A general method for synchronizing an integer-order chaotic system and a fractional-order chaotic system

Si Gang-Quan(司刚全), Sun Zhi-Yong(孙志勇)†, and Zhang Yan-Bin(张彦斌)

State Key Laboratory of Electrical Insulation and Power Equipment, School of Electrical Engineering, Xi'an Jiaotong University, Xi'an 710049, *China*

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This paper investigates the synchronization between integer-order and fractional-order chaotic systems. By introducing fractional-order operators into the controllers, the addressed problem is transformed into a synchronization one among integer-order systems. A novel general method is presented in the paper with rigorous proof. Based on this method, effective controllers are designed for the synchronization between Lorenz systems with an integer order and a fractional order, and for the synchronization between an integer-order Chen system and a fractional-order Liu system. Numerical results, which agree well with the theoretical analyses, are also given to show the effectiveness of this method.

Keywords: chaos synchronization, integer-order chaotic system, fractional-order chaotic system, fractional calculus

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1. Introduction

Fractional calculus, which can be dated back to the late seventeenth century, has been of considerable importance and popularity over the past several decades.[1] A particular feature is that engineers and scientists have developed many new models that involve fractional differential equations, among which the study of chaotic systems with fractional orders is one of the hottest research areas. Up to now, people have studied the fractional-order nonlinear dynamics of many famous chaotic systems, such as the fractional-order Lorenz system,[2] the fractional-order Rössler system,[3] the fractional-order Chen system,[4] and the fractional-order Liu system.[5]

Chaos synchronization has received much attention from researchers in recent years. The synchronization problem among fractional-order systems, as well as among integer-order systems, has been intensively studied. One can refer to Refs. [6]–[14] and references cited therein for some relevant work.

The above-mentioned papers are mainly concerned with the synchronization scheme when the drive and response systems are of identical orders, i.e., all integer orders or identical fractional orders. Recently, some researchers started to investigate the synchronization problem when the drive and response systems were of non-identical orders. It was found that in such a case synchronization can also be achieved. Zhou *et al.*[15,16] first studied the synchronization between a fractional-order system and an integer-order system. Based on the concept of tracking control, effective controllers were designed in Refs. [15] and [16] to achieve the synchronization. Later, via similar methods, they further studied the function projective synchronization between an integer-order Lorenz system and a fractional-order Chen system.[17] In Ref. [18], Odibat studied the synchronization of Chen systems and the synchronization of Rössler systems when the drive and response systems were of different orders. More recently, Jia *et al.*[19] proposed a novel method based on Lyapunov stability theory and numerical differentiation to realize synchronization between a fractional-order unified system and an integer-order Chen system. However, the method proposed in Ref. [19], though somewhat simpler than those in Refs. [15] and [16], has some drawbacks. One of them that should be mentioned is that the synchronization error in Ref. [19] does not converge to zero even if the control gains have been adopted to as large as 1000. Since the synchronization between chaotic systems with non-identical orders is a new domain in the research field of chaos synchronization, no other reference papers relevant to this topic have been reported to our knowledge.

In this paper, we consider the synchronization case when an integer-order system is chosen as the drive system and a fractional-order system serves as the response system. Based on several properties on

†Corresponding author. E-mail: susn.zhi.yong@stu.xjtu.edu.cn

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fractional calculus, we propose a novel general method to accomplish chaos synchronization in such a case. This method is similar but not identical to that in Ref. [18]. Furthermore, this paper presents a generalized and extended form of the work in Ref. [18]. By introducing fractional-order operators into the controllers, the addressed problem is transformed into a synchronization problem among integer-order systems. The designed controllers in the response system based on this method involve fractional calculus operators, which are different from the controllers adopted in the synchronization schemes between chaotic systems with identical orders as reported in most of the relevant papers.

This paper is organized as follows. Section 2 gives the definition and some basic properties on fractional calculus. In Section 3 we discuss the formulation problem and then present the general method together with its proof. In Section 4, we consider two examples to illustrate the effectiveness of this method. The controllers for the synchronization between the integer-order Lorenz system and the commensurate fractional-order Lorenz system, and that between the integer-order Chen system and the incommensurate fractional-order Liu system, are well designed based on the proposed method. Moreover, the numerical results are also given in Section 4, which agree well with the analytical results. Finally, conclusions in Section 5 close this article.

2. Some basic definitions and properties of fractional calculus

Fractional calculus is the field of mathematical analysis that deals with the use of integrals and derivatives of arbitrary order. At present there are several definitions of fractional calculus. The two most commonly used are the Riemann–Liouville (RL) definition and the Caputo definition.[1] In practical applications, researchers prefer the Caputo definition to the RL definition, since the Caputo definition involves initial conditions that are physically interpretable and measurable. In this paper we consider the Caputo definition.

The Caputo definition is given as[1]

$$ {}_a^*D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau \qquad (1) $$

for $n-1 < \alpha < n$, $n \in \mathbf{N}$, where $\Gamma(\cdot)$ is the gamma function. In the following analysis, the lower terminal a is set to be zero and we use the notation D^α as the simplified form of ${}_0^*D_t^\alpha$ $(t > 0)$.

