Influence of lane change on stability analysis for two-lane traffic flow*

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This paper deals mainly with the influence of lane changing behaviours on the stability of two-lane traffic flow under a periodic boundary condition. Following the description of an optimal velocity model for two vehicle groups and the derivation of their stability conditions, the feedback signals, which involve information about vehicles from both lanes acting on the two-lane traffic system, are introduced into the optimal velocity model. The control signals play a role in alleviating the traffic jam only if the traffic state is in congestion, and their role will vanish if the traffic state is in the steady state. The numerical simulations show that lane changing behaviours can break the steady state of two-lane traffic flow and aggravate the traffic disturbance, but the control method would successfully suppress the traffic jam eventually, which implies that the conclusions obtained here have certain theoretical and practical significance.

Keywords: stability analysis, optimal velocity model, lane changing behaviours, feedback signals

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1. Introduction

Traffic problems have attracted much attention with considerable interest for decades. In order to understand the complex phenomena of traffic flow, various vehicular traffic models have been proposed in recent decades and have been developed by using the representation on the macroscopic[1−7] or microscopic scale.[8−14] Here we concentrate mainly on the optimal velocity (OV) model proposed by Bando *et al.*[14] on the microscopic scale, where each vehicle is described by a simple differential model and each driver controls the velocity based on the OV function. In that paper, the traffic congestion under periodic conditions was presented and a simple stability condition for the OV model was derived. Since then, many researchers have explored the OV model from different aspects and utilized it to analyse various traffic density waves so as to obtain stability conditions in different situations.[15−24] As to the stability analysis for two-lane traffic flow, Kurata and Nagatani[25] and Nagai *et al.*[26] have studied the spatiotemporal dynamics of jam transition induced by a bus or blockage in a two-lane highway. Tang *et al.*[27,28] recently put forward a new two-lane car-following model with consideration of the lateral distance and potential lane changing, which has improved the stability of traffic flow. Obviously, all of the above have demonstrated that most scholars have investigated different factors of the stability condition for a single-lane or multi-lane OV model.

Moreover, as to the control of the traffic disturbance or traffic jam, Kerner[29] proposed a congested pattern control approach to the allevaition of traffic congestion at a highway bottleneck and drew some important conclusions. In addition, with the development of intelligent transportation systems (ITS), advanced traffic control systems have also developed greatly, such as ramp metering, adaptive speed limit, and dynamic and individual route guidance. In particular, adaptive cruise control (ACC) systems have been implemented to improve road capacity and reduce traffic congestion.[30−32] In addition, what deserves our attention is that Konishi *et al.*[33] proposed various versions of the decentralized delayed feedback control (DDFC) method and used them to alleviate traffic jams in single lane traffic flow. Zhao and Gao[34] presented another control method to avoid traffic jams and the computer simulations illustrate that the disorder state in a traffic system could be suppressed by the control method. Little attention has been paid to the control of traffic jams and the design of control

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systems for two-lane traffic flow from the viewpoint of control theory. However, recently, Chen *et al.*[35] extended the single-lane feedback control model to the two-lane case and simulation results indicated that no serious traffic jams occur when the feedback signal is introduced, but they did not give the OV model for two-lane traffic flow and neglected the friction interference from the neighbouring lane. Moreover, when the lane changing behaviours are considered, the reason why the delayed feedback control method[33] was not chosen is that once a vehicle transfers to another lane, its past headway and lateral distance have no significance in the feedback adjustment of the vehicle motion. Therefore, in this paper, we extend the form of control signals[34] to the two-lane case and consider the friction interference from the neighbouring lane, where the control signals are mainly from current velocity differences.

Moreover, congestion mechanism and traffic jam avoidance for two-lane traffic flow with lane changing behaviours should be regarded as an important issue, because i) two-lane traffic flow with lane changing behaviours is more like real traffic; ii) it is crucial to reveal the interaction mechanism between the vehicles from different lanes (i.e. friction interference), which is an important factor for the design of the control system; and iii) adopting our control systems to avoid traffic jams for two-lane traffic flow can contribute much to the development of the economy and the pollution prevention.

The remainder of this paper is organized as follows. In Section 2, we explain the OV model with lane changing behaviours for two-lane traffic flow and analyse the stability conditions. In Section 3, the feedback control method is introduced to compress traffic jams in the two-lane road. Numerical simulations are presented in Section 4 to confirm the theoretical results. Finally, conclusions are drawn from the present study in Section 5.

2. Optimal velocity model

2.1. Description of model

In this paper, in order to describe the dynamic behaviours of two vehicle groups in a circular two-lane road, we can extend the single-lane OV model to the two-lane case. Moreover, it is well known that the vehicle motions are influenced not only by the block interference from the front vehicles in the same lane but also by the friction interference from vehicles in the neighbouring lane, which should not be neglected when two-lane or multi-lane traffic flow is modeled. Now we consider the two-lane case as follows (see Fig. 1).

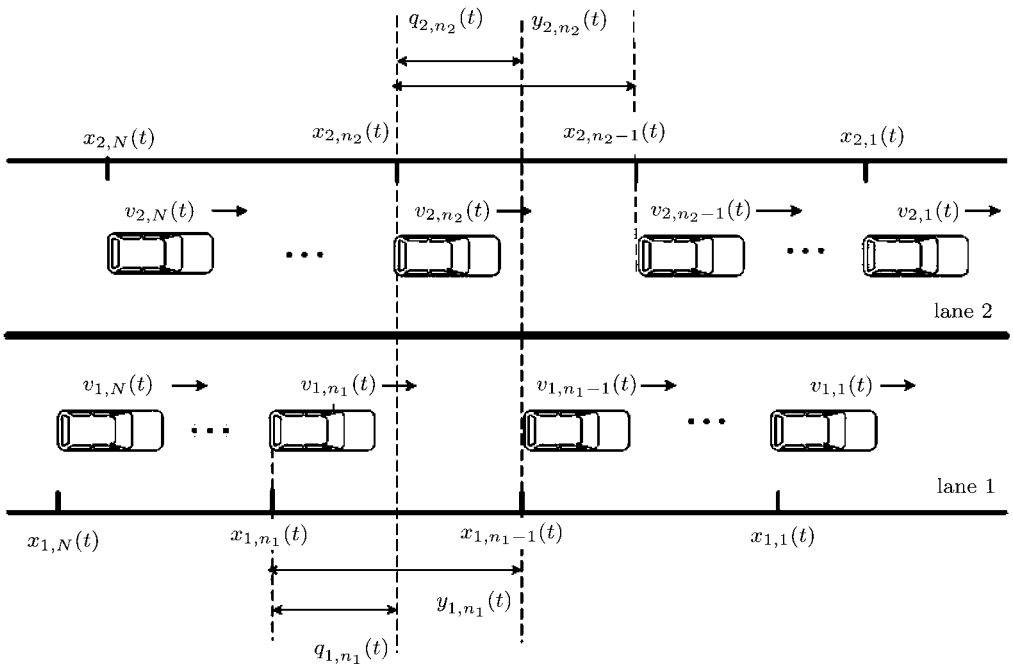


Fig. 1. Location of vehicles.

From Fig. 1, it is obviously known that a vehicle optimal velocity is determined by the headway (i.e. $y_{l,n_l}(t)$) and lateral distance (i.e. $q_{l,n_l}(t)$), which reflect the block and friction interferences, respectively. Therefore, we can give the following two-lane OV model:

$$\begin{cases} \dfrac{\mathrm{d}v_{l,n_l}(t)}{\mathrm{d}t} = a_l\{V_l(y_{l,n_l}(t), q_{l,n_l}(t)) - v_{l,n_l}(t)\}, \\ \dfrac{\mathrm{d}y_{l,n_l}(t)}{\mathrm{d}t} = v_{l,n_l-1}(t) - v_{l,n_l}(t), \\ \dfrac{\mathrm{d}q_{l,n_l}(t)}{\mathrm{d}t} = v^f_{l,n_l}(t) - v_{l,n_l}(t), \quad (n_l = 1, \cdots, N_l), \end{cases} \tag{1}$$

where $v_{l,n_l}(t)$ is the velocity of vehicle n_l at time t in lane l, $V_l(y_{l,n_l}(t), q_{l,n_l}(t))$ is the OV function in lane l, $y_{l,n_l}(t)$ is the headway between two vehicles $(n_l - 1)$ and n_l in the same lane l at time t, $q_{l,n_l}(t)$ is the lateral distance (i.e., the distance between the vehicle n_l in lane l and the closest vehicle in the neighbouring lane in front of this vehicle at time t), $v^f_{l,n_l}(t)$ is the velocity of the closest vehicle in the neighbouring lane in front of the vehicle n_l at time t, $a_l > 0$ is the driver sensitivity to the difference between the optimal and current velocities in the lane l.

