A new lattice hydrodynamic traffic flow model with a consideration of multi-anticipation effect*

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We present a new multi-anticipation lattice hydrodynamic model based on the traffic anticipation effect in the real world. Applying the linear stability theory, we obtain the linear stability condition of the model. Through nonlinear analysis, we derive the modified Korteweg-de Vries equation to describe the propagating behaviour of a traffic density wave near the critical point. The good agreement between the simulation results and the analytical results shows that the stability of traffic flow can be enhanced when the multi-anticipation effect is considered.

Keywords: traffic flow, lattice hydrodynamic model, simulation

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1. Introduction

To study the traffic problem of modern society, an increasing number of traffic models,[1−9] including cellular automaton models, car-following models, gas kinetic models, and hydrodynamic models, have been presented and many important results have been obtained. The lattice hydrodynamic model proposed by Nagatani,[10] which is a simplified version of the macroscopic hydrodynamic model, has successfully depicted the formation of the density waves and revealed its transition mechanism as the car following model did. Furthermore, inspired by the original lattice model,[10] some researchers further considered other factors in the lattice hydrodynamic model to enhance its abilities in describing real traffic.[11−17]

However, these lattice hydrodynamic models mentioned above cannot be used to study the complex phenomena resulting from anticipation behaviours since this effect was not considered in those models. In fact, along with the remarkable development of the wireless communication technology in intelligent transportation system (ITS), information about traffic flow on road ahead is obtainable. Depending on this information, drivers can anticipate the traffic state and then adjust the speed to the optimal state in advance. Hence, anticipation behaviours will lead to some complex phenomena. However, few scholars have studied the anticipation effect in lattice hydrodynamic models.

In view of the reasons stated above, in this paper, we propose a new lattice hydrodynamic model by considering the effect of multi-anticipation behaviours. In the following section, the new lattice hydrodynamic model will be introduced. In Section 3, linear stability analysis is conducted. In Section 4, by means of the nonlinear analysis, we derive the modified Korteweg-de Vries (mKdV) equation near the critical point and obtain its kink–antikink solitons solution to describe traffic jams. In Section 5, numerical simulation is carried out to validate the analytic results. In Section 6, we draw some conclusions from the presented study.

2. Models

In 1993, Kerner and Konhauser[18] presented a very charming high-order continuum traffic flow model (for short, K & K model) with a partial differential equation of the form

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1}$$

$$\partial_t v + v\partial_x v = [V_{\rm e}(\rho) - v]/\tau - (c_0^2/\rho)\partial_x \rho + (\mu/\rho)\,\partial_x^2 v, \tag{2}$$

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where $\partial_t = \partial/\partial t$ and $\partial_x = \partial/\partial x$, ρ denotes the traffic density, v represents the space mean velocity, $V_e(\rho)$ is the equilibrium speed, and c_0 and μ are the viscosity coefficient and anticipation coefficient, respectively. The right-hand side of Eq. (2) includes three terms. The first denotes a relaxation to equilibrium state. The second is an anticipation term, which reflects that the driver adaptively adjusts his velocity according to the observed traffic density ahead. The final term describes the viscosity factor, which represents the presumed tendency to adjust one's speed to that of the surrounding traffic. It is not easy to identify the values of c_0 and μ, and to obtain its density wave equation.

For the convenience of analysing the evolution of density wave in traffic flow, in 1998, Nagatani[10] first proposed the simplified version of the hydrodynamic model as follows,

$$\partial_t \rho + \rho_0 \partial_x(\rho v) = 0, \tag{3}$$

$$\partial_t(\rho v) = a\rho_0 V(\rho(x+\delta)) - a\rho v, \tag{4}$$

where ρ_0 denotes the average density, v is the optimal velocity function, and a is the sensitivity of a driver, $\rho(x+\delta)$ represents the local density at position $x+\delta$ at time t, δ denotes the average headway, which means $\delta = 1/\rho_0$, local density is expressed as $\rho(x+\delta) = 1/h(x,t)$, where $h(x,t)$ is the headway. Equation (3) is the continuity equation, while equation (4), which corresponds to Eq. (2), is the evolution equation, but the anticipation term and viscosity term are omitted for simplicity. In fact, according to actual traffic, the anticipation term in K & K model is essential. However, this factor was not considered in the lattice hydrodynamic model.

