Knotted pictures of the Bell bases on the surface of a trivial torus

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By means of the method of torus knot theory, this paper gives the complete processes of obtaining the knotted pictures of four Bell bases from the knotted pictures of four basic two qubit states.

Keywords: knotted picture, torus knot theory, Bell bases, surface of trivial torus

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1. Introduction

Knot theory is a powerful mathematical tool for researching into physical phenomena.[1−3] Although quantum entanglement in quantum mechanics and classical twist in knot theory are two basically different phenomena, we have found that there is an intimate relationship between quantum entanglement and knot theory.[4−6] By comparing the covariance correlation tensor in the theory of quantum networks and the Alexander relation matrix in the theory of knot crystals, we have found that there is a one-to-one correspondence between four Bell bases and four oriented links of the linkage 4_1 in knot theory.[4,5] We have also found a one-to-one correspondence between GHZ states and the oriented links.[6] Hence, we have used the classical language of knot theory to describe the property of the algebraic structure of quantum entanglement, and revealed the interrelation of two seemingly different phenomena. Furthermore, we have given the knotted picture of the complete quantum measurement process of quantum teleportation.[7] The description of quantum mechanics in terms of classical knot theory is a newly developing branch of quantum mechanics, i.e., topological quantum mechanics. Its significance and application are still in the searching and progressing state. Based on our previous studies on the correspondence between Bell bases and the linkage 4_1 in knot theory,[4,5] in this paper we shall show that there exists a very simple and clear way of obtaining the linkage 4_1 by using the method of torus knot theory. The significance of this work is that by means of this method we can easily study the complex problems about the correspondences between n-qubit ($n > 2$) quantum states and their corresponding knotted pictures. Thus, this method provides an alternative and elegant way to study quantum entanglement.

It is well known that for the system consisting of two spin –1/2 particles 1 and 2, there are four well-known entangled Bell bases, which are[4,5]

$$\begin{aligned}
|\,\Phi^+\rangle_{12} &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle_{12}+|\downarrow\downarrow\rangle_{12}),\\
|\,\Phi^-\rangle_{12} &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle_{12}-|\downarrow\downarrow\rangle_{12}),\\
|\,\Psi^+\rangle_{12} &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_{12}+|\downarrow\uparrow\rangle_{12}),\\
|\,\Psi^-\rangle_{12} &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_{12}-|\downarrow\uparrow\rangle_{12}).
\end{aligned} \tag{1}$$

Previously, we have introduced a unified expression $|\,Bell\rangle_{12}^{i}$ to express the above four Bell bases[7]

$$\begin{aligned}
i = 1 &\longleftrightarrow |\,\Phi^+\rangle_{12},\\
i = 2 &\longleftrightarrow |\,\Phi^-\rangle_{12},\\
i = 3 &\longleftrightarrow |\,\Psi^+\rangle_{12},\\
i = 4 &\longleftrightarrow |\,\Psi^-\rangle_{12}.
\end{aligned} \tag{2}$$

A two qubit state is usually represented by $|\,c, w\rangle$, the left qubit is called the controlled qubit, the right qubit is called the worked qubit. Bell bases $|\,Bell\rangle_{12}^{i}$ are two qubit states, in which 1 corresponds to the controlled qubit, 2 corresponds to the worked qubit. Hence, we can rewrite $|\,Bell\rangle_{12}^{i}$ as $|\,Bell\rangle_{cw}^{i}$. The values of the qubit are taken to be zero and one for $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively. Therefore, for the four basic two qubit states we have

$$|\uparrow\uparrow\rangle = |\,0, 0\rangle,$$

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$$
\begin{aligned}
|\downarrow\uparrow\rangle &= |\,1,0\rangle,\\
|\uparrow\downarrow\rangle &= |\,0,1\rangle,\\
|\downarrow\downarrow\rangle &= |\,1,1\rangle.
\end{aligned} \tag{3}
$$

The linkage 4_1 corresponding to $\{|\ Bell\rangle_{12}^{i}\}$ are shown in Fig. 1.[8]

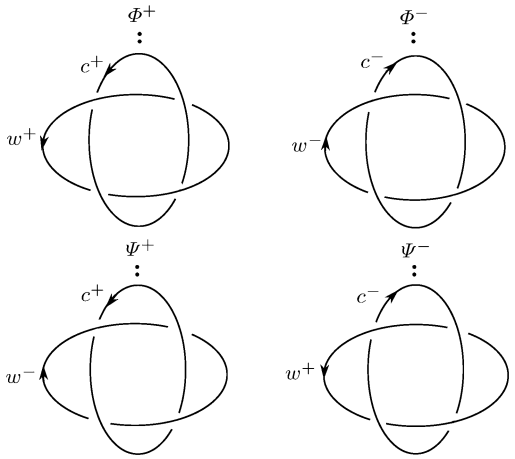


Fig. 1. The knotted pictures of the four Bell bases $\{|\ Bell\rangle_{cw}^{i}\}$.

