Modified (2+1)-dimensional displacement shallow water wave system and its approximate similarity solutions*

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Recently, a new (2+1)-dimensional shallow water wave system, the (2+1)-dimensional displacement shallow water wave system (2DDSWWS), was constructed by applying the variational principle of the analytic mechanics in the Lagrange coordinates. The disadvantage is that fluid viscidity is not considered in the 2DDSWWS, which is the same as the famous Kadomtsev–Petviashvili equation and Korteweg–de Vries equation. Applying dimensional analysis, we modify the 2DDSWWS and add the term related to the fluid viscidity to the 2DDSWWS. The approximate similarity solutions of the modified 2DDSWWS (M2DDSWWS) is studied and four similarity solutions are obtained. For the perfect fluids, the coefficient of kinematic viscosity is zero, then the M2DDSWWS will degenerate to the 2DDSWWS.

Keywords: modified (2+1)-dimensional displacement shallow water wave system, viscidity, approximate similarity solutions, Kadomtsev–Petviashvili equation

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1. Introduction

Modern soliton theory is widely applied in almost all fields of physics such as field theory, condensed matter, plasma physics, fluid mechanics, optics, and particle and nuclear physics.[1−5] Moreover, in other scientific and technological fields, such as communications, chemistry, geography, atmospheric dynamics, and biology, the theory is widely employed as well. As a shallow water wave equation, the well-known Korteweg–de Vries (KdV) equation[6−8]

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

plays a key role in soliton theory.

However, when the KdV soliton is applied to describing the real observations for the shallow water solitary waves, there exist two disadvantages: 1) The KdV soliton can only move in one direction (right or left) while the real solitary waves can move in both directions. 2) The velocity of the KdV soliton is slightly larger than that actually observed in shallow water solitary waves. Some alternative descriptions, such as two types of the Boussinesq equations, are used to overcome the first disadvantage.[9] However, neither of the Boussinesq systems can overcome the second problem. The fact is that one of the models possesses larger velocity than the KdV equation, and the other one possesses the same velocity as the KdV system.

Customarily, the Euler coordinates are applied when people describe the shallow water waves. However, the Lagrange coordinates and the displacement of the studied system are often used in traditional analytic mechanics. Using the Lagrange coordinates and the variational principle of the analytic mechanics,[10] Zhong and Yao have derived the (1+1)-dimensional equation

$$q_{tt} - \frac{h^2}{3}q_{xxtt} + gh(3q_x - 1)q_{xx} = 0 \tag{2}$$

to describe the shallow water waves, where $q \equiv q(x,t)$ denotes the horizontal displacement of the shallow water system. It is interesting that the solitary wave solution of Eq. (2) can move in both directions and the velocity of the solitary wave is slightly less than that of the KdV soliton.

Similar to the (1+1)-dimensional case, some (2+1)-dimensional models such as the Kadomtsev–

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Petviashvili (KP) equation[11,12]

$$(u_\tau + 6uu_\xi + u_{\xi\xi\xi})_\xi + u_{\eta\eta} = 0, \tag{3}$$

are used to describe the (2+1)-dimensional shallow water waves. For the KP equation, there is another new problem besides the previous two (moving in a single direction and having the larger velocity than the real shallow water waves), i.e., x and y are nonsymmetric in the model equation. Recently, Ref. [13] has extended the method to research on (2+1)-dimensional shallow water waves and constructed a novel (2+1)-dimensional shallow water wave system called (2+1)-dimensional displacement shallow water wave system (2DDSWWS) by applying a variation principle of analytic mechanics in the Lagrange coordinate as

$$\begin{aligned} s_{tt} &+ gh(3s_x + w_y - 1)s_{xx} + gh(s_x + w_y - 1)w_{xy} \\ &- \frac{h^2}{3}(s_x + w_y)_{xtt} = 0, \end{aligned} \tag{4a}$$

$$\begin{aligned} w_{tt} &+ gh(3w_y + s_x - 1)w_{yy} + gh(s_x + w_y - 1)s_{xy} \\ &- \frac{h^2}{3}(s_x + w_y)_{ytt} = 0. \end{aligned} \tag{4b}$$