Considering the above reason, in this paper, we develop a new model called the multi-anticipation lattice hydrodynamic (MALH) model by introducing the anticipation term into the right-hand side of Eq. (4). Thus the evolution equation can be changed to

$$\begin{aligned}\partial_t(\rho v) =\ & a\rho_0 V(\rho(x+\delta)) - a\rho v \\ & + ka\sum\nolimits_{m=1}^{n} \beta_m \partial_x Q\,[\,\rho\,(x+m\delta)],\end{aligned} \tag{5}$$

where β_m is the weighted coefficient of the flux gradient $\partial_x Q\,[\,\rho\,(x+m\delta)]$ at site m, Q denotes the flux. The term $\sum_{m=1}^{n} \beta_m \partial_x Q\,[\,\rho\,(x+m\delta)]$ represents the anticipation term, which reflects that the driver adjusts his car according to the observed varying flux on n numbers of preceding sites, and k is a sensitivity coefficient of anticipation term. The ideal of the model is that the variation in traffic flow ρv at position x is determined by the optimal flow $\rho_0 V(\rho(x+\delta))$ at points $x+\delta$, actual flow ρv at position x, and the varying flux $\sum_{m=1}^{n} \beta_m \partial_x Q\,[\,\rho\,(x+m\delta)]$ on n number of preceding sites. When $k=0$, the extended model reduces into Nagatani's lattice model.[10]

It is well known that the effect of the sites ahead on vehicle motion decreases gradually as the distance between the considered site and that ahead increases. So in this paper, we select the weighted function tentatively as $\beta_m = (1/5)^{m-1}$.

With dimensionless space x (let $\tilde{x} = x/\delta$, and $\tilde{x}$ is indicated as x hereafter), the equation of motion Eq. (5) with the anticipation term is modified and expressed as

$$\begin{aligned}\partial_t \rho v =\ & a\rho_0 V(\rho(x+1)) - a\rho v \\ & + k\rho_0 a \sum_{m=1}^{n} \beta_m [Q(\rho(x+m)) \\ & - Q(\rho(x+(m-1)))].\end{aligned} \tag{6}$$

The corresponding lattice version of the above system consisting of Eqs. (3) and (6) can be rewritten as

$$\partial_t \rho_j + \rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = 0, \tag{7}$$

$$\begin{aligned}\partial_t(\rho_j v_j) =\ & a\rho_0 V(\rho_{j+1}) - a\rho_j v_j \\ & + ka\rho_0 \sum_{m=1}^{n} \beta_m [Q(\rho_{j+m}) \\ & - Q(\rho_{j+m-1})].\end{aligned} \tag{8}$$

Combining with the fundamental law of traffic flow ($Q = \rho v$), and discretizing the time variable in Eqs. (7) and (8), we obtain

$$\begin{aligned}&\rho_j(t+\tau) - \rho_j(t) + \tau\rho_0[\,\rho_j(t)v_j(t) \\ &\quad - \rho_{j-1}(t)v_{j-1}(t)] = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}&\rho_j(t+\tau)v_j(t+\tau) \\ &= a\rho_0 V(\rho_{j+1}(t)) + k\rho_0 \sum_{m=1}^{n} \beta_m[\,\rho_{j+m}(t)v_{j+m}(t) \\ &\quad - \rho_{j+m-1}(t)v_{j+m-1}(t)].\end{aligned} \tag{10}$$

Eliminating the speed v in Eqs. (9) and (10), we obtain the density equations as follows,

$$\begin{aligned}&\rho_j(t+2\tau) - \rho_j(t+\tau) + \tau\rho_0^2[V(\rho_{j+1}) - V(\rho_j)] \\ &- k\rho_0 \sum_{m=1}^{n} \beta_m[\Delta\rho_{j+m-1}(t+\tau) \\ &- \Delta\rho_{j+m-1}(t)] = 0,\end{aligned} \tag{11}$$

where $\Delta\rho_{j+m-1} = \rho_{j+m} - \rho_{j+m-1}$.