In Ref. [9], we have pointed out that for the single qubit state, the knotted pictures of two basic states $|\uparrow\rangle$ and $|\downarrow\rangle$ are the circles with counter clockwise and clockwise orientations respectively. Naturally, we shall use two disconnected circles to represent four basic two qubit states $|\ c,w\rangle^{i}$. We let the inner circle represent the controlled qubit, the outer circle represent the worked qubit, the counter clockwise orientation corresponds to the qubit with value zero, $|\uparrow\rangle$, whereas the clockwise orientation corresponds to the qubit with value one, $|\downarrow\rangle$. Then we readily obtain the knotted pictures $\{|\ c,w\rangle^{i}\}$ of the four basic two qubit states as shown in Fig. 2.[10]

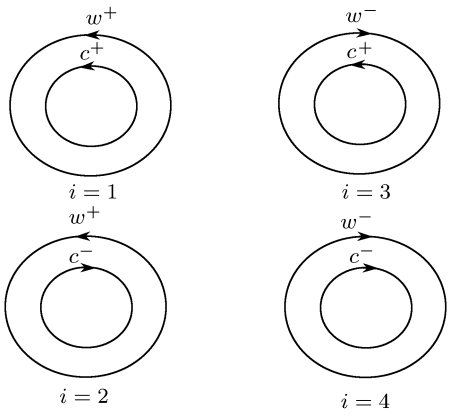


Fig. 2. Knotted pictures of the four basic two qubit states.

From Figs. 1 and 2, we can clearly see that each single qubit state corresponds to a trivial knot (orientated circle), and the different entangled two qubit states correspond to the resulting links of the combined trivial knots with different kinds of twist, namely, entangled states correspond to links with different kinds of twist. For $n \geq 2$ qubit states, the corresponding links have $n \geq 2$ components. This is the characteristic of our work, which is quite different from previous works of other authors, e.g., Li *et al.*[11] used one component link, the trefoil knot, to represent $n = 3$ qubit states.

All of the above-mentioned knotted pictures are supported on a plane. We have introduced five kinds of surgical operations: cutting, glue, twisting, untwisting and denominator operations.[10] In terms of these five surgical operations, we have successfully obtained four oriented links of the linkage 4_1 in knot theory from the corresponding knotted pictures of the four basic two qubit states, and *vice versa*. However, the process is rather complex, not so simple that we can easily grasp it. In this paper, we will use the method of torus knot[12] in knot theory to represent all the knotted pictures on the surface of a trivial torus. The knotted pictures of four basic qubit states are simply represented as four kinds of two oriented parallel lines on the surface of a cylinder with circular cross section. Then twisting the cylinder an angle of 4π, and using the method of constructing a trivial torus, we can easily obtain the four oriented links of the linkage 4_1 on the trivial torus.

The contents of this paper are organized as follows: in Section 2 we give a brief review of our work obtaining four Bell bases from the corresponding four basic two qubit states under the combined action of Hardmard gate and CNOT gate. In Section 3 we introduce the fundamental concepts of the torus knot (or link), and the basic properties of the (q,r) torus knot. In Section 4 we give the complete processes of obtaining the linkage 4_1 from the knotted pictures of four basic two qubit states.

2. Combined action of the Hadamard gate and the CNOT gate on the basic two qubit states

In Ref. [10] we have discussed how to obtain the four Bell bases $|\ Bell\rangle_{cw}^{i}$ by the combined application of the Hadamard gate and the CNOT gate on the four basic two qubit states $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. Furthermore, we have given the knotted picture

of the process of obtaining $\mid Bell\rangle^{i}_{cw}$ from the basic two qubit states $\mid c,w\rangle^{i}$ by using the knotted pictures of the Hadamard gate and the CNOT gate.