The general travelling wave solution of the model is obtained by means of an elliptic integral. In particular, for a special selection of one integral constant, the elliptic integral solution can be explicitly expressed by means of the Jacobi elliptic function which is the generalization of the solitary wave solution. The traditional (2+1)-dimensional shallow water wave models, such as the KP equation, are usually given in the Euler coordinates. The soliton solution of the KP equation, which is nonsymmetric for space variables x and y, can move only in one direction (the positive or negative x direction). It possesses a velocity slightly larger than that actually observed in the shallow water waves. The solitary wave solution of the new model is symmetric in the x- and y-directions and possesses a smaller velocity. Furthermore, the KP equation can also be derived from the 2DDSWWS by using the weak two-dimensional (2D) long waves assumed property. In light of the above consideration, the 2DDSWWS may be a more considerable and applicable model to describe (2+1)-dimensional shallow water waves and other physical phenomena.[13] The symmetries of the 1DDSWWS and 2DDSWWS have been studied in Refs. [14] and [15], respectively.

It is noted that the common fundamental assumption for the KdV equation, the KP equation, 1DDSWWS and 2DDSWWS is that water is an inviscous fluid of constant density. The 2DDSWWS cannot describe the viscous fluids exactly. We can modify the 2DDSWWS and add the terms related to the viscidity of the considered fluid. It is an important task of this paper to propose the modified 2DDSWWS (M2DDSWWS).

There have been many methods developed to research partial differential equations (PDEs). Studying similarity reductions of PDEs is remarkable in nonlinear physics.[16−18] Recently, several symmetry-based perturbation methods have been developed to deal with the perturbation properties of PDEs, such as the approximate Lie point symmetry method, the approximate conditional symmetry (ACS) method, and the approximate generalized conditional symmetry (AGCS) approach.[19,20]

2. Governing equations

Dimensional analysis (also called the factor-label method or the unit factor method) is a problem-solving method using the fact that any number or expression can be multiplied by one without changing its value. It is a useful technique. To modify the 2DDSWWS, we will recur to the dimensional analysis. From Eqs. (4), it is easy to know that the SI unit of every term in the equations is $\mathrm{m}\cdot\mathrm{s}^{-2}$. The corresponding dimension is $L\cdot T^{-2}$. Let us consider the incompressible Navier–Stokes equation in the form of[21]

$$\boldsymbol{v}_t + \boldsymbol{v}\cdot\bigtriangledown\boldsymbol{v} - \nu\bigtriangleup\boldsymbol{v} + \bigtriangledown p = 0, \tag{5}$$

where $\boldsymbol{v}$ is the velocity vector field, p is the pressure and ν is the coefficient of kinematic viscosity. The kinematic viscosity coefficient satisfies

$$\nu = \frac{\mu}{\rho}, \tag{6}$$

where μ is the coefficient of dynamic viscosity in units of $\mathrm{Pa}\cdot\mathrm{s}$, so the SI unit of ν is $\mathrm{m}^2\cdot\mathrm{s}^{-1}$. For (2+1)-dimensional fluid, Eq. (5) becomes

$$u_t + uu_x + vu_y + p_x - \nu(u_{xx} + u_{yy}) = 0, \tag{7a}$$

$$v_t + uv_x + vv_y + p_y - \nu(v_{xx} + v_{yy}) = 0. \tag{7b}$$

It is easy to know that the dimension of every term in Eqs. (7) is also $L\cdot T^{-2}$. The terms related to viscosity are $\nu\bigtriangleup u$ and $\nu\bigtriangleup v$ in Eqs. (7a) and (7b), respectively. Noticing that s_t and w_t in Eqs. (4) are equivalent to u and v in Eqs.(7) respectively, we can modify the 2DDSWWS to the form of

$$s_{tt} + gh(3s_x + w_y - 1)s_{xx} + gh(s_x + w_y - 1)w_{xy}$$

$$-\frac{h^2}{3}(s_x+w_y)_{xtt}-\nu(s_{txx}+s_{tyy})=0, \quad (8a)$$

$$w_{tt}+gh(3w_y+s_x-1)w_{yy}+gh(s_x+w_y-1)s_{xy} -\frac{h^2}{3}(s_x+w_y)_{ytt}-\nu(w_{txx}+w_{tyy})=0. \quad (8b)$$

Equations (7) can be treated as perturbed PDEs when the coefficient of kinematic viscosity ν is small. Similar to (2+1)-dimension, we can propose the modified 1DDSWWS (M1DDSWWS) as

$$q_{tt}-\frac{h^2}{3}q_{xxtt}+gh(3q_x-1)q_{xx}-\nu q_{txx}=0. \quad (9)$$

It is obvious that the M1DDSWWS and M2DDSWWS will reduce back to 1DDSWWS and 2DDSWWS respectively, when the fluids considered are perfect fluids.