Moreover, the OV function has been given by[14]

$$\begin{aligned} &V_l(y_{l,n_l}(t), q_{l,n_l}(t)) = V_l(\bar{y}_l(t)) \\ &= \tanh(\bar{y}_l(t) - h^d_l) + \tanh(h^d_l), \end{aligned}$$

where $\bar{y}_l(t) = \alpha^y_l y_{l,n_l}(t) + \alpha^q_l q_{l,n_l}(t)$ is the comprehensive distance reflecting both block and friction interferences, with α^y_l and α^q_l being the weights of $y_{l,n_l}(t)$ and $q_{l,n_l}(t)$ in lane l, respectively. $\alpha^y_l + \alpha^q_l = 1$ and $\alpha^y_l \geq \alpha^q_l$, h^d_l is the desired comprehensive distance in the lane l.

In addition, if the desired velocity of the vehicle in lane l is v^*_l, the desired comprehensive distance is $\bar{y}^*_l = F^{-1}_l(v^*_l)$. And once weight coefficients α^y_l and α^q_l and desired headway y^*_l are determined, the desired lateral distance q^*_l is confirmed. Therefore, the final steady state of the whole vehicular system can be expressed as

$$(v^*_l, y^*_l, q^*_l)^{\mathrm{T}}. \tag{2}$$

2.2. Introduction of lane changing rules

In order to describe the lane changing behaviours of vehicles in a circular two-lane road, we should introduce proper lane-changing rules. Some types of lane-changing rules were presented in Refs. [36] and [37]. These rules can be independently used in modeling the lane-changing behaviours. In this paper, we select one of them to demonstrate the application in our model. Tthe rule is described as follows.

Figure 2 illustrates that vehicle n_2 in lane 2 will conduct lane changing behaviour if the following conditions are met,

$$y_{l,n_l}(t) < 2h^f_l, \tag{3}$$

$$y_{l,n_l}(t) < q_{l,n_l}(t), \tag{4}$$

$$b_{l,n_l}(t) > h^b_l, \tag{5}$$

where inequalities (3) and (4) are incentive criteria for a lane-changing decision, inequality (5) is the security criterion, $b_{l,n_l}(t)$ is the distance between the vehicle n_l in lane l and the closest vehicle in the neighbouring lane behind this vehicle at time t, h^f_l is the safety distance in lane l, and h^b_l is the safety distance in the neighbouring lane.

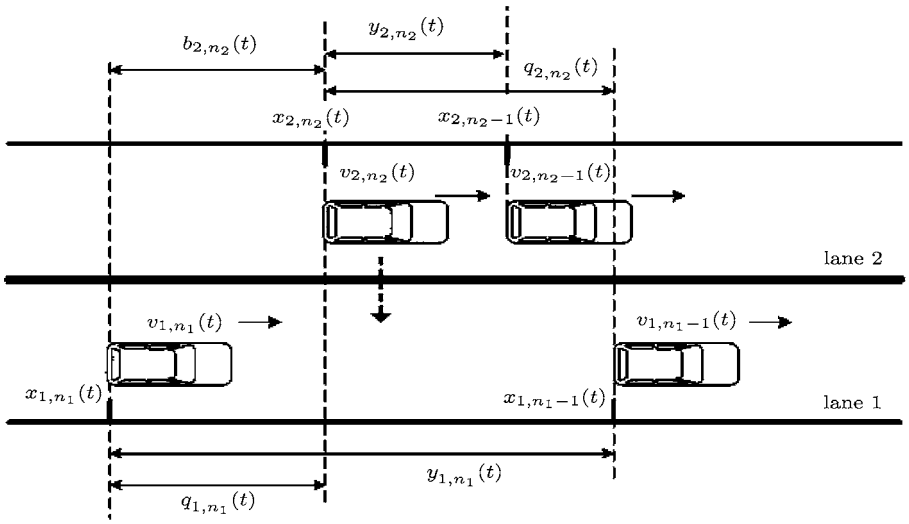


Fig. 2. Situation of lane changing.

2.3. Stability analysis

In order to obtain stability conditions for two-lane traffic flow, first let vehicular dynamics (1) be linearized around steady state (2), and then the linearized vehicular dynamics in lane l can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}v^o_{l,n_l}(t)}{\mathrm{d}t} = a_l\{\Lambda^y_l y^o_{l,n_l}(t) + \Lambda^q_l q^o_{l,n_l}(t) - v^o_{l,n_l}(t)\}, \\ \dfrac{\mathrm{d}y^o_{l,n_l}(t)}{\mathrm{d}t} = v^o_{l,n_l-1}(t) - v^o_{l,n_l}(t), \\ \dfrac{\mathrm{d}q^o_{l,n_l}(t)}{\mathrm{d}t} = v^{fo}_{l,n_l}(t) - v^o_{l,n_l}(t) + \Delta v^*, \\ (n_l = 1, \cdots, N_l), \end{cases} \tag{6}$$

where

$$\Lambda^y_l = \left.\frac{\partial V_l(y_{l,n_l}(t), q_{l,n_l}(t))}{\partial y_{l,n_l}(t)}\right|_{y_{l,n_l}(t)=y^*_l, q_{l,n_l}(t)=q^*_l},$$

and

$$\Lambda^q_l = \left.\frac{\partial V_l(y_{l,n_l}(t), q_{l,n_l}(t))}{\partial q_{l,n_l}(t)}\right|_{y_{l,n_l}(t)=y^*_l, q_{l,n_l}(t)=q^*_l},$$

$y^o_{l,n_l}(t) = y_{l,n_l}(t) - y^*_l$, $q^o_{l,n_l}(t) = q_{l,n_l}(t) - q^*_l$, $v^o_{l,n_l}(t) = v_{l,n_l}(t) - v^*_l$, and $v^{fo}_{l,n_l}(t) = v^f_{l,n_l}(t) - v^*_{Al}$, v^*_{Al} is the desired velocity of vehicles in the neighbouring lane, $\Delta v^* = (v^*_{Al} - v^*_l)$ is the desired velocity difference between vehicles from different lanes.

From the viewpoint of control theory, vehicular dynamics (6) can be rewritten as a linear time-invariant system, that is,

$$\begin{aligned} \begin{bmatrix} \dfrac{\mathrm{d}v^o_{l,n_l}(t)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y^o_{l,n_l}(t)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}q^o_{l,n_l}(t)}{\mathrm{d}t} \end{bmatrix} &= \begin{bmatrix} -a_l & a_l\Lambda^y_l & a_l\Lambda^q_l \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v^o_{l,n_l}(t) \\ y^o_{l,n_l}(t) \\ q^o_{l,n_l}(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v^o_{l,n_l-1}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v^{fo}_{l,n_l}(t) \\ &+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Delta v^*. \end{aligned}$$

After Laplace transformation, we have

$$\begin{aligned} s \begin{bmatrix} V^o_{l,n_l}(s) \\ Y^o_{l,n_l}(s) \\ Q^o_{l,n_l}(s) \end{bmatrix} &= M \begin{bmatrix} V^o_{l,n_l}(s) \\ Y^o_{l,n_l}(s) \\ Q^o_{l,n_l}(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} V^o_{l,n_l-1}(s) \\ &+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V^{fo}_{l,n_l}(s) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Delta V^*, \end{aligned} \tag{7}$$

where

$$\begin{aligned} &M = \begin{bmatrix} -a_l & a_l\Lambda^y_l & a_l\Lambda^q_l \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \\ &V^o_{l,n_l}(s) = L(v^o_{l,n_l}(t)), \quad Y^o_{l,n_l}(s) = L(y^o_{l,n_l}(t)), \\ &Q^o_{l,n_l}(s) = L(q^o_{l,n_l}(t)), \quad V^o_{l,n_l-1}(s) = L(v^o_{l,n_l-1}(t)), \\ &V^{fo}_{l,n_l}(s) = L(v^{fo}_{l,n_l}(t)), \quad \Delta V^* = L(\Delta v^*), \end{aligned}$$

and $L(\cdot)$ denotes the Laplace transform.