The matrix expression of the Hadamard gate is

$$H=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{4}$$

The matrix expressions of basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ are

$$|\uparrow\rangle=\begin{pmatrix}1\\ 0\end{pmatrix},\qquad |\downarrow\rangle=\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{5}$$

Application of H on $|\uparrow\rangle$ and $|\downarrow\rangle$ gives

$$\begin{aligned} H\,|\uparrow\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle), \end{aligned} \tag{6}$$

$$\begin{aligned} H\,|\downarrow\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}(|\uparrow\rangle-|\downarrow\rangle). \end{aligned} \tag{7}$$

Since we only apply the Hadamard gate to the controlled qubit, hereafter we use the symbol H_c instead of H to represent the Hadamard gate.

The result of the action of the CNOT gate on a two qubit state $\mid c,w\rangle$ is

$$\mid c,w\rangle \longrightarrow \mid c,c\oplus w\rangle, \tag{8}$$

where the operation $\oplus$ means mod 2 addition, i.e., $1\oplus 1=0$. Hence, when the value of the controlled qubit equals zero, the action of the CNOT gate does not change the value of the worked qubit, i.e., $\mid 0,w\rangle \longrightarrow \mid 0,0\oplus w\rangle = \mid 0,w\rangle$. Then we have

$$\begin{aligned} CNOT\,|\uparrow\uparrow\rangle &= CNOT\mid 0,0\rangle\\ &= \mid 0,0\oplus 0\rangle\\ &= \mid 0,0\rangle\\ &= |\uparrow\uparrow\rangle,\\ CNOT\,|\uparrow\downarrow\rangle &= CNOT\mid 0,1\rangle\\ &= \mid 0,0\oplus 1\rangle\\ &= \mid 0,1\rangle\\ &= |\uparrow\downarrow\rangle. \end{aligned} \tag{9}$$

i.e., when the controlled qubit is $|\uparrow\rangle$, the CNOT gate has no effect on the two qubit state $\mid c,w\rangle$. On the contrary, when the value of the controlled qubit equals one, the action of the CNOT gate does change the value of the worked qubit, $\mid 1,w\rangle \longrightarrow \mid 1,1\oplus w\rangle$. That is, the value of the worked qubit changes from zero to one or from one to zero. Then we have

$$\begin{aligned} CNOT\,|\downarrow\uparrow\rangle &= CNOT\mid 1,0\rangle\\ &= \mid 1,1\oplus 0\rangle\\ &= \mid 1,1\rangle\\ &= |\downarrow\downarrow\rangle,\\ CNOT\,|\downarrow\downarrow\rangle &= CNOT\mid 1,1\rangle\\ &= \mid 1,1\oplus 1\rangle\\ &= \mid 1,0\rangle\\ &= |\downarrow\uparrow\rangle. \end{aligned} \tag{10}$$

The action of the CNOT gate is schematically represented by Fig. 3.

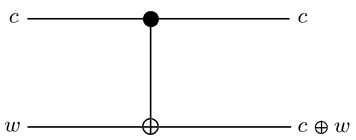


Fig. 3. Schematic diagram of the action of the CNOT gate.

Since the CNOT gate only changes the value of the worked qubit, and the value of the worked qubit depends on the value of the controlled qubit, hereafter we use the notation $CNOT_{w(c)}$ instead of $CNOT$.

From the combined action of the Hadamard gate and the CNOT gate on the two qubit state $\mid c,w\rangle$, we have obtained the four Bell bases $\mid Bell\rangle^{i}_{cw}$. Firstly we apply H_c, then we apply $CNOT_{w(c)}$ gate to the resulting two qubit state $\mid H_c c,w\rangle$, i.e., we obtain $CNOT_{w(c)} \mid H_c c,w\rangle$. The combined action of the Hadamard gate H_c and the CNOT gate $CNOT_{w(c)}$ on the two qubit state $\mid c,w\rangle$ can be schematically represented by Fig. 4.