3. Approximate symmetry

Perturbation analysis plays an important role in physics, especially in finding approximate analytical solutions to perturbed partial differential equations. In what follows, we will try to find the approximate similarity solutions of Eqs. (8). According to the perturbation theory, the solution can be be expressed as a series containing a small parameter, namely,

$$s=\sum_{j=0}^{\infty}s_j\nu^j=s_0+s_1\nu+s_2\nu^2+\cdots, \quad (10)$$

$$w=\sum_{j=0}^{\infty}w_j\nu^j=w_0+w_1\nu+w_2\nu^2+\cdots \quad (11)$$

Here, s_j and w_j are functions of x, y and t. Substituting Eqs. (10) and (11) into Eqs. (8), and identifying the coefficients of all the different powers of ν, we obtain arbitrary ranges of the expansion

$$\nu^k\ (k=0,1,2,\cdots):$$

$$s_{k,tt}-hgw_{k,xy}-\frac{1}{3}h^2s_{k,xxtt}-\frac{1}{3}h^2w_{k,xytt} -hgs_{k,xx}-(s_{k-1,xxt}+s_{k-1,yyt}) +hg\sum_{j=0}^{k}[w_{j,xy}w_{k-j,y}+s_{j,x}w_{k-j,xy} +3s_{j,x}s_{k-j,xx}+s_{j,xx}s_{k-j,y}]=0, \quad (12a)$$

$$w_{k,tt}-\frac{1}{3}h^2s_{k,xytt}-hgs_{k,xy}-hgw_{k,yy} -\frac{1}{3}h^2w_{k,yytt}-(w_{k-1,xxt}+w_{k-1,yyt}) +hg\sum_{j=0}^{k}[s_{j,x}s_{k-j,xy}+3w_{j,y}w_{k-j,yy} +s_{j,x}w_{k-j,yy}+s_{j,xy}w_{k-j,y}]=0, \quad (12b)$$

where $s_{-1}=0$ and $w_{-1}=0$. It is noted that Eqs. (12) with $k=0$ are just the original unperturbed equations.

To study the Lie point symmetry reductions of Eqs. (12), we can search for the transformations in the vector form of

$$V=X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+T\frac{\partial}{\partial t}+S_k\frac{\partial}{\partial s_k}+W_k\frac{\partial}{\partial w_k}, \quad (k=0,1,2); \quad (13)$$

where X, Y, T, S_k, and W_k are functions of x, y, t, s_0, s_1, s_2, w_0, w_1, and w_2. Expression (13) means that Eqs. (12) are invariant under the following transformations:

$$\{x,y,t,s_0,w_0,s_1,w_1,s_2,w_2\}\longrightarrow \{x+\epsilon X,y+\epsilon Y,t+\epsilon T,s_0+\epsilon S_0,w_0+\epsilon W_0,s_1 +\epsilon S_1,w_1+\epsilon W_1,s_2+\epsilon S_2,w_2+\epsilon W_2\}, \quad (14)$$

where ϵ is an infinitesimal parameter and the parameters X, Y, T, S_0, W_0, S_1, W_1, S_2, and W_2 are all functions of x, y, t, s_0, w_0, s_1, w_1, s_2 and w_2.