After derivation of Eq. (7), we have the following transfer relationship,

$$\begin{aligned} V^o_{l,n_l}(s) = &\frac{a_l\Lambda^y_l}{s(s+a_l) + a_l(\Lambda^y_l + \Lambda^q_l)} V^o_{l,n_l-1}(s) \\ &+ \frac{a_l\Lambda^q_l}{s(s+a_l) + a_l(\Lambda^y_l + \Lambda^q_l)} V^{fo}_{l,n_l}(s) \\ &+ \frac{a_l\Lambda^q_l}{s(s+a_l) + a_l(\Lambda^y_l + \Lambda^q_l)} \Delta V^*. \end{aligned} \tag{8}$$

In order to deduce the main relationship of backward propagation of velocity fluctuation (i.e., the velocity fluctuations of the vehicle $(n_l - 1)$ in lane l and the closest vehicle in the neighbouring lane in front of vehicle n_l are propagated to the velocity fluctuation of vehicle n_l in lane l), we assume that their maximum disturbances are the same in magnitude, then the first two terms on the right-hand side of Eq. (8) are combined as follows,[33]

$$V^o_{l,n_l}(s) = G_l(s) V^o_{l,n_l-1}(s),$$

where the transfer function $G_l(s)$ in the lane l is $[a_l(\Lambda^y_l + \Lambda^q_l)]/[s^2 + a_l s + a_l(\Lambda^y_l + \Lambda^q_l)]$, and the characteristic polynomial $d_l(s)$ in lane l is $s^2 + a_l s + a_l(\Lambda^y_l + \Lambda^q_l)$.

From Definition 1 in Ref. [33], it is known that if the characteristic polynomial $d_l(s)$ is stable and H_∞ norm of transfer function $G_l(s)$ is equal to or less than 1, then traffic jams will not occur. What deserves our attention is that the stability of the characteristic polynomials is the necessary and sufficient condition for a steady state (2) to exist, and H_∞ norm of transfer functions should be equal to or less than 1 so as to guarantee that the velocity fluctuation will not be amplified when propagated backwards. Therefore, after derivation, we can obtain the stability condition as $a_l \geq 2(\Lambda^y_l + \Lambda^q_l)$.

Moreover, the relationship of backward propagation of velocity fluctuation from the desired velocity difference to the velocity fluctuation of vehicle n_l in lane l is analysed here. Let $T_l(s) = a_l\Lambda_l^q/(s^2 + a_l s + a_l(\Lambda_l^y + \Lambda_l^q))$, and if $a_l \geq 2(\Lambda_l^y + \Lambda_l^q)$ we can easily obtain $\|T_l(s)\|_\infty < 1$, which is to say that the velocity disturbance due to Δv^* would not be amplified when propagated backwards.

For the aforementioned analyses, the stability conditions can be abbreviated as follows.

Lemma If the stability conditions are met for both lanes (i.e., $a_l \geq 2(\Lambda_l^y + \Lambda_l^q)$), the traffic jams will never occur in the two-lane road; otherwise, once the stability condition is not met for either lane, the velocity fluctuation in the lane would be propagated backwards and to the neighbouring lane with an increase in disturbance, which results in traffic jams eventually.

3. Stability analysis with control signals

In order to alleviate the traffic jams, which may be caused by traffic accidents or traffic bottlenecks, a control signal term, $u_{l,n_l}(t)$, is added to vehicular dynamics (1),

$$\begin{cases} \dfrac{\mathrm{d}v_{l,n_l}(t)}{\mathrm{d}t} = a_l\{V_l(y_{l,n_l}(t), q_{l,n_l}(t)) - v_{l,n_l}(t)\} \\ \qquad\qquad + u_{l,n_l}(t), \\ \dfrac{\mathrm{d}y_{l,n_l}(t)}{\mathrm{d}t} = v_{l,n_l-1}(t) - v_{l,n_l}(t), \\ \dfrac{\mathrm{d}q_{l,n_l}(t)}{\mathrm{d}t} = v_{l,n_l}^f(t) - v_{l,n_l}(t), \quad (n_l = 1, \cdots, N_l), \end{cases} \tag{9}$$

where control signal $u_{l,n_l}(t)$ is

$$\begin{aligned} u_{l,n_l}(t) = {} & k_l^y(v_{l,n_l-1}(t) - v_{l,n_l}(t)) \\ & + k_l^q(v_{l,n_f}^f(t) - v_{l,n_l}(t)), \end{aligned} \tag{10}$$

k_l^y and k_l^q are the feedback gains. Such a control signal is composed of two aspects (i.e., the extension of control signals in Ref. [34]): one is the velocity difference between vehicles $(n_l - 1)$ and n_l in lane l, the other is the velocity difference between vehicle n_l in lane l and the closest vehicle in the neighbouring lane in front of vehicle n_l. Moreover, if the velocity of vehicle $(n_l - 1)$ or $v_{l,n_f}^f(t)$ is larger than that of vehicle n_l, vehicle n_l will receive positive feedback. Otherwise, vehicle n_l will receive negative feedback.

Around steady state (2), vehicle dynamics (9) with control signals (10) can be written as

$$\begin{cases} \dfrac{\mathrm{d}v_{l,n_l}^o(t)}{\mathrm{d}t} = a_l\{\Lambda_l^y y_{l,n_l}^o(t) \\ \qquad\qquad + \Lambda_l^q q_{l,n_l}^o(t) - v_{l,n_l}^o(t)\} + u_{l,n_l}, \\ \dfrac{\mathrm{d}y_{l,n_l}^o(t)}{\mathrm{d}t} = v_{l,n_l-1}^o(t) - v_{l,n_l}^o(t), \\ \dfrac{\mathrm{d}q_{l,n_l}^o(t)}{\mathrm{d}t} = v_{l,n_l}^{fo}(t) - v_{l,n_l}^o(t) + \Delta v^*, \\ (n_l = 1, \cdots, N_l). \end{cases} \tag{11}$$

And the form of control signal can be changed into

$$u_{l,n_l}(t) = k_l^y(\mathrm{d}y_{l,n_l}^o(t)/\mathrm{d}t) + k_l^q(\mathrm{d}q_{l,n_l}^o(t)/\mathrm{d}t).$$

In addition, in order to deduce the structure of the control system, equation (11) can be given as

$$\begin{aligned} \begin{bmatrix} \dfrac{\mathrm{d}v_{l,n_l}^o(t)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y_{l,n_l}^o(t)}{\mathrm{d}t} \\ \dfrac{\mathrm{d}q_{l,n_l}^o(t)}{\mathrm{d}t} \end{bmatrix} = {} & M \begin{bmatrix} v_{l,n_l}^o(t) \\ y_{l,n_l}^o(t) \\ q_{l,n_l}^o(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_{l,n_l-1}^o(t) \\ & + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_{l,n_l}^{fo}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_{l,n_l}(t) \\ & + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Delta v^*. \end{aligned}$$

After Laplace transformation, we have

$$\begin{aligned} s \begin{bmatrix} V_{l,n_l}^o(s) \\ Y_{l,n_l}^o(s) \\ Q_{l,n_l}^o(s) \end{bmatrix} = {} & M \begin{bmatrix} V_{l,n_l}^o(s) \\ Y_{l,n_l}^o(s) \\ Q_{l,n_l}^o(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} V_{l,n_l-1}^o(s) \\ & + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V_{l,n_l}^{fo}(s) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U_{l,n_l}(s) \\ & + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Delta V^*, \end{aligned} \tag{12}$$

$$U_{l,n_l}(s) = s[k_l^y Y_{l,n_l-1}^o(s) + k_l^q Q_{l,n_l}^o(s)]. \tag{13}$$

After derivation, we have the following relationship,

$$\begin{bmatrix} V_{l,n_l}^o(s) \\ Y_{l,n_l}^o(s) \\ Q_{l,n_l}^o(s) \end{bmatrix} = R(s) \begin{bmatrix} V_{l,n_l-1}^o(s) \\ V_{l,n_l}^{fo}(s) \\ \Delta V^* \\ U_{l,n_l}(s) \end{bmatrix},$$

where the expression of $R(s)$ is given in appendix A. Then, the control system can be described as shown in Fig. 3.