In this paper, we adopt the optimal velocity function as follows,[13,19]

$$V(\rho_j) = \tanh\left(\frac{2}{\rho_0} - \frac{\rho_j}{\rho_0^2} - \frac{1}{\rho_c}\right) + \tanh\left(\frac{1}{\rho_c}\right), \quad (12)$$

where $\rho_c = 0.2$ is the critical density. Note that equation (12) has a turning point at $\rho_j = \rho_c$ when $\rho_0 = \rho_c$.

3. Linear stability analysis

To investigate the influence of multi-anticipation behaviour on the traffic flow, linear stability theory can be applied to the extended model as follows. First, we define the state of the uniform traffic flow with constant density ρ_0 and optimal velocity $V(\rho_0)$. The solution of the homogeneous traffic flow is given as

$$\rho_j(t) = \rho_0 \quad \text{and} \quad v_j(t) = V(\rho_0). \quad (13)$$

Suppose that $y_j(t)$ is a small deviation from the steady state flow,

$$\rho_j(t) = \rho_0 + y_j(t). \quad (14)$$

Substituting Eq. (14) into Eq. (11) and linearizing them, we obtain

$$\begin{aligned} & y_j(t+2\tau) - y_j(t+\tau) + \tau\rho_0^2 V' \Delta y_j(t) \\ & - k\rho_0 \sum_{m=1}^{n} \beta_m [\Delta y_{j+m-1}(t+\tau) \\ & - \Delta y_{j+m-1}(t))] = 0, \end{aligned} \quad (15)$$

where

$$\Delta y_j = y_{j+1}(t) - y_j(t), \text{ and } V' = [\mathrm{d}V(\rho_j)/\mathrm{d}\rho_j]|_{\rho_j=\rho_0}.$$

Expanding $y_j(t) = \exp(\mathrm{i}kj + zt)$ into the Fourier models, where $z = z_1\mathrm{i}k + z_2(\mathrm{i}k)^2 + \cdots$, and inserting them into Eq. (15), we obtain the first-order and second-order terms of coefficients in the expression of z_1 and z_2, respectively, as follows,

$$z_1 = -\rho_0^2 V', \quad (16)$$

$$z_2 = -\frac{3}{2}\tau(\rho_0^2 V')^2 - \frac{1}{2}\rho_0^2 V' - k\rho_0^3 V' \sum_{m=1}^{n} \beta_m. \quad (17)$$

If z_2 is negative, the uniformly steady-state flow becomes unstable for long-wavelength modes, while the uniform flow is stable when z_2 is positive, thus the neutral stable criterion for this steady state is given by

$$\tau = -\frac{1 + 2k\rho_0 \sum_{m=1}^{n} \beta_m}{3\rho_0^2 V'}. \quad (18)$$

For a small disturbance with long wavelengths, the homogeneous traffic flow is stable under the condition

$$\tau < -\frac{1 + 2k\rho_0 \sum_{m=1}^{n} \beta_m}{3\rho_0^2 V'}. \quad (19)$$

As $k = 0$, the result of the stable condition is the same as that of Nagatani's model,[10]

$$\tau < -\frac{1}{3\rho_0^2 V'}. \quad (20)$$

The neutral stability lines (solid lines) in parameter space (ρ, a) are shown in Fig. 1 for the MALH model ($k = 0.4$, $n = 1, 2, 3$) and Nagatani's model[10] ($k = 0$). In Fig. 1, the dot lines represent the co-existing curves obtained from the solution of the mKdV equation (see Section 4). From Fig. 1, we can see that:

(i) the stability region in the MALH model is larger than that in the Nagatani's model,[10] which means that the stability of traffic flow is strengthened when the multi-anticipation effect is considered;

(ii) the more sites ahead are considered, the more critical points and the neutral stability curves are lowered for the MALH model, which means that the traffic flow becomes more stable;

(iii) the information of two sites ahead is enough for the MALH model to suppress the traffic jam effectively, that is to say, $n = 2$ is the optimal state in the MALH model.

