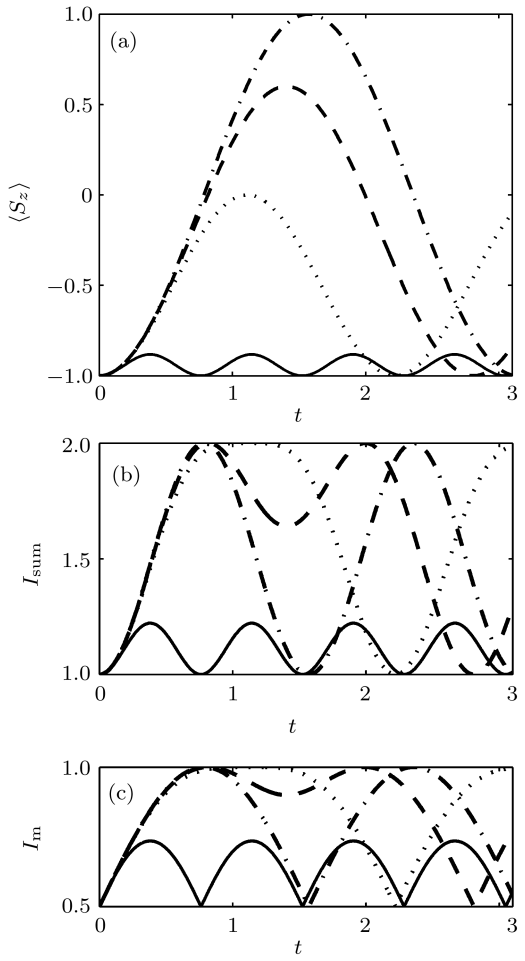


Fig. 1. (a) $\langle S_z \rangle$, (b)$I_{\rm sum}$, and (c) $I_{\rm m}$ versus t, with parameter $\chi_1 = 1$. Ω taking 0 (dashed dotted lines), 0.5 (dashed lines), 1 (dotted lines) and 4 (solid lines).

4.2. Two-axis countertwisting model with an external field

The Hamiltonian for the two-axis countertwisting model with an external field can be described as

$$H_2 = \frac{\chi_2}{2\mathrm{i}}(S_+^2 - S_-^2) + \Omega S_z. \tag{42}$$

The exact diagonalization of the Hamiltonian provides the following two available eigenstates

$$\phi_{2\pm} = \frac{1}{A_{2\pm}} \begin{pmatrix} 1 \\ 0 \\ \mathrm{i}\left(\Omega \pm \sqrt{\chi_2^2 + \Omega^2}\right)/\chi_2 \end{pmatrix}, \tag{43}$$

where $A_{2\pm}$ are normalization factors given by

$$A_{2\pm} = \chi_2 / \left[2\left(\chi_2^2 + \Omega^2 \pm \Omega\sqrt{\chi_2^2 + \Omega^2}\right)\right]^{1/2}. \tag{44}$$

The states ϕ_{2+} and ϕ_{2-} correspond to the maximal energy $E_{2+} = \sqrt{\chi_2^2 + \Omega^2}$ and the minimum energy $E_{2-} = -\sqrt{\chi_2^2 + \Omega^2}$, respectively. It is easy to obtain expansion coefficients $c_{2\pm}$ and the state vectors as

$$c_{2\pm} = \frac{\mp \mathrm{i}\chi_2}{2\sqrt{\chi_2^2 + \Omega^2}}, \tag{45}$$

$$\psi_2(t) = (a_2(t) \;\; 0 \;\; b_2(t))^{\rm T}, \tag{46}$$

$$a_2(t) = \frac{\chi_2 \sin\left(\sqrt{\chi_2^2 + \Omega^2}t\right)}{\sqrt{\chi_2^2 + \Omega^2}}, \tag{47}$$

$$b_2(t) = \frac{\Omega \mathrm{i} \sin\left(\sqrt{\chi_2^2 + \Omega^2}t\right)}{\sqrt{\chi_2^2 + \Omega^2}} - \cos\left(\sqrt{\chi_2^2 + \Omega^2}t\right). \tag{48}$$

Following the similar procedures leading to Eq. (41), we can obtain the expectation value $\langle S_z \rangle$ at any time t as

$$\langle S_z \rangle = \frac{1}{\chi_2^2 + \Omega^2}\left[-\Omega^2 - \chi_2^2 \cos\left(2\sqrt{\chi_2^2 + \Omega^2}t\right)\right]. \tag{49}$$

We find that the result is the same as that of the previous model. This is interesting and unexpected.