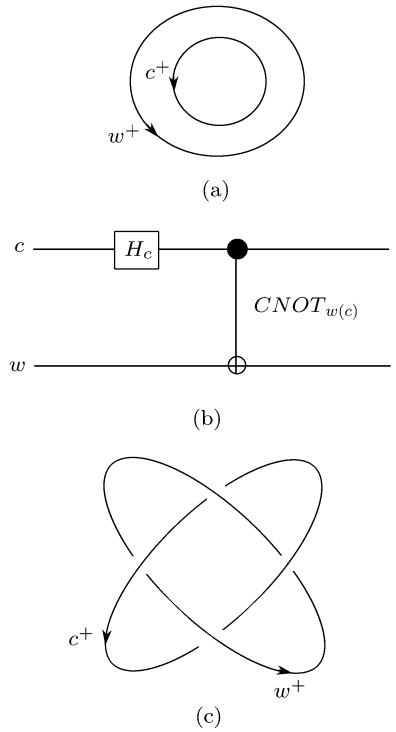


Fig. 4. The combined action of the Hadamard gate and the CNOT gate on the two qubit state $\mid c, w\rangle$: (a) knotted picture of $|c, w\rangle$, (b) schematic diagram of the combined action of the Hadamard gate and the CNOT gate, (c) knotted picture of $|Bell\rangle_{c^+w^+}$.

From Eqs. (6) and (7), we readily see

$$\begin{aligned} & CNOT_{w(c)}H_c\,|\uparrow\uparrow\rangle_{cw} \\ &= CNOT_{w(c)}\frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle)_c\,|\uparrow\rangle_w \\ &= CNOT_{w(c)}\frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle+|\downarrow\uparrow\rangle)_{cw} \\ &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle+|\downarrow\downarrow\rangle)_{cw} \\ &= \mid \Phi^+\rangle_{cw} \\ &= \mid Bell\rangle^1_{cw}. \end{aligned} \tag{11}$$

Similarly, we have

$$\begin{aligned} & CNOT_{w(c)}H_c\,|\downarrow\uparrow\rangle_{cw} \\ &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle-|\downarrow\downarrow\rangle)_{cw} \\ &= \mid \Phi^-\rangle_{cw} \\ &= \mid Bell\rangle^2_{cw}, \end{aligned} \tag{12}$$

$$\begin{aligned} & CNOT_{w(c)}H_c\,|\uparrow\downarrow\rangle_{cw} \\ &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle)_{cw} \\ &= \mid \Psi^+\rangle_{cw} \\ &= \mid Bell\rangle^3_{cw}, \end{aligned} \tag{13}$$

$$\begin{aligned} & CNOT_{w(c)}H_c\,|\downarrow\downarrow\rangle_{cw} \\ &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)_{cw} \\ &= \mid \Psi^-\rangle_{cw} \\ &= \mid Bell\rangle^4_{cw}. \end{aligned} \tag{14}$$

We can introduce a unified expression $\mid c, w\rangle^i$ to represent the four basic two qubit states $|\uparrow\uparrow\rangle_{cw}$, $|\downarrow\uparrow\rangle_{cw}$, $|\uparrow\downarrow\rangle_{cw}$, and $|\downarrow\downarrow\rangle_{cw}$, where $i = 1$ corresponds to $|\uparrow\uparrow\rangle_{cw}$, $i = 2$ corresponds to $|\downarrow\uparrow\rangle_{cw}$, $i = 3$ corresponds to $|\uparrow\downarrow\rangle_{cw}$, and $i = 4$ corresponds to $|\downarrow\downarrow\rangle_{cw}$. Using the unified expressions $\mid c, w\rangle^i$ and $\mid Bell\rangle^i_{cw}$, we can write Eqs. (11)–(14) with the following equation:

$$CNOT_{w(c)}H_c \mid c, w\rangle^i = \mid Bell\rangle^i_{cw}. \tag{15}$$

Using the language of knot theory, we can write Eq. (15) in the following form:

$$\{CNOT_{w(c)}\}\{H_c\}\{\mid c, w\rangle^i\} = \{\mid Bell\rangle^i_{cw}\}, \tag{16}$$

where $\{CNOT_{w(c)}\}$ and $\{H_c\}$ are the knotted pictures of $CNOT_{w(c)}$ gate and H_c gate respectively, and $\{\mid c, w\rangle^i\}$ and $\{\mid Bell\rangle^i_{cw}\}$ are the knotted pictures of the basic two qubit states $\mid c, w\rangle^i$ and the Bell bases $\mid Bell\rangle^i_{cw}$ respectively. We have already found $\{\mid Bell\rangle^i_{cw}\}$ in Ref. [2], which are four oriented links of the linkage 4_1 in knot theory. Since $\{CNOT_{w(c)}\}$ depends on the controlled qubit, we must consider $\{CNOT_{w(|\uparrow\rangle_c)}\}$ and $\{CNOT_{w(|\downarrow\rangle_c)}\}$ for the cases $|\uparrow\rangle_c$ and $|\downarrow\rangle_c$ separately . In Ref. [10] we have already given the explicit expressions of $\{H_c\}$, $\{CNOT_{w(|\uparrow\rangle_c)}\}$ and $\{CNOT_{w(|\downarrow\rangle_c)}\}$ in terms of surgical operations described in knot theory. Since these expressions are somewhat complex, we would not repeat them here.