The symmetry of a generalized equation $u_t=K(x,t,u)$ is defined as $\sigma\equiv\sigma(x,t)$ satisfying[15]

$$\frac{\partial}{\partial\epsilon}\left[(u+\epsilon\sigma)_t-K(x,t,u+\epsilon\sigma)\right]\big|_{\epsilon=0}=0, \quad (15)$$

with u restricted by equation $u_t=K(x,t,u)$. For Eqs. (12) with $k=0,1,2$, the Lie point symmetry components σ_i $(i=1,2,3,4,5,6)$ are in the forms of

$$\begin{aligned}\sigma_1&=Xs_{0,x}+Ys_{0,y}+Ts_{0,t}-S_0,\\ \sigma_2&=Xw_{0,x}+Yw_{0,y}+Tw_{0,t}-W_0,\\ \sigma_3&=Xs_{1,x}+Ys_{1,y}+Ts_{1,t}-S_1,\\ \sigma_4&=Xw_{1,x}+Yw_{1,y}+Tw_{1,t}-W_1,\\ \sigma_5&=Xs_{2,x}+Ys_{2,y}+Ts_{2,t}-S_2,\\ \sigma_6&=Xw_{2,x}+Yw_{2,y}+Tw_{2,t}-W_2. \end{aligned} \quad (16)$$

Then, substituting the symmetry components (16) into the symmetry equations and removing the coefficients of different derivatives of the functions (s_0, s_1, s_2, w_0, w_1, and w_2) with the help of the original system (12), we have the determinate equations for the undetermined functions X, Y, T, S_0, W_0, S_1, W_1, S_2, and W_2. Finally, solving the determinate equations, we obtain

$$\begin{aligned}&T=C_1,\quad X=C_2,\quad Y=C_3,\\ &S_0=F_1+F_2,\quad W_0=F_3+F_4,\\ &S_1=\frac{1}{2}F_{1,yy}t^2+F_8t+C_4yt+C_5t+F_{12},\end{aligned}$$

$$S_2 = \frac{1}{6}F_{1,yyyy}t^3 + \frac{1}{2}F_{8,yy}t^2 + F_6 + F_7,$$
$$W_1 = \frac{1}{2}F_{3,xx}t^2 + F_5 t + C_6 xt + C_7 t + F_{11},$$
$$W_2 = \frac{1}{6}F_{3,xxxx}t^3 + \frac{1}{2}F_{5,xx}t^2 + F_9 t + F_{10}, \quad (17)$$

where C_1–C_7 are arbitrary constants, F_3, F_4, F_5, F_9, F_{10}, and F_{11} are arbitrary functions of x; F_1, F_2, F_6, F_7, F_{12}, and F_{14} are arbitrary functions of y. The concrete forms of symmetry components σ_i $(i = 1, 2, 3, 4, 5, 6)$ can be written as

$$\begin{aligned}
\sigma_1 &= C_2 s_{0,x} + C_3 s_{0,y} + C_1 s_{0,t} - F_1 t - F_2,\\
\sigma_2 &= C_2 w_{0,x} + C_3 w_{0,y} + C_1 w_{0,t} - F_3 t - F_4,\\
\sigma_3 &= C_2 s_{1,x} + C_3 s_{1,y} + C_1 s_{1,t} - \frac{1}{2}F_{1,yy}\\
&\quad - F_{14}t - C_4 ty - C_5 t - F_{12},\\
\sigma_4 &= C_2 w_{1,x} + C_3 w_{1,y} + C_1 w_{1,t} - \frac{1}{2}F_{3,xx}\\
&\quad - F_{13}t - C_6 tx - C_7 t - F_{11},\\
\sigma_5 &= C_2 s_{2,x} + C_3 s_{2,y} + C_1 s_{2,t} - \frac{1}{6}F_{1,yyyy}t^3\\
&\quad - \frac{1}{2}F_{14,yy}t^2 - F_6 t - F_7,\\
\sigma_6 &= C_2 w_{2,x} + C_3 w_{2,y} + C_1 w_{2,t} - \frac{1}{6}F_{3,xxxx}t^3\\
&\quad - \frac{1}{2}F_{13,xx}t^2 - F_9 t - F_{10}. \quad (18)
\end{aligned}$$