4.3. Mixed model of the one-axis twisting and two-axis countertwisting with an external field

Further, we consider a general model, and call it the mixed model of the one-axis twisting and the two-axis countertwisting models with an external field. The Hamiltonian reads

$$H_3 = \chi_1 S_x^2 + \frac{\chi_2}{2\mathrm{i}}(S_+^2 - S_-^2) + \Omega S_z. \tag{50}$$

The two available eigenstates are

$$\phi_{3\pm} = \frac{1}{A_{3\pm}} \begin{pmatrix} \frac{\left(\Omega \pm \sqrt{\chi^2+\Omega^2}\right)(\chi_1 - \mathrm{i}\chi_2)}{\chi^2} \\ 0 \\ 1 \end{pmatrix} \tag{51}$$

with the normalization factors

$$A_{3\pm} = \chi / \left[2\left(\chi^2 + \Omega^2 \pm \Omega\sqrt{\chi^2+\Omega^2}\right)\right]^{1/2}, \tag{52}$$

where $\chi^2 = \chi_1^2 + \chi_2^2$. The two corresponding available eigenvalues are

$$E_{3\pm} = \chi_1 \pm \sqrt{\chi^2+\Omega^2}. \tag{53}$$

From the initial condition, we can obtain expansion coefficients $c_{3\pm}$ and the final state as

$$c_{3\pm} = \frac{\sqrt{\chi^2+\Omega^2} \mp \Omega}{2\sqrt{\chi^2+\Omega^2}}, \tag{54}$$

$$\psi_3(t) = (a_3(t) \;\; 0 \;\; b_3(t))^{\mathrm{T}}, \tag{55}$$

$$a_3(t) = \frac{(\chi_1 - \mathrm{i}\chi_2)(\mathrm{e}^{-\mathrm{i}E_{3+}t} - \mathrm{e}^{-\mathrm{i}E_{3-}t})}{2\sqrt{\chi^2+\Omega^2}}, \tag{56}$$

$$b_3(t) = \frac{(\mathrm{e}^{-\mathrm{i}E_{3+}t} + \mathrm{e}^{-\mathrm{i}E_{3-}t})}{2} - \frac{(\mathrm{e}^{-\mathrm{i}E_{3+}t} - \mathrm{e}^{-\mathrm{i}E_{3-}t})}{2\Omega^{-1}\sqrt{\chi^2+\Omega^2}}. \tag{57}$$

Further, we can obtain the expectation value as

$$\langle S_z \rangle = \frac{1}{\chi^2+\Omega^2}\left[-\Omega^2 - \chi^2 \cos\left(2\sqrt{\chi^2+\Omega^2}t\right)\right]. \tag{58}$$

From Eqs. (41), (49), and (58), we can see that for the three models, expectation value $\langle S_z \rangle$ has the same form. This result shows that the discussion of the first model is still valid for the third model. A difference is that χ^2 used here is equal to $\chi_1^2 + \chi_2^2$. It should be noted that in the three models, generically, those states that possess the same $\langle S_z \rangle$ are different in phase. As an example, for the $\Omega = 0$ case at t_{m},

$$\psi_1(t) = \begin{pmatrix} \frac{-1-\mathrm{i}}{2} \\ 0 \\ \frac{1-\mathrm{i}}{2} \end{pmatrix}, \quad \psi_2(t) = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix},$$

$$\psi_3(t) = \begin{pmatrix} \frac{-\chi_2 - \mathrm{i}\chi_1}{\sqrt{2}\chi}\mathrm{e}^{\frac{-\mathrm{i}\chi_1\pi}{4\chi}} \\ 0 \\ \frac{1}{\sqrt{2}}\mathrm{e}^{\frac{-\mathrm{i}\chi_1\pi}{4\chi}} \end{pmatrix}. \tag{59}$$

5. Conclusion

We have investigated the MSI and the TSI for symmetric pure states. For the two-qubit pure states, we present the relations among the TSI, the MSI, the linear entropy, the concurrence, and the spin squeezing parameter, and find that the expectation value $\langle S_z \rangle$ plays a key role. According to $\langle S_z \rangle^2$, we can easily obtain those quantities above. Further, we apply the results to the two-qubit system implemented in three nonlinear interaction models. We find that $\langle S_z \rangle$ has the same form and its evolution cycle is determined by the energy space between the second excited state and the ground state. If and only if the strength of the external field is equal to or less than that of the nonlinear interaction, the maximal entangled state can occur.

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