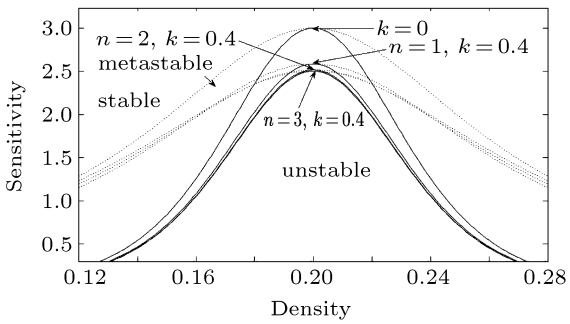


Fig. 1. Phase diagram in parameter space (ρ, a).

4. Nonlinear analysis and mKdV equation

In this section, the reductive perturbation method is applied to Eq. (11) and the system behaviour near the critical point is deeply analysed. We introduce slow scales for space variable j and time variable t, and define slow variable X and T as follows,

$$X = \varepsilon(j + bt) \quad \text{and} \quad T = \varepsilon^3 t, \quad 0 < \varepsilon \leq 1, \quad (21)$$

where b is a constant to be determined. Let

$$\rho_j(t) = \rho_c + \varepsilon R(X, T). \quad (22)$$

Substituting Eqs. (21) and (22) into Eq. (11) and making the Taylor expansion to the fifth order of ε lead to the following expression,

$$\varepsilon^2(b+\rho_{\rm c}^2V')\partial_X R+\varepsilon^3\left(\frac{3}{2}b^2\tau+\frac{\rho_{\rm c}^2V'}{2}-k\rho_{\rm c}b\right)\partial_X^2 R$$
$$+\varepsilon^4\left\{\partial_T R+\left[\frac{7b^3\tau^2}{6}+\frac{\rho_{\rm c}^2V'}{6}-\frac{3k\rho_{\rm c}b}{6}\sum_{m=1}^{n}\beta_m(b\tau+2m-1)\right]\partial_X^3 R+\frac{\rho_{\rm c}^2V'''}{6}\partial_X R^3\right\}$$
$$+\varepsilon^5\left\{(3b\tau-k\rho_{\rm c})\partial_X\partial_T R+\left[\frac{5b^4\tau^3}{8}+\frac{\rho_{\rm c}^2V'}{24}-\frac{k\rho_{\rm c}}{24}\sum_{m=1}^{n}\beta_m(4b^3\tau^2+6b^2\tau(2m-1)\right.$$
$$\left.+4b(3m^2-3m+1))\right]\partial_X^4 R+\frac{\rho_{\rm c}^2V'''}{12}\partial_X^2 R^3\right\}=0, \tag{23}$$

where $V'=[{\rm d}V(\rho_j)/{\rm d}\rho_j]|\rho_j=\rho_{\rm c}$ and $V'''=[{\rm d}^3V(\rho_j)/{\rm d}\rho_j^3]|\rho_j=\rho_{\rm c}$. Near the critical point $(\rho_{\rm c},a_{\rm c})$, $\tau=(1+\varepsilon^2)\tau_{\rm c}$, taking $b=-\rho_{\rm c}^2V'$ and eliminating the second order and third order terms of ε from Eq. (23), we obtain

$$\varepsilon^4[\partial_T R-g_1\partial_X^3 R+g_2\partial_X R^3]+\varepsilon^5[\,g_3\partial_X^2 R+g_4\partial_X^4 R+g_5\partial_X^2 R^3]=0, \tag{24}$$