From Fig. 3, it is obviously known that the control system is a typical closed-loop feedback control system. Obviously, the desired velocity difference and the velocity disturbances of vehicle $(n_l - 1)$ and the closest vehicle in the neighbouring lane in front of vehicle n_l are the input signals, the comprehensive signals of block and friction interferences (i.e. $U_{l,n_l}(s)$) are the feedback control signals, and the output signals include the disturbances of velocity, headway and lateral distance of vehicle n_l. In the following, we aim to acquire the proper feedback gains for such a control system, which would make three output signals vanish gradually.

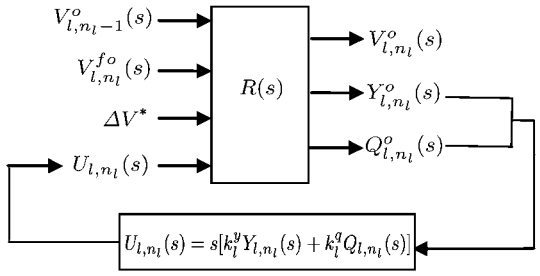


Fig. 3. Diagram of the control system for the n_l-th vehicle of lane l.

In the other way, if the control signals are written as

$$\begin{aligned} u_{l,n_l}(t) = {} & k^y_l(v^o_{l,n_l-1}(t) - v^o_{l,n_l}(t)) \\ & + k^q_l(v^{fo}_{l,n_l}(t) - v^o_{l,n_l}(t) + \Delta v^*), \end{aligned}$$

after Laplace transform, we have

$$\begin{aligned} U_{l,n_l}(s) = {} & k^y_l(V^o_{l,n_l-1}(s) - V^o_{l,n_l}(s)) + k^q_l(V^{fo}_{l,n_l}(s) \\ & - V^o_{l,n_l}(s) + \Delta V^*). \end{aligned}$$

Then equation (12) can be transformed into

$$\begin{aligned} s\begin{bmatrix} V^o_{l,n_l}(s) \\ Y^o_{l,n_l}(s) \\ Q^o_{l,n_l}(s) \end{bmatrix} = {} & M^* \begin{bmatrix} V^o_{l,n_l}(s) \\ Y^o_{l,n_l}(s) \\ Q^o_{l,n_l}(s) \end{bmatrix} + \begin{bmatrix} k^y_l \\ 1 \\ 0 \end{bmatrix} V^o_{l,n_l-1}(s) \\ & + \begin{bmatrix} k^q_l \\ 0 \\ 1 \end{bmatrix} V^{fo}_{l,n_l}(s) + \begin{bmatrix} k^q_l \\ 0 \\ 1 \end{bmatrix} \Delta V^*, \end{aligned}$$

where

$$M^* = \begin{bmatrix} -a_l - k^y_l - k^q_l & a_l\Lambda^y_l & a_l\Lambda^q_l \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Therefore, after derivation, equation (13) can be rewritten as

$$\begin{aligned} V^o_{l,n_l}(s) = {} & H_{11}(s)V^o_{l,n_l-1}(s) + H_{12}(s)V^{fo}_{l,n_l}(s) \\ & + H_{13}(s)\Delta V^*, \\ Y^o_{l,n_l}(s) = {} & H_{21}(s)V^o_{l,n_l-1}(s) + H_{22}(s)V^{fo}_{l,n_l}(s) \\ & + H_{23}(s)\Delta V^*, \\ Q^o_{l,n_l}(s) = {} & H_{31}(s)V^o_{l,n_l-1}(s) + H_{32}(s)V^{fo}_{l,n_l}(s) \\ & + H_{33}(s)\Delta V^*, \end{aligned}$$

where the expressions of H_{11}, H_{12}, $H_{13}, \ldots, H_{33}$ are given in Appendix B.

Here, we study mainly the transfer relationship of velocity fluctuation

$$\begin{aligned} V^o_{l,n_l}(s) = {} & H_{11}(s)V^o_{l,n_l-1}(s) + H_{12}(s)V^{fo}_{l,n_l}(s) \\ & + H_{13}(s)\Delta V^*. \end{aligned}$$

It is assumed that the maximum velocity disturbances in both lanes are the same. Then we obtain the following main transfer relationship of velocity disturbance in lane l,

$$V^o_{l,n_l}(s) = G^*_l(s)V^o_{l,n_l-1}(s),$$

where transfer function $G^*_l(s)$ in lane l is

$$G^*_l(s) = \frac{(k^y_l + k^q_l)s + a_l(\Lambda^y_l + \Lambda^q_l)}{s^2 + (a_l + k^y_l + k^q_l)s + a_l(\Lambda^y_l + \Lambda^q_l)},$$

the characteristic polynomial $d^*_l(s)$ in lane l is $s^2 + (a_l + k^y_l + k^q_l)s + a_l(\Lambda^y_l + \Lambda^q_l)$.

According to our analysis, the traffic jams will never occur in the controlled system if the characteristic polynomial $d^*_l(s)$ is stable and $\|G^*_l(s)\|_\infty \leq 1$. No traffic jams can be realized by adjusting parameters k^y_l and k^q_l in the controlled system.

First of all, in order to make $d_l(s)^*$ stable, according to the Hurwitz stability criterion, we should confirm that $a_l + k^y_l + k^q_l > 0$ and $a_l(\Lambda^y_l + \Lambda^q_l) > 0$. In addition, because of the OV function characteristics of monotonic increase (i.e., $\Lambda^y_l > 0$ and $\Lambda^q_l > 0$) and $a_l > 0$, the condition for $d_l(s)^*$ to be stable is $k^y_l + k^q_l > -a_l$. Then, consider $\|G^*_l(s)\|_\infty \leq 1$, that is,

$$\|G^*_l(s)\|_\infty = \sup_{\omega\in[0,+\infty)} |G^*_l(j\omega)| \leq 1,$$

where

$$\begin{aligned} |G^*_l(\mathrm{j}\,\omega)| = {} & \sqrt{G^*_l(\mathrm{j}\,\omega)G^*_l(-\mathrm{j}\,\omega)} \\ = {} & \sqrt{\frac{[a_l(\Lambda^y_l + \Lambda^q_l)]^2 + [(k^y_l + k^q_l)\omega]^2}{[a_l(\Lambda^y_l + \Lambda^q_l) - \omega^2)]^2 + [(a_l + k^y_l + k^q_l)\omega]^2}} \\ \leq {} & 1 \end{aligned}$$

for $\omega \in [0, +\infty)$.

Let

$$g(\omega) = \frac{[a_l(\Lambda_l^y + \Lambda_l^q)]^2 + [(k_l^y + k_l^q)\omega]^2}{[a_l(\Lambda_l^y + \Lambda_l^q) - \omega^2)]^2 + [(a_l + k_l^y + k_l^q)\omega]^2}.$$

Obviously, $g(0) = 1$. Then we should make $g(\omega) \leq 1$ for $\omega \in (0, +\infty)$, that is,

$$\omega^2 + (a_l + k_l^y + k_l^q)^2 - 2a_l(\Lambda_l^y + \Lambda_l^q) - (k_l^y + k_l^q)^2 \geq 0. \quad (14)$$

It is easily known that the sufficient condition for Eq. (14) is given as $(a_l + k_l^y + k_l^q)^2 - 2a_l(\Lambda_l^y + \Lambda_l^q) - (k_l^y + k_l^q)^2 \geq 0$ (i.e. $(k_l^y + k_l^q) \geq (\Lambda_l^y + \Lambda_l^q) - a_l/2$).