The vector field can be written as

$$\begin{aligned}
V &= C_1\frac{\partial}{\partial t} + C_2\frac{\partial}{\partial x} + C_3\frac{\partial}{\partial y} + (F_1 t + F_2)\frac{\partial}{\partial s_0}\\
&\quad + (F_3 t + F_4)\frac{\partial}{\partial w_0}\\
&\quad + \left(\frac{1}{2}F_{1,yy}t^2 + F_{14}t + C_4 yt + C_5 t + F_{12}\right)\frac{\partial}{\partial s_1}\\
&\quad + \left(\frac{1}{2}F_{3,xx}t^2 + F_{13}t + C_6 xt + C_7 t + F_{11}\right)\frac{\partial}{\partial w_1}\\
&\quad + \left(\frac{1}{6}F_{1,yyyy}t^3 + \frac{1}{2}F_{14,yy}t^2 + F_6 t + F_7\right)\frac{\partial}{\partial s_2}\\
&\quad + \left(\frac{1}{6}\frac{\mathrm{d}^4 F_3}{\mathrm{d}x^4}t^3 + \frac{1}{2}\frac{\mathrm{d}^2 F_{13}}{\mathrm{d}x^2}t^2 + F_9 t + F_{10}\right)\frac{\partial}{\partial w_2}.
\end{aligned}$$

The corresponding generators are

$$\begin{aligned}
&V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = \frac{\partial}{\partial y}, \quad V_4 = yt\frac{\partial}{\partial s_1},\\
&V_5 = xt\frac{\partial}{\partial w_1}, \quad V_7 = t\frac{\partial}{\partial w_1}, \quad V_8 = F_2\frac{\partial}{\partial s_0},\\
&V_9 = F_1 t\frac{\partial}{\partial s_0} + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}t^2\frac{\partial}{\partial s_1} + \frac{t^3}{6}F_{1,yyyy}\frac{\partial}{\partial s_2},\\
&V_{10} = F_4\frac{\partial}{\partial w_0}, \quad V_{11} = F_6 t\frac{\partial}{\partial s_2},\\
&V_{12} = F_3 t\frac{\partial}{\partial w_0} + \frac{1}{2}F_{3,xx}t^2\frac{\partial}{\partial w_1} + \frac{1}{6}F_{3,xxx}t^3\frac{\partial}{\partial w_2},\\
&V_{13} = F_7 t\frac{\partial}{\partial s_2}, \quad V_{14} = F_9 t\frac{\partial}{\partial w_2},\\
&V_{15} = F_{10}t\frac{\partial}{\partial w_2}, \quad V_{16} = F_{11}\frac{\partial}{\partial w_1}, \quad V_{17} = F_{12}\frac{\partial}{\partial s_1},\\
&V_{18} = tF_{13}\frac{\partial}{\partial w_1} + \frac{1}{2}F_{13,xx}t^2\frac{\partial}{\partial w_2},\\
&V_{19} = \frac{1}{2}F_{14,yy}t^2\frac{\partial}{\partial s_2} + tF_{14}\frac{\partial}{\partial s_1}. \quad (19)
\end{aligned}$$

And these generators constitute an enclosed Lie algebra.

4. Approximate similarity solutions

In this section, we will find approximate similarity reductions of the 2DDSWWS with the help of symmetries of the 2DDSWWS. In order to find the symmetry reductions, we have to solve the characteristic equations

$$\begin{aligned}
\frac{\mathrm{d}x}{X} &= \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}s_0}{S_0} = \frac{\mathrm{d}w_0}{W_0} = \frac{\mathrm{d}s_1}{S_1}\\
&= \frac{\mathrm{d}w_1}{W_1} = \frac{\mathrm{d}s_2}{S_2} = \frac{\mathrm{d}w_2}{W_2}, \quad (20)
\end{aligned}$$

where X, Y, T, S_0, W_0, S_1, W_1, S_2, and W_2 are given by Eqs. (17). From the symmetries presented in the last section, correspondingly, we obtain the following similarity solutions.

4.1. $C_2 \neq 0$, $C_3 \neq 0$, C_1 arbitrary

Identifying the symmetry components $\{\sigma_i, i = 1, 2, 3, 4, 5, 6\}$ expressed by Eqs. (19), we conclude that the following equations exist:

$$\begin{aligned}
s_0 &= \frac{C_1}{C_3^2}\int F_1 y\mathrm{d}y + \frac{\eta}{C_3}\int F_1\mathrm{d}y\\
&\quad + \frac{1}{C_3}\int F_2\mathrm{d}y + S_0(\xi,\eta),\\
w_0 &= \frac{C_1}{C_2^2}\int F_3 x\mathrm{d}x + \frac{tC_2 - C_1 x}{C_2^2}\int F_3\mathrm{d}x\\
&\quad + \frac{1}{C_2}\int F_4\mathrm{d}x + W_0(\xi,\eta),\\
s_1 &= \frac{C_1^2}{2C_3^3}\int F_{1,yy}y^2\mathrm{d}y + \frac{C_1}{C_3^2}\int F_{14}y\mathrm{d}y\\
&\quad + \frac{tC_3 - C_1 y}{C_3^2}\int F_{14}\mathrm{d}y\\
&\quad + \frac{1}{C_3}\int F_{12}\mathrm{d}y - \frac{(C_1^2 y^2 - t^2 C_3^2)F_{1,y}}{2C_3^3}\\
&\quad + S_1(\xi,\eta) + \frac{C_1(C_1 y - tC_3)F_1}{C_3^3},\\
w_1 &= \frac{C_1}{C_2^2}\int F_{13} x\mathrm{d}x + \frac{1}{C_2}\int F_{11}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\frac{tC_2-C_1x}{C_2^2}\int F_{13}\mathrm{d}x+W_1(\xi,\eta)\\
&+\frac{C_1^2}{2C_2^3}\int F_{3,xx}x^2\mathrm{d}x-\frac{(C_1^2x^2-t^2C_2^2)F_{3,x}}{2C_2^3}\\
&+\frac{C_1F_3(-tC_2+C_1x)}{C_2^3},\\
s_2 =\ &\frac{C_1^2}{2C_3^3}\int F_{14,yy}y^2\mathrm{d}y+\frac{1}{C_3}\int F_7\mathrm{d}y\\
&+\frac{\eta}{C_3}\int F_6\mathrm{d}y+\frac{C_1}{C_3^2}\int F_6y\mathrm{d}y\\
&+\frac{1}{6}\frac{(C_1y+\eta C_3)^3F_{1,yyy}}{C_3^4}+\frac{C_1^2(C_1y+\eta C_3)F_{1,y}}{C_3^4}\\
&-\frac{C_1(C_1^2F_1+F_{14}C_3^2\eta)}{C_3^4}\\
&-\frac{1}{2}\frac{C_1(C_1y+\eta C_3)^2F_{1,yy}}{C_3^4}\\
&+\frac{1}{2}\frac{\eta(\eta C_3+2C_1y)F_{14,y}}{C_3^2}+S_2(\xi,\eta),\\
w_2 =\ &\frac{1}{C_2}\int F_{10}\mathrm{d}x-\frac{-tC_2+C_1x}{C_2^2}\int F_9\mathrm{d}x\\
&+\frac{C_1}{C_2^2}\int F_9x\mathrm{d}x+\frac{C_1^2tF_{3,x}}{C_2^3}\\
&+\frac{C_1^2}{2C_2^3}\int F_{13,xx}x^2\mathrm{d}x-\frac{C_1t^2F_{3,xx}}{2C_2^2}+\frac{t^3F_{3,xxx}}{6C_2}\\
&-\frac{(C_1^2x^2-t^2C_2^2)F_{13,x}}{2C_2^3}+W_2(\xi,\eta)\\
&+\frac{C_1(-tC_2+C_1x)F_{13}}{C_2^3}-\frac{C_1^3F_3}{C_2^4},
\end{aligned}
\tag{21}
$$

where the group invariants ξ and η are determined by

$$\xi=\frac{xC_3-C_2y}{C_3},\quad \eta=\frac{tC_3-C_1y}{C_3}.$$

4.2. $C_2\neq 0$, $C_3=0$, C_1 arbitrary

Identifying the expressions of symmetry components $\{\sigma_i, i=1,2,3,4,5,6\}$ with $C_3=0$, we obtain the second type of similarity solution