where

$$g_1=-\frac{7b^3\tau_{\rm c}^2}{6}-\frac{\rho_{\rm c}^2V'}{6}+\frac{k\rho_{\rm c}b}{2}\sum_{m=1}^{n}\beta_m(b\tau_{\rm c}+2m-1),\quad g_2=\frac{\rho_{\rm c}^2V'''}{6},\quad g_3=\frac{3}{2}b^2\tau_{\rm c},$$
$$g_4=-\frac{1}{6}(3b\tau_{\rm c}-k\rho_{\rm c}\sum_{m=1}^{n}\beta_m)\left[(7b^3\tau_{\rm c}^2+\rho_{\rm c}^2V')-3bk\rho_{\rm c}\sum_{m=1}^{n}\beta_m(b\tau_{\rm c}+2m-1)\right]$$
$$+\frac{5}{8}b^4\tau^3+\frac{\rho_{\rm c}^2V'}{24}-\frac{k\rho_{\rm c}}{24}\sum_{m=1}^{n}\beta_l\left[4b^3\tau_{\rm c}^2+6b^2\tau_{\rm c}(2m-1)+4b(3m^2-3m+1)\right]$$
$$g_5=\frac{1-6b\tau_{\rm c}+2k\rho_{\rm c}\sum\limits_{m=1}^{n}\beta_m}{12}\rho_{\rm c}^2V'''.$$

In order to derive the regularized equation, we make the following transformations for Eq. (24),

$$T'=g_1T,\quad R=\sqrt{\frac{g_1}{g_2}}R'. \tag{25}$$

We have the standard mKdV equation with the $O(\varepsilon)$ correction term as follows,

$$\partial_T' R'-\partial_X^3 R'+\partial_X R'^3+\varepsilon M[R']=0, \tag{26}$$

where

$$M[R']=\frac{1}{g_1}\left[g_3\partial_X^2 R'+g_4\partial_X^4 R'+\frac{g_1g_5}{g_2}\partial_X^2 R'^3\right]. \tag{27}$$

Ignoring the $O(\varepsilon)$ term, we obtain the mKdV equation with the kink–antikink soliton solution,

$$R_0'(X,T')=\sqrt{c}\tanh\sqrt{\frac{c}{2}}(X-cT'). \tag{28}$$

With the method described in Ref. [20], we obtain the selected velocity C,

$$C=\frac{5g_2g_3}{2g_2g_4-3g_1g_5}. \tag{29}$$

Hence, we obtain the kink–antikink soliton solution as follows,

$$\rho_j(t)=\rho_{\rm c}+\sqrt{\frac{g_1C}{g_2}\left(\frac{\tau}{\tau_{\rm c}}-1\right)}\tanh\sqrt{\frac{C}{2}\left(\frac{\tau}{\tau_{\rm c}}-1\right)}$$
$$\times\left[j+\left(1-Cg_1\left(\frac{\tau}{\tau_{\rm c}}-1\right)\right)t\right]. \tag{30}$$

The kink–antikink solution represents co-existing phases, which consist of the freely moving phase at low density and the jammed phase at high density. The densities corresponding to the freely moving phase and the jammed phase are given by $\rho_j=\rho_{\rm c}+A$ and $\rho_j=\rho_{\rm c}-A$, respectively. When $k=0$, the result is the same as that in Nagatani's model. Thus we can obtain the co-existing curves in the (ρ,a) parameter space (see Fig. 1). From Fig. 1, we can see that the space is divided into three regions: the stability region, the metastable region, and the unstable region. In the stable region, traffic flow is stable and a traffic jam will not appear. However, in the metastable and unstable region, traffic flow is unstable and the density waves will occur when small disturbances are

added on the uniform traffic flow. In Fig. 2, the propagating backward kink–antikink density wave appears, which is in good agreement with the analytical ones.

5. Numerical simulation

To validate the above theoretical results, numerical simulation was carried out for the extended model described by Eq. (11), in which the periodic boundary condition is adopted. The initial conditions are given as $\rho_j(0) = \rho_0 = 0.25, \rho_j(1) = \rho_j(0) = 0.25$, for $j \neq 49, 50$, $\rho_j(1) = 0.25 - 0.1$ for $j = 49$ and $\rho_j(1) = 0.25 + 0.1$ for $j = 50$, where the total number of sites is $N = 100$, and the sensitivity is $a = 2.48$. The simulation results are shown in Fig. 2.