Moreover, as for the relationship of backward propagation of desired velocity difference to the velocity fluctuation of vehicle n_l in lane l is analysed as follows,

$$H_{13}(s) = \frac{k_l^q s + a_l \Lambda_l^q}{s(s + a_l + k_l^y + k_l^q) + a_l(\Lambda_l^y + \Lambda_l^q)}.$$

Once the condition (i.e. $(k_l^y + k_l^q) \geq (\Lambda_l^y + \Lambda_l^q) - a_l/2$) is met, we easily have $\|H_{13}(s)\|_\infty < 1$, which indicates that velocity disturbance due to Δv^* will not be amplified when propagated backwards.

For all the above analyses, a theorem can be derived as follows.

Theorem 1 Under the condition that the traffic disturbance has caused the traffic jams in the two-lane road, the feedback control signals should be added to vehicles so as to suppress the traffic jams, and the feedback gains should meet the conditions: $(k_l^y + k_l^q) \geq (\Lambda_l^y + \Lambda_l^q) - a_l/2$ and $k_l^y + k_l^q > -a_l$.

4. Numerical simulations

In this paper, the boundary condition is the periodic boundary condition. The desired velocities of vehicles in both lanes are $v_1^* = 0.9354$ and $v_2^* = 0.9468$, respectively, and the desired comprehensive distances are $h_1^d = 1.7$ and $h_2^d = 1.8$ respectively. Meanwhile, we can set that $\alpha_1^y = 0.7$, $\alpha_1^q = 0.3$, $y_1^* = 2$, $q_1^* = 1$ in lane 1 and $\alpha_2^y = 0.8$, $\alpha_2^q = 0.2$, $y_2^* = 2$, $q_2^* = 1$ in lane 2. $h_1^b = h_2^f = 0.5$. In addition, the length of the circular two-lane highway is 200. Let N_l vehicles be distributed along lane l uniformly, where N_l is an even number (i.e., $N_1 = N_2 = 100$). The initial positions of these vehicles in both lanes are $x_{1,1}(0) = 199, x_{1,n_1}(0) = 199 - 2(n_1 - 1)$; $x_{2,1}(0) = 200, x_{2,n_2}(0) = 200 - 2(n_2 - 1)$. Obviously, we know that $y_{l,n_l}(0) = 2$ and $q_{l,n_l}(0) = 1$ for $l = 1$ and 2. However, because of the unsteadiness of traffic flow, the headway of the 50th vehicle decreases suddenly for a small period of time as follows, if $(20 \leq t \leq 30)$,

$$y_{1,50}(t) = y_{1,50}(t)/3,$$
$$y_{1,51}(t) = y_{1,51}(t) + 2y_{1,50}(t)/3.$$

Besides, when a vehicle carries out lane changing, its headway is set to be 5 so as to record its lane changing behaviour. For the above experimental scenario, we construct some numerical experiments with or without control signals and with or without lane changing behaviours.

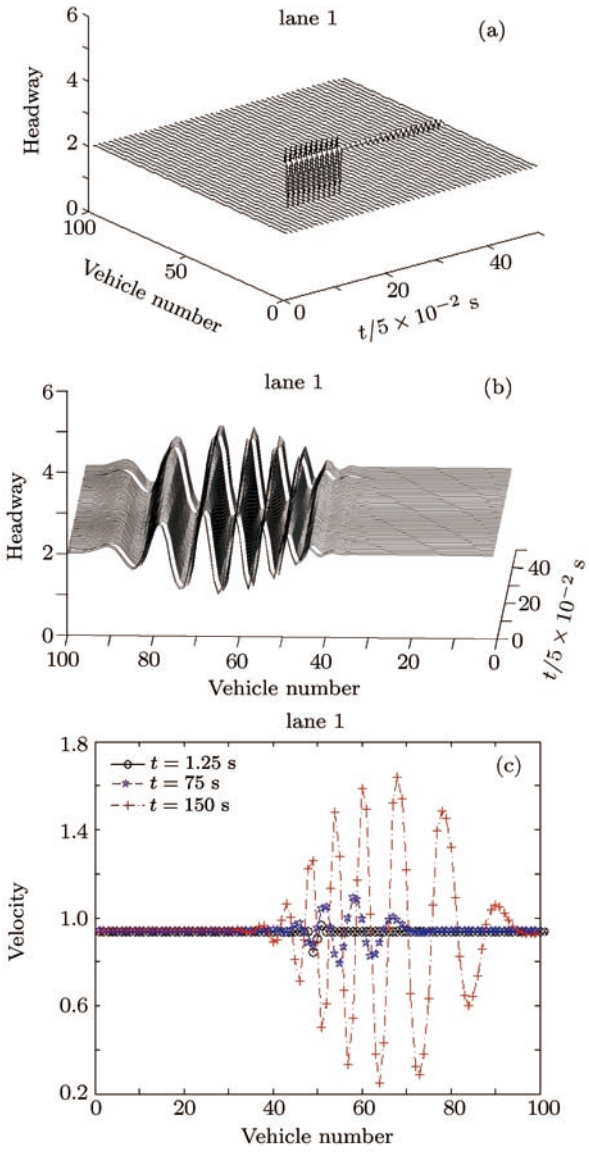


Fig. 4. Diagrams of headway and velocity fluctuations without lane changing behaviours. (a) Headway fluctuation in the first 2.5 s. (b) Headway fluctuation in the last 2.5 s. (c) Velocity fluctuation at different time points.

4.1. Numerical experiments without control signals

(i) Sensitivities of drivers in both lanes are $a_1 = 1$ and $a_2 = 1.5$ respectively.

When the stability conditions in Lemma are not met, figures 4 and 5 illustrate that the traffic disturbance, because of the reduction in 50-th vehicle headway, is amplified when propagated backwards, which accords well with the theoretical results in Lemma. In addition, figure 5(a) indicates that four vehicles have carried out lane changing. And by comparing Fig. 4 with Fig. 5, it is known that lane changing behaviours have a significant effect on the running state of two vehicle groups, which would intensify the velocity and headway fluctuations of vehicles.

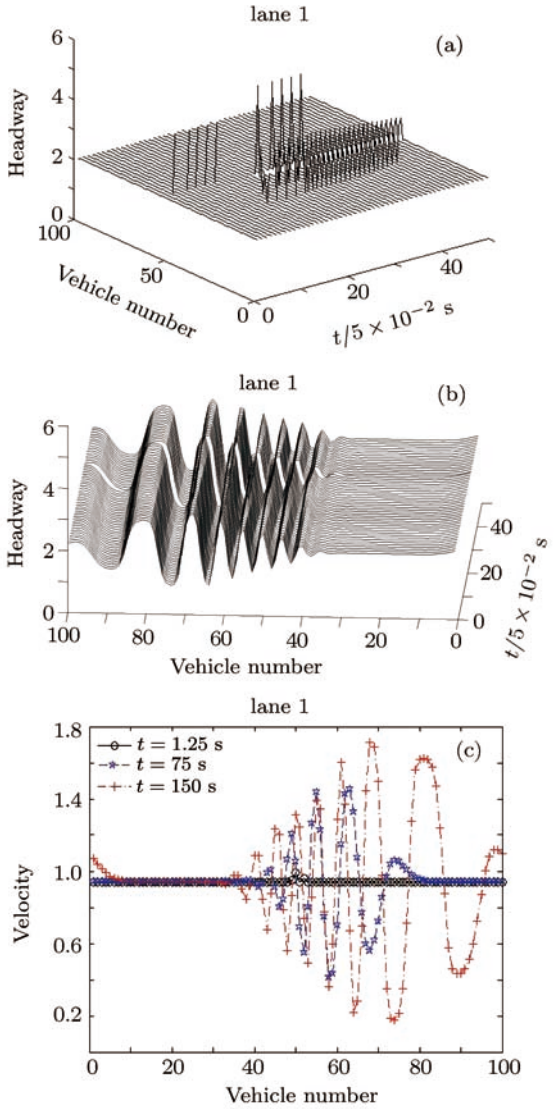


Fig. 5. Diagrams of headway and velocity fluctuations with lane changing behaviours. (a) Headway fluctuation in the first 2.5 s. (b) Headway fluctuation in the last 2.5 s. (c) Velocity fluctuation at different time points.

(ii) Sensitivities of drivers in both lanes are $a_1 = 2$ and $a_2 = 2.5$ respectively.