$$
\begin{aligned}
s_0 =\ &\frac{1}{2}\frac{F_1x^2C_1}{C_2^2}+\frac{F_1\xi\ x}{C_2}+\frac{F_2x}{C_2}+S_0(\xi,\eta),\\
w_0 =\ &\frac{C_1}{C_2^2}\int F_3x\mathrm{d}x+\frac{\xi}{C_2}\int F_3\mathrm{d}x\\
&+\frac{1}{C_2}\int F_4\mathrm{d}x+W_0(\xi,\eta),\\
s_1 =\ &\left(\frac{x\xi^2}{2C_2}+\frac{C_1x^2\xi}{2C_2^2}+\frac{C_1^2x^3}{6C_2^3}\right)F_{1,yy}+\frac{F_{14\xi x}}{C_2}\\
&+S_1(\xi,\eta)+\frac{x(2C_2F_{12}+C_1xF_{14})}{2C_2^2},\\
w_1 =\ &\frac{\xi}{C_2}\int F_{13}\mathrm{d}x+\frac{C_1^2}{2C_2^3}\int F_{3,xx}x^2\mathrm{d}x\\
&+\frac{C_1}{C_2^2}\int F_{13}x\mathrm{d}x+\frac{\int F_{11}\mathrm{d}x}{C_2}\\
&+\left(\frac{\xi^2}{2C_2}+\frac{C_1x\xi}{C_2^2}\right)F_{3,x}-\frac{C_1F_3\xi}{C_2^2}+W_1(\xi,\eta),\\
s_2 =\ &\frac{1}{24C_2^4}x(C_1x+2C_2\xi)(2C_2^2\xi^2\\
&+2C_1C_2\xi x+C_1^2x^2)F_{1,yyyy}+\frac{F_7x}{C_2}\\
&+\frac{1}{6C_2^3}x(C_1^2x^2+3C_1C_2\xi x+3C_2^2\xi^2)F_{14,yy}\\
&+S_2(\xi,\eta)+\frac{F_6x(C_1x+2C_2\xi)}{2C_2^2},\\
w_2 =\ &\frac{C_1}{C_2^2}\int F_9x\mathrm{d}x+\frac{C_1^2}{2C_2^3}\int F_{13,xx}x^2\mathrm{d}x\\
&+\frac{C_2t-C_1x}{C_2^2}\int F_9\mathrm{d}x+\frac{\int F_{10}\mathrm{d}x}{C_2}\\
&+\frac{C_1^2tF_{3,x}}{C_2^3}+W_2(\xi,\eta)-\frac{C_1t^2F_{3,xx}}{2C_2^2}+\frac{F_{3,xxx}t^3}{6C_2}\\
&-\frac{(C_1^2x^2-C_2^2t^2)F_{13,x}}{2C_2^3}\\
&+\frac{C_1(C_1C_2F_{13}x-C_1^2F_3-C_2^2F_{13}t)}{C_2^4},
\end{aligned}
\tag{22}
$$

where ξ and η are group invariants given by

$$\xi=\frac{C_2t-C_1x}{C_2},\quad \eta=y.$$

It is easy to know that $Y=C_3=0$ leads to y being a group invariant.

4.3. $C_3\neq 0$, $C_2=0$, C_1 arbitrary

Letting $C_2=0$ and letting the symmetry components $\{\sigma_i,\ i=1,2,3,4,5,6\}$ be zero, we obtain