Figure 2 shows the space–time evolution of the density after a sufficiently long time $t = 10^4$ for the MALH model. Pattern (a) with $k = 0$ corresponds to that of Nagatani's model. The patterns (b)–(d) exhibit the time evolution of the density profile of the MALH model for various values of $n = 1, 2, 3$ at $k = 0.4$. Comparing the patterns (a) and (b) under the same conditions, since the effect of the multi-anticipation is considered, the stability of the MALH model is better than that of Nagatani's model with the same sensitivity, which demonstrates that multi-anticipation behaviour could be added into the lattice hydrodynamic model to enhance the stability of traffic flow. From patterns (b)–(d), one can find that as the value of n increases, the amplitude of the density wave weakens gradually. In particular, in patterns (c) and (d), the traffic flow is uniform over the whole space as $n = 2, 3$, which means that just considering the information of two preceding sites is enough to suppress the traffic congestion quickly and efficiently in the MALH model. Therefore, the results of simulation are in agreement with those of theoretical analysis.

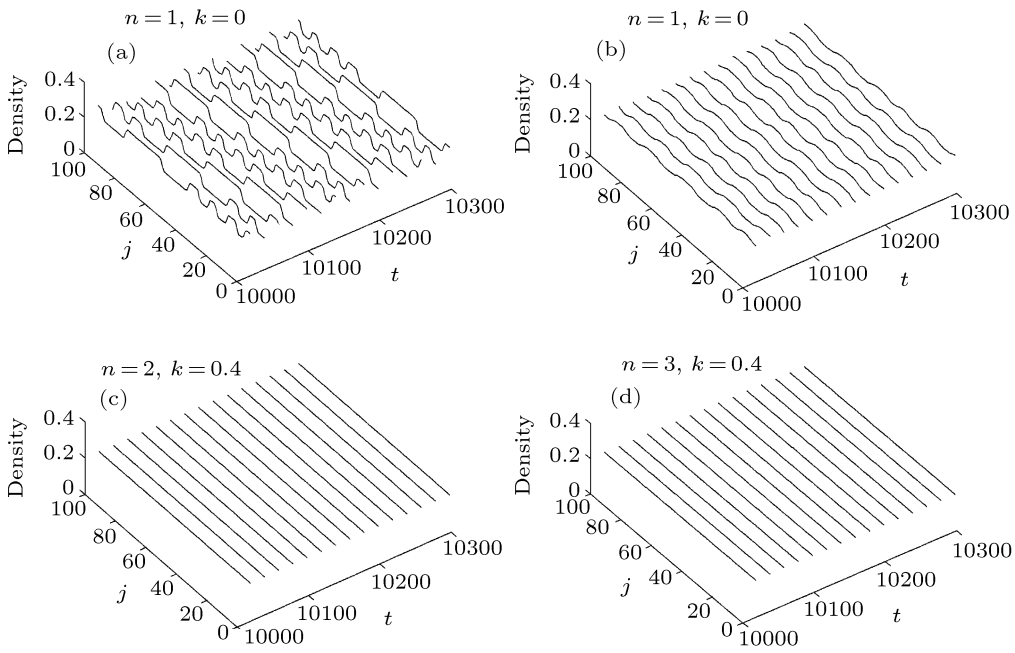


Fig. 2. Space–time evolution of the density after $t = 10000$. The pattern (a) for the Nagatani's model, the patterns (b), (c), and (d) for the MALH model.

6. Conclusion

In recent years, wireless communication technology has been applied in ITS widely, and thus much more information is available for drivers in ITS environments than ever before. With this information, drivers can anticipate the traffic situation ahead. However, few existing lattice hydrodynamic models directly study the anticipation factor. In this paper, we presented a new lattice hydrodynamic model (MALH model) considering the effect of multi-anticipation on single lane freeway traffic. We have worked out the stability criterion and derived the mKdV equation near the critical point to describe traffic jams. Numerical simulation results show that the stability of traffic flow is enhanced by introducing this new con-

sideration. The analytical results are in good agreement with the simulation results.

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