When the stability conditions in Lemma are met for the coupled lane, without the consideration of lane changing behaviours, the two vehicle groups can adjust themselves to their steady states gradually although the headway fluctuation appears in the beginning. However, the whole steady states are broken when the lane changing behaviours are involved and some vehicles would stay in a fluctuation state in the end (cf. Figs. 6 and 7).

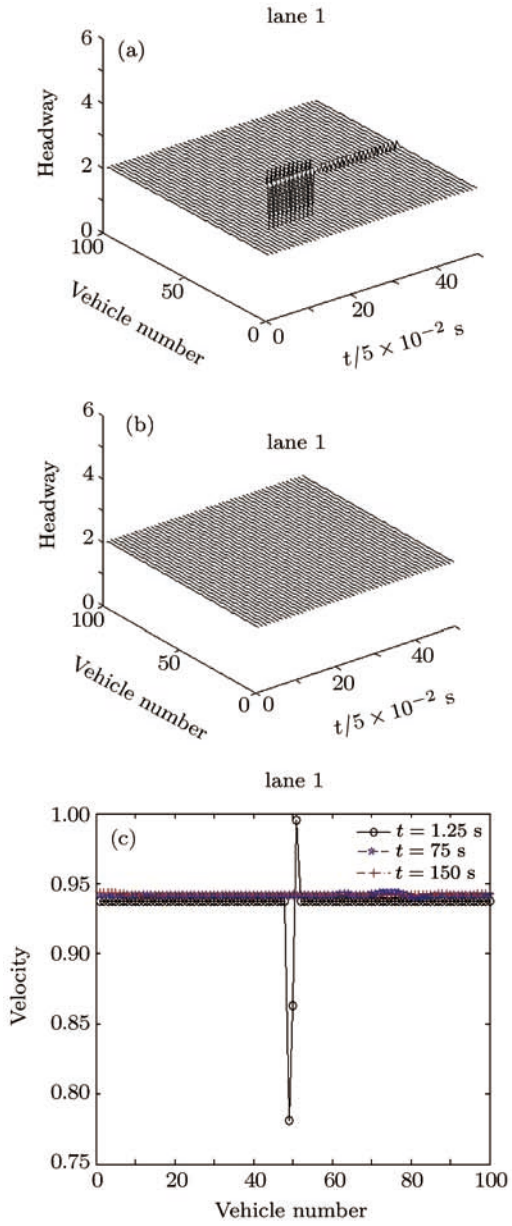


Fig. 6. Diagrams of headway and velocity fluctuations without lane changing behaviours. (a) Headway fluctuation in the first 2.5 s. (b) Headway fluctuation in the last 2.5 s. (c) Velocity fluctuation at different time points.

What deserves our attention is that when the sensitivity coefficients of drivers in both lanes are increased, the frequency of lane changing behaviours and the number of vehicles in the oscillation state become smaller (cf. Figs. 7 and 8), which implies that the more sensitive the drivers, the more easily they can adapt themselves to the road conditions.

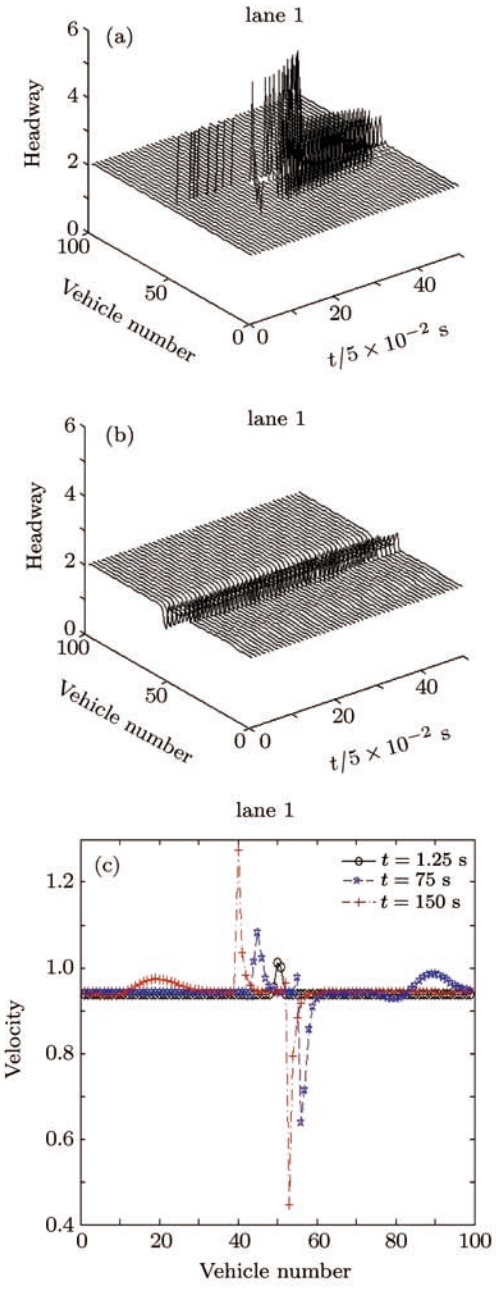


Fig. 7. Diagrams of headway and velocity fluctuations with lane changing behaviours. (a) Headway fluctuation in the first 2.5 s. (b) Headway fluctuation in the last 2.5 s. (c) Velocity fluctuation at different time points.

(iii) The sensitivities of drivers in both lanes are $a_1 = 4$ and $a_2 = 4.5$ respectively.

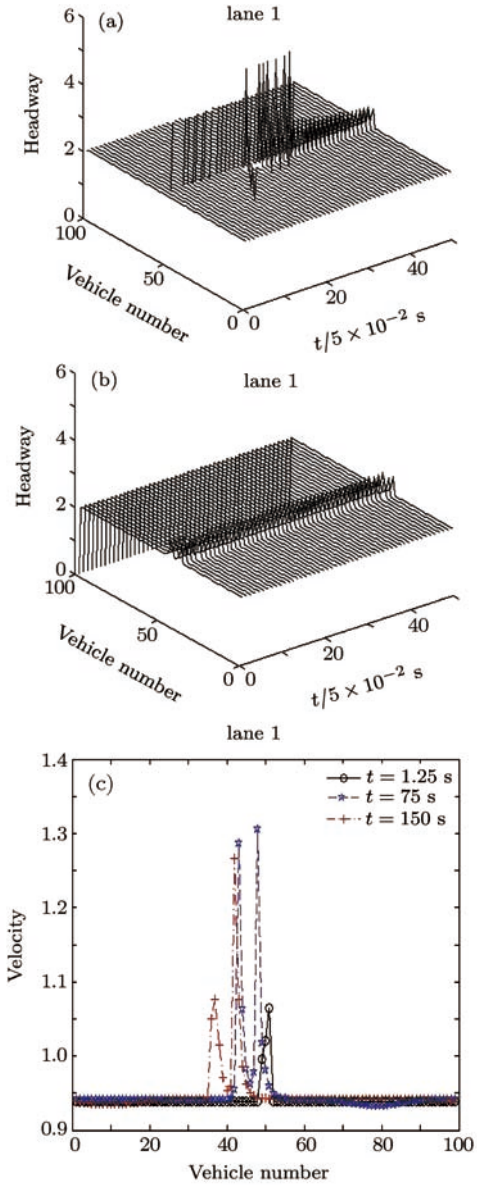


Fig. 8. Diagrams of headway and velocity fluctuations with lane changing behaviours. (a) Headway fluctuation in the first 2.5 s. (b) Headway fluctuation in the last 2.5 s. (c) Velocity fluctuation at different time points.

4.2. Numerical simulations with control signals

(I) Feedback gains: $k_1^y = 0.3$, $k_1^q = 0.2$, $k_2^y = 0.15$, $k_2^q = 0.1$, $a_1 = 1$, and $a_2 = 1.5$.

(II) Feedback gains: $k_1^y = 0.3$, $k_1^q = 0.2$, $k_2^y = 0.15$, $k_2^q = 0.1$, $a_1 = 1$, and $a_2 = 1.5$.