3. Torus knots[11]

A knot(or link) is a torus knot if it is equivalent to a knot (or a link) that can be drawn without any points of intersection on the trivial torus. The trivial torus is a solid T obtained by rotating around the y-axis the circle m: $(x-2)^2+y^2=1$ on the xy-plane, which has as its centre of (2,0) and radius of 1 unit, as shown in Fig. 5.[11]

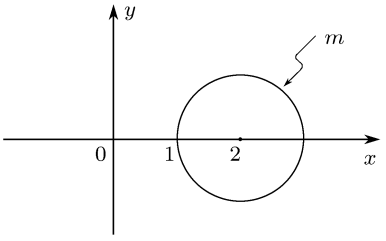


Fig. 5. Obtaining trivial torus by rotating around the y-axis the circle m: $(x-2)^2+y^2=1$.

An alternative way to construct the trivial torus is to take a primary cylinder with the unit circle C_1 as its base and the unit circle C_2 as its top (Fig. 6(a)), then bend the primary cylinder (Fig. 6(b)) and glue C_1 and C_2 together so that the central axis of the primary cylinder C becomes the trivial knot (Fig. 6(c)).[11]

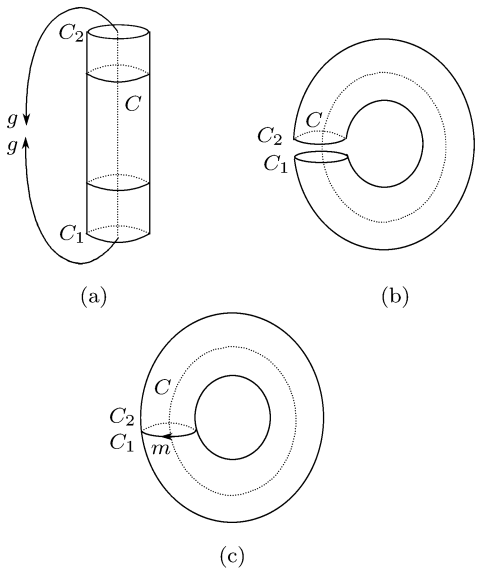


Fig. 6. An alternative way to construct the trivial torus. (a) Primary cylinder with the unit circle C_1 as its base and the unit circle C_2 as its top, (b) bending of the primary cylinder such that the central axis C becomes a circular arc, (c) gluing C_1 and C_2 together such that C becomes a trivial knot (circle).

Now we choose the above primary cylinder with a height of 1 unit and a base of a unit circle in xy-plane, then assign to the base C_1 and to the top C_2 the r points ($r = 0, 1, \ldots, r-1$). At base C_1, there are points $A_0 = (1, 0, 0)$, $A_1 = (\cos(2\pi/r), \sin(2\pi/r), 0)$, $\ldots$, $A_{r-1} = (\cos(2(r-1)\pi/r),\ \sin(2(r-1)\pi/r), 0)$; at top C_2, there are points $B_0 = (1, 0, 1)$, $B_1 = (\cos(2\pi/r),\ \sin(2\pi/r), 1)$, $\ldots$, $B_{r-1} = (\cos(2(r-1)\pi/r),\ \sin(2(r-1)\pi/r), 1)$. Connect the points A_k and B_k ($k = 0, 1, \ldots, r-1$) on the primary cylinder by the segments α_k. We have r parallel segments on the surface of the primary cylinder, as shown in Fig. 7(a).