$$
\begin{aligned}
s_0 =\ &\frac{C_1}{C_3^2}\int yF_1\mathrm{d}y-\frac{-tC_3+C_1y}{C_3^2}\int F_1\mathrm{d}y+\frac{1}{C_3}\int F_2\mathrm{d}y+S_0(\xi,\eta),\\
w_0 =\ &-\frac{1}{2}\frac{F_3C_1y^2}{C_3^2}+\frac{F_3yt}{C_3}+\frac{F_4y}{C_3}+W_0(\xi,\eta),\\
s_1 =\ &\frac{\eta}{C_3}\int F_{14}\mathrm{d}y+\frac{1}{C_3}\int F_{12}\mathrm{d}y+\frac{C_1}{C_3^2}\int yF_{14}\mathrm{d}y+\frac{1}{2}\frac{C_1^2}{C_3^3}\int y^2F_{1,yy}\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{\eta(2C_1y+\eta C_3)F_{1,y}}{C_3^2}+S_1(\xi,\eta)-\frac{C_1\eta F_1}{C_3^2},\\
w_1 =& \frac{y(C_1^2y^2+3C_1y\eta C_3+3\eta^2C_3^2)F_{3,xx}}{6C_3^3}+W_1(\xi,\eta)+\frac{F_{11}y}{C_3}+\frac{F_{13}y(2\eta C_3+C_1y)}{2C_3},\\
s_2 =& \frac{1}{C_3}\int F_7\mathrm{d}y+\frac{C_1}{C_3^2}\int yF_6\mathrm{d}y+\frac{1}{2}\frac{C_1^2}{C_3^3}\int y^2F_{14,yy}\mathrm{d}y+\frac{\eta}{C_3}\int F_6\mathrm{d}y\\
&+\frac{C_1^2(C_1y+\eta C_3)F_{1,y}}{C_3^4}-\frac{1}{2}\frac{C_1(C_1y+\eta C_3)^2F_{1,yy}}{C_3^4}+\frac{1}{6}\frac{(C_1y+\eta C_3)^3F_{1,yyy}}{C_3^4}\\
&+\frac{1}{2}\frac{\eta(2C_1y+\eta C_3)F_{14,y}}{C_3^2}+S_2(\xi,\eta)-\frac{C_1(C_1^2F_1+F_{14}C_3^2\eta)}{C_3^4},\\
w_2 =& \frac{y(2\eta C_3+C_1y)(2\eta^2C_3^2+2C_1y\eta C_3+C_1^2y^2)F_{3,xxxx}}{24C_3^4}+W_2(\xi,\eta)\\
&+\frac{y(C_1^2y^2+3C_1y\eta C_3+3\eta^2C_3^2)F_{13,xx}}{6C_3^3}+\frac{y(2\eta C_3+C_1y)F_9}{2C_3^2}+\frac{F_{10}y}{C_3},
\end{aligned}
\tag{23}
$$

where group invariants ξ and η are determined by

$$
\begin{aligned}
\xi &= x,\\
\eta &= \frac{tC_3-C_1y}{C_3}.
\end{aligned}
$$

Obviously, $X=C_2=0$ leads to x being invariant.

4.4. $C_2=0$, $C_3=0$, $C_1\neq 0$

Substituting $C_2=0$ and $C_3=0$ into Eqs. (24) and letting the six symmetry components $\{\sigma_i,\ i=1,2,3,4,5,6\}$ be zero, one can obtain the following final type of similarity solution:

$$
\begin{aligned}
s_0 =& \frac{t(F_1t+2F_2)}{2C_1}+S_0(\xi,\eta),\\
w_0 =& \frac{t(F_3t+2F_4)}{2C_1}+W_0(\xi,\eta),\\
s_1 =& \frac{t(F_{1,yy}t^2+3tF14+6F12)}{6C_1}+S_1(\xi,\eta),\\
w_1 =& \frac{t(F_{3,yy}t^2+3tF_{13}+6F_{11})}{6C_1}+W_1(\xi,\eta),\\
s_2 =& \frac{t(F_{1,yyyy}t^3+4F_{14,yy}t^2+12F_6t+24F_7)}{24C_1}\\
&+S_2(\xi,\eta),\\
w_2 =& \frac{t(F_{3,yyyy}t^3+4F_{13,xx}t^2+12F_9t+24F_{10})}{24C_1}\\
&+W_2(\xi,\eta),
\end{aligned}
\tag{24}
$$

where group invariants ξ and η are determined by $\xi=x,\eta=y$. Obviously, $X=Y=0$ leads to x and y being invariants.

5. Summary and discussion

Recently, Ref. [13] constructed the 2DDSWWS by applying variation principle in analytic mechanics in the Lagrange coordinates. Compared with some traditional (2+1)-dimensional shallow water wave systems such as the KP description in the Euler coordinates, the 2DDSWWS has its own advantages and may be a more considerable and applicable model to describe (2+1)-dimensional shallow water waves and other physical phenomena. Its disadvantage is that the 1DDSWWS and 2DDSWWS cannot describe the viscous fluids exactly. Modifying the 1DDSWWS and 2DDSWWS, we propose the M1DDSWWS and M2DDSWWS in this paper. They can describe the viscous fluids. For the fluid with small coefficient of kinematic viscosity, we can study the approximate similarity solutions of the M2DDSWWS. Four similarity solutions are obtained. For the perfect fluids, the coefficient of kinematic viscosity is zero, and the M1DDSWWS and the M2DDSWWS will degenerate to 1DDSWWS and 2DDSWWS, respectively.

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