When stability conditions in Lemma are not fulfilled but control signals are added to vehicles in the road, the traffic disturbance, because of the reduction in the 50th vehicle's headway, cannot be amplified but compressed efficiently, and the two vehicle groups would return to their steady states immediately (cf. Fig. 9). However, when lane changing is introduced,

the headway and velocity fluctuations are aggravated and the arrival of the steady state will be delayed. Correspondingly, feedback signals are also magnified (cf. Figs. 9(d) and 10(d)), which illustrates that the lane changing behaviours have produced a significant influence on stability of the controlled traffic system.

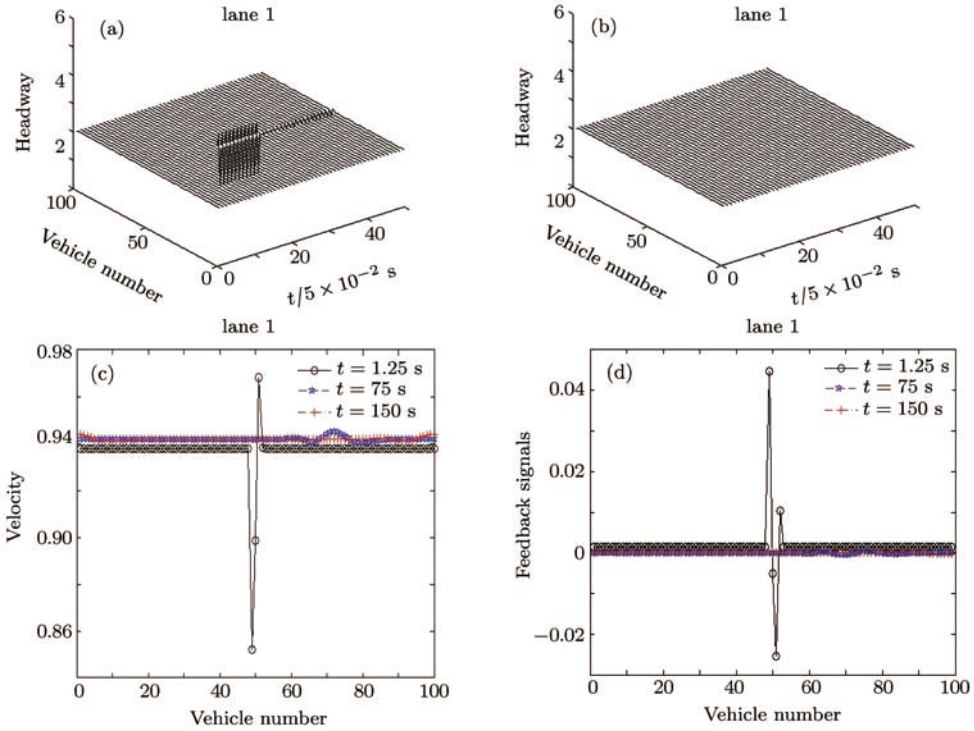


Fig. 9. Diagrams of headway and velocity fluctuations without lane changing behaviours. (a) Headway fluctuation in the first 2.5 s. (b) Headway fluctuation in the last 2.5 s. (c) Velocity fluctuation at different time points. (d) Feedback signals at different time points.

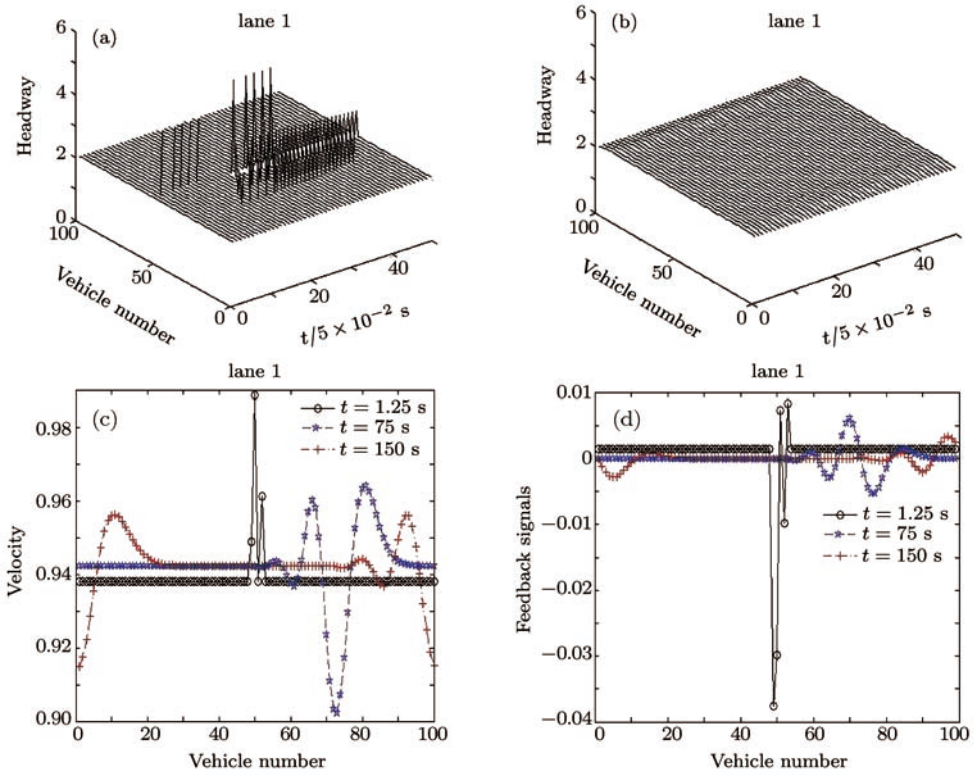


Fig. 10. Diagrams of headway and velocity fluctuations with lane changing behaviours. (a) Headway fluctuation in the first 2.5 s. (b) Headway fluctuation in the last 2.5 s. (c) Velocity fluctuation at different time points. (d) Feedback signals at different time points.

(III) Feedback gains: $k_1^y = 0.6$, $k_1^q = 0.4$, $k_2^y = 0.3$, $k_2^q = 0.2$, $a_1 = 1$, and $a_2 = 1.5$.

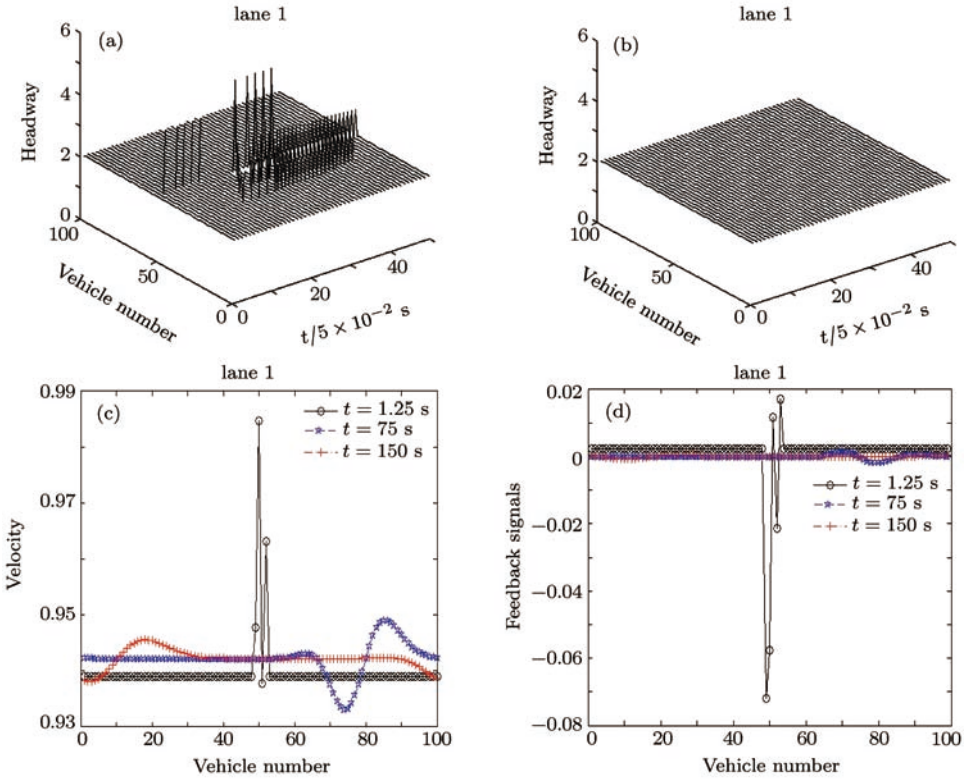


Fig. 11. Diagrams of headway and velocity fluctuations with lane changing behaviours. (a) Headway fluctuation in the first 2.5 s. (b) Headway fluctuation in the last 2.5 s. (c) Velocity fluctuation at different time points. (d) Feedback signals at different time points.