Next, keeping the base C_1 fixed, we give the whole primary cylinder a twist by rotating the top about the z-axis by an angle of $2\pi q/r$ (q is either a positive or a negative integer), as shown in Fig. 7(b) and Fig. 7(c) for positive q and negative q respectively. Then we use the glue operation as that shown in Fig. 6(b) to identify the point $(x, y, 0)$ of C_1 to the point $(x, y, 1)$ of C_2. Finally, we create a single trivial torus T, with the r segments $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ that have been transformed into a knot (or link) on its surface.[11] This knot (or link) is called a (q, r)-torus knot (or link) and is denoted by $K_{q,r}$, when $q = 4$, $r = 2$, i.e., there are originally $r = 2$ parallel segments on the surface of the primary cylinder and the twisting angle is $2\pi q/r = 4\pi$, the knotted picture is shown in Fig. 8.

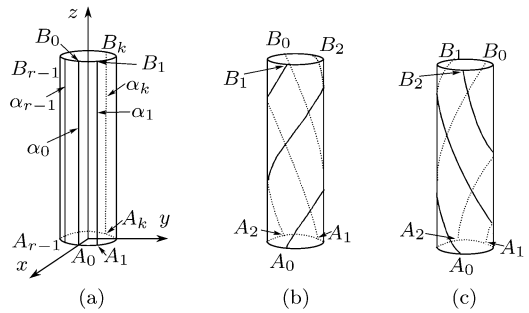


Fig. 7. Operations applied on the primary cylinder. (a) Assigning to the base C_1 the r points A_0, $A_1, \ldots$, A_{r-1}, and assigning to the top C_2 the r points B_0, B_1, $\ldots, B_{r-1}$; (b) twisting the whole primary cylinder by rotating the top about the z-axis by an angle of $2\pi q/r (q > 0)$; (c) twisting the whole primary cylinder by rotating the top about the z-axis by an angle of $2\pi q/r (q < 0)$.

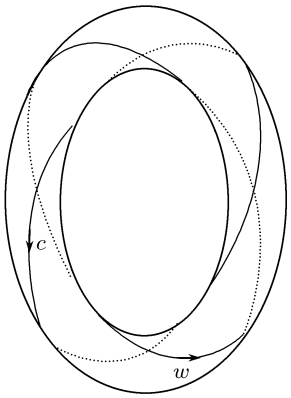


Fig. 8. Knotted picture of $K_{4,2}$.

We again consider the circle m in Fig. 5, and further suppose K is the boundary of a disk that lies on T, as shown in Fig. 9.

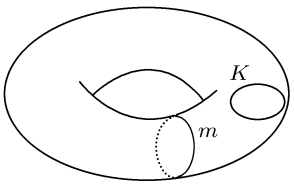


Fig. 9. Circle m in Fig. 5 and the boundary of a disk K on the trivial torus.

These trivial knots m and K are not torus knots, and we consider them to be corresponding to the case $r = 0$, and say that m is a (1,0) torus knot $K_{1,0}$, whereas we call K a (0,0) torus knot $K_{0,0}$.

In the particular case of $q = 0$, i.e., the twisting angle $2\pi q/r = 0$, obviously the resulting torus knot

$K_{0,2}$ is two disconnected circles, i.e. two trivial knots. Since every segment can have two possible orientations, upward and downward, two orientated parallel segments have four different kinds of orientation. Correspondingly, there are four different kinds of orientation for the two trivial knots, a counterclockwise circle corresponds to the upward segment, whereas a clockwise circle corresponds to the downward segment. Clearly $K_{0,2}$ are just the knotted pictures of the four basic two qubit states. Figure 10 shows the knotted picture of $K_{0,2}$ for the case $i = 1$.

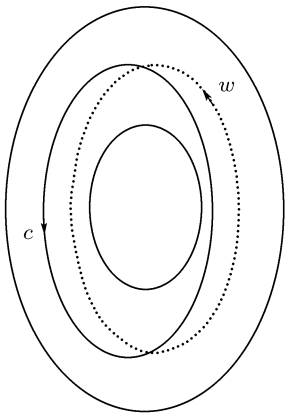


Fig. 10. Knotted picture of $K_{0,2}$.

As mentioned above, when applying the surgical operation, twisting an angle $2\pi q/r = 4\pi$, we can easily obtain $K_{4,2}$ from $K_{0,2}$ just by representing it in terms of two parallel segments on the surface of the cylinder.

It is very interesting to point out that when we quit the trivial torus, to our surprise, we obtain the linkage 4_1. Namely, when we support the linkage 4_1 on the trivial torus we will readily obtain $K_{4,2}$. Hence, we conclude that the linkage 4_1 corresponds to $K_{4,2}$, which is just a special case of $K_{q,r}$.

4. The complete process of obtaining the linkage 4_1 from the theory of a torus knot

In Section 3 we have described the process of obtaining $K_{4,2}$ from $K_{0,2}$. Since on the surface of the plane, the former corresponds to the linkage 4_1, the latter corresponds to two trivial knots, i.e., the knotted pictures of the four basic two qubit states, we readily obtain the complete processes of obtaining the linkage 4_1 from the theory of a torus knot as follows:

Step 1 Take the primary cylinder with a height of 1 unit and a base of a unit circle C_1 in xy-plane and the unit circle C_2 as its top, as shown in Fig. 6(a).

Step 2 Assign the two points A_0 and A_1 to the base C_1 and the two points B_0 and B_1 to the top C_2, where $A_0 = (1, 0, 0)$, $A_1 = (-1, 0, 0)$, $B_0 = (1, 0, 1)$, $B_1 = (-1, 0, 1)$. Connect the points A_k and B_k $(k = 0, 1)$ on the primary cylinder by the segments α_k. Hence, we have two parallel segments on the surface of the cylinder. There are four different kinds of two orientated segments corresponding to four basic two qubit states. For example, we choose $i = 1$, as shown in Fig. 11(a).

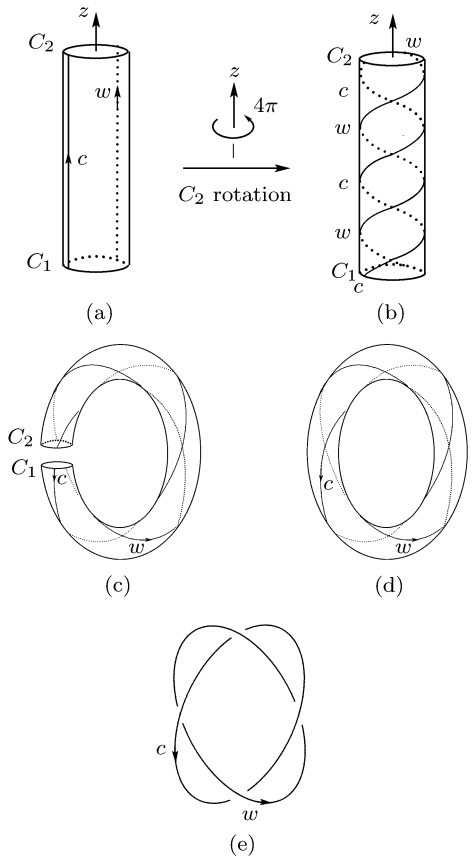


Fig. 11. The complete process of obtaining the linkage 4_1. (a) Choosing $r = 2$, for case $i = 1$, both the lines c and w are oriented upward; (b) twisting an angle 4π, i.e., $q = 4$; (c) bending the cylinder; (d) completing the gluing operation; (e) quitting the trivial torus and projecting the torus link on a plane.

Step 3 Give the whole primary cylinder a twist by rotating the top about the z-axis by an angle of 4π, as shown in Fig. 11(b).

Step 4 Use the glue operation as that shown in Fig. 6(b) to identify the point $(x, y, 0)$ to the point $(x, y, 1)$, then we obtain a trivial torus, on whose surface we have the torus link $K_{4,2}$, which is composed of two entangled orientated knots, as shown in Figs. 11(c) and 11(d).

Step 5 Quit the trivial torus and project the torus link $K_{4,2}$ on a plane, finally the linkage 4_1 is

obtained. For the case $i = 1$, we just obtain the knotted picture of the Bell basis $|\Phi^+\rangle_{cw} = (|\uparrow\uparrow\rangle_{cw} + |\downarrow\downarrow\rangle_{cw})/\sqrt{2}$, as shown in Figs. 11(d) and 11(e).

Therefore, using the above five steps we can easily obtain knotted pictures of Bell bases. This method is much more vivid and simple than the method using the two quantum gates: the Hadamard gate and the CNOT gate. Furthermore, we can easily generalize this method to the case of n qubits. For example, when $n = 3$, we have three trivial knots, which would correspond to three parallel orientated lines on the surface of the above-mentioned primary cylinder. Similarly, for some problems, such as the problems treated by Refs. [13] and [14], we can easily obtain their knotted pictures by some generalizations of our methods. Detailed discussion will be given in the forthcoming papers.

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