(IV) Feedback gains: $k_1^y = 0.9$, $k_1^q = 0.6$, $k_2^y = 0.45$, $k_2^q = 0.3$, $a_1 = 1$, and $a_2 = 1.5$.

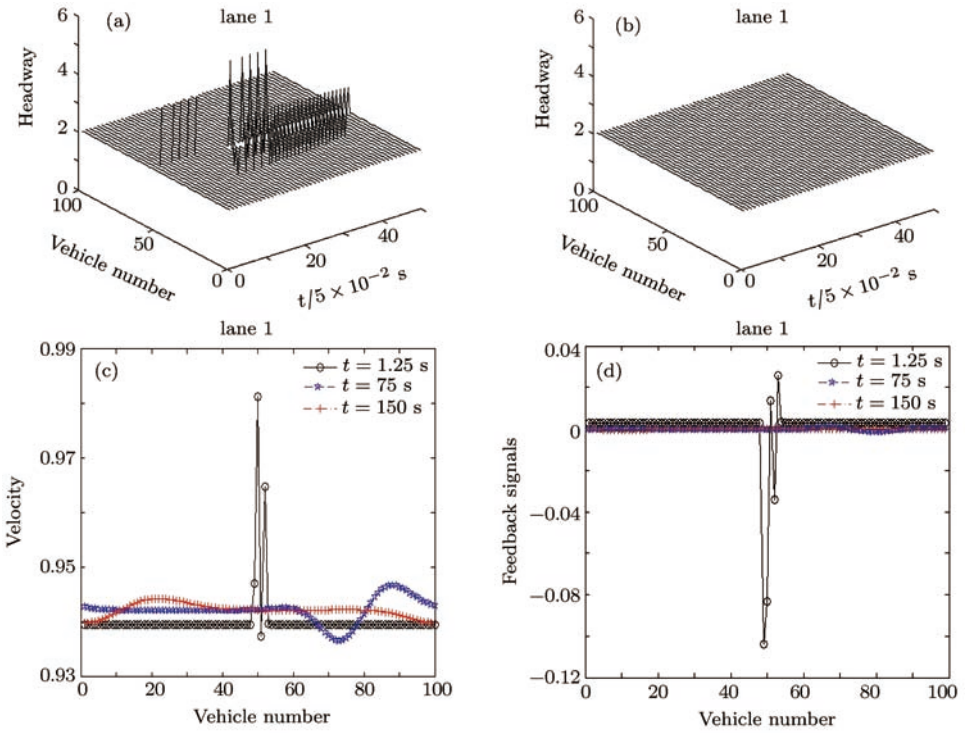


Fig. 12. Diagrams of headway and velocity fluctuations with lane changing behaviours. (a) Headway fluctuation in the first 2.5 s. (b) Headway fluctuation in the last 2.5 s. (c) Velocity fluctuation at different time points. (d) Feedback signals at different time points.

Figures 10–12 also show that with the consideration of lane changing behaviours, the velocity fluctuations are very significant. However, they are all propagated backwards with a decline in vibration amplitudes, and finally the velocity and headway fluctuations become very small, which implies that the control method can alleviate the traffic fluctuation to some extent although the lane changing behaviours are introduced. In addition, it is worth mentioning that the velocity and headway fluctuations are decreasing when the feedback gains are enlarged (cf. Figs. 10–12), which indicates that the traffic disturbance can be suppressed successfully only if the feedback gains are large enough.

5. Conclusions

This paper deals mainly with the influence of lane changing behaviours on the stability of two vehicle groups with or without control signals. First, without control signals, the stability conditions for both lanes are derived, which would make the two vehicle groups return to their own steady states gradually when lane changing behaviours are not considered. Then, when control signals are added to all vehicles in the road, we also obtain the stability conditions under which control signals can alleviate the traffic disturbance in the road efficiently no matter whether lane changing is involved.

Finally, the numerical experiments are conducted and we can draw some important conclusions as follows. I) without control signals, vehicles from different lanes closely interact with each other. That is, when the stability conditions in Lemma are not met, the headway disturbance in one lane can be propagated backwards and to the neighbouring lane with an increase in disturbance, which results in traffic jams eventually. II) Without control signals, lane changing behaviours will break the steady states or intensify the velocity and headway fluctuations of vehicles although the stability conditions in Lemma are met. III) With the consideration of control signals, although the conditions of feedback gains in Theorem 1 are met, the traffic fluctuation will be alleviated only to some extent due to lane changing behaviours. However, the traffic jams can be suppressed ultimately only if the feedback gains are large enough, which implies that lane changing behaviours also generate some influence on the stability of the controlled two-lane traffic flow. Above all, the conclusions accord well with our theoretical results, which indicates that the present paper has certain theoretical and practical significance.

Appendix A

The expression of $R(s)$ reads

$$R(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) & H_{13}(s) & H_{14}(s) \\ H_{21}(s) & H_{22}(s) & H_{23}(s) & H_{24}(s) \\ H_{31}(s) & H_{32}(s) & H_{33}(s) & H_{34}(s) \end{bmatrix},$$

where

$$\begin{aligned}
H_{11}(s) &= \frac{a_l \Lambda_l^y}{s(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)}, \\
H_{12}(s) &= \frac{a_l \Lambda_l^q}{s(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)}, \\
H_{13}(s) &= \frac{a_l \Lambda_l^q}{s(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)}, \\
H_{14}(s) &= \frac{s}{s(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)}, \\
H_{21}(s) &= \frac{s(s+a_l) + a_l \Lambda_l^q}{s^2(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)s}, \\
H_{22}(s) &= \frac{-a_l \Lambda_l^q}{s^2(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)s}, \\
H_{23}(s) &= \frac{-a_l \Lambda_l^q}{s^2(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)s}, \\
H_{24}(s) &= \frac{-1}{s(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)}, \\
H_{31}(s) &= \frac{-a_l \Lambda_l^y}{s^2(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)s}, \\
H_{31}(s) &= \frac{s(s+a_l) + a_l \Lambda_l^y}{s^2(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)s}, \\
H_{33}(s) &= \frac{s(s+a_l) + a_l \Lambda_l^y}{s^2(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)s}, \\
H_{34}(s) &= \frac{-1}{s(s+a_l) + a_l(\Lambda_l^y + \Lambda_l^q)}.
\end{aligned}$$

Appendix B

The expressions of H_{ij} $(i, j = 1, 2, 3)$ are give below.

$$H_{11}(s) = \frac{k_l^y s + a_l \Lambda_l^y}{s(s + a_l + k_l^y + k_l^q) + a_l(\Lambda_l^y + \Lambda_l^q)}, \quad H_{12}(s) = H_{13}(s) = \frac{k_l^q s + a_l \Lambda_l^q}{s(s + a_l + k_l^y + k_l^q) + a_l(\Lambda_l^y + \Lambda_l^q)},$$

$$H_{21}(s)=\frac{s(s+a_l+k_l^q)+a_l\Lambda_l^q}{s^2(s+a_l+k_l^y+k_l^q)+a_l(\Lambda_l^y+\Lambda_l^q)s},\quad H_{22}(s)=H_{23}(s)=\frac{-k_l^q s-a_l\Lambda_l^q}{s^2(s+a_l+k_l^y+k_l^q)+a_l(\Lambda_l^y+\Lambda_l^q)s},$$
$$H_{31}(s)=\frac{-k_l^y s-a_l\Lambda_l^y}{s^2(s+a_l+k_l^y+k_l^q)+a_l(\Lambda_l^y+\Lambda_l^q)s},\quad H_{32}(s)=H_{33}(s)=\frac{s(s+a_l+k_l^y)+a_l\Lambda_l^y}{s^2(s+a_l+k_l^y+k_l^q)+a_l(\Lambda_l^y+\Lambda_l^q)s}.$$

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