Some basic properties of fractional derivatives and integrals used in the paper are listed as follows.[1,20]

(i) For $\alpha = n$, where n is an integer, the operation $D^\alpha f(t)$ gives the same result as classical integer order n. Particularly, when $\alpha = 1$, the operation $D^\alpha f(t)$ is the same as the ordinary derivative, i.e., $D^1 f(t) = \mathrm{d}f(t)/\mathrm{d}t$; when $\alpha = 0$, the operation $D^\alpha f(t)$ is the identity operation: $D^0 f(t) = f(t)$.

(ii) Fractional differentiation and fractional integration are linear operations:

$$ D^\alpha[af(t) + bg(t)] = aD^\alpha f(t) + bD^\alpha g(t). \qquad (2) $$

(iii) For $\alpha \geq 0$, the following equation holds

$$ D^\alpha D^{-\alpha} f(t) = D^0 f(t) = f(t), \qquad (3) $$

which means that the fractional differentiation operator is a left inverse to the fractional integration operator of the same order α.

(iv) The Laplace transform formula for the Caputo fractional derivative is as follows:

$$ L\{D^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad (\alpha > 0, n-1 < \alpha \leq n). \qquad (4) $$

Particularly, when $\alpha \in (0, 1]$, we have $L\{D^\alpha f(t)\} = s^\alpha F(s) - s^{\alpha-1} f(0)$. The Laplace transform of the fractional integral satisfies

$$ L\{D^{-\alpha} f(t)\} = s^{-\alpha} L\{f(t)\} = s^{-\alpha} F(s), \ (\alpha > 0). \qquad (5) $$

3. Problem formulation and synchronization scheme

Consider the case that the integer-order chaotic system serves as the drive system and the fractional-order chaotic system is the response system, which are given as

$$ \dot{\boldsymbol{x}} = f(\boldsymbol{x}) \rightarrow \text{drive system}, \qquad (6) $$

and

$$ D^{\boldsymbol{\alpha}} \boldsymbol{y} = g(\boldsymbol{y}) + U(\boldsymbol{x}, \boldsymbol{y}) \rightarrow \text{response system}, \qquad (7) $$

where $\boldsymbol{\alpha}$ is an $n \times 1$ vector which denotes the fractional orders for each state of the response system, i.e., $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^{\mathrm{T}}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$ are the n-dimensional state vectors for drive and response systems, respectively; $f, g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ are continuous vector functions and $U(\boldsymbol{x}, \boldsymbol{y})$ is the controller that should be designed later.

Decompose the integer-order drive system (6) and the fractional-order response system (7) as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + F(\boldsymbol{x}), \tag{8}$$

and

$$D^{\boldsymbol{\alpha}}\boldsymbol{y} = \boldsymbol{B}\boldsymbol{y} + G(\boldsymbol{y}) + U(\boldsymbol{x}, \boldsymbol{y}), \tag{9}$$

where $\boldsymbol{A}, \boldsymbol{B} \in \mathbf{R}^{n\times n}$ are the parameter vectors for the linear parts in the drive system and the response system, respectively; $F, G : \mathbf{R}^n \to \mathbf{R}^n$ are the nonlinear parts of the drive system and the response system, respectively.

We define the state errors as $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x}$ if $\boldsymbol{e}(t) \to 0$ as $t \to \infty$, then we say that the integer-order drive system and the fractional-order response system achieve synchronization.

Remark 1 If the fractional orders in Eq. (7) satisfy $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, system (7) is called a commensurate order system, otherwise system (7) indicates an incommensurate order system.

Remark 2 When $\boldsymbol{A} = \boldsymbol{B}$ and $F(\cdot) = G(\cdot)$, the problem is reduced to a synchronization problem between identical systems with integer order and fractional order. In Ref. [18] some relevant research has been done where the author designed effective controllers to realize the synchronization between an integer-order Chen system and a fractional-order Chen system. Next, we propose a generalized method that is applicable in such a case as well as other cases when the drive system and response system are nonidentical not only in derivative orders but also in system descriptions.

We decompose the controller function $U(\boldsymbol{x}, \boldsymbol{y})$ into two sub-controllers $U_{\rm I}(\boldsymbol{x}, \boldsymbol{y})$ and $U_{\rm II}(\boldsymbol{x}, \boldsymbol{y})$, i.e., $U(\boldsymbol{x}, \boldsymbol{y}) = U_{\rm I}(\boldsymbol{x}, \boldsymbol{y}) + U_{\rm II}(\boldsymbol{x}, \boldsymbol{y})$ and propose the following form for the sub-controller $U_{\rm I}(\boldsymbol{x}, \boldsymbol{y})$

$$U_{\rm I}(\boldsymbol{x}, \boldsymbol{y}) = (D^{-(1-\boldsymbol{\alpha})} - 1)[\,g(\boldsymbol{y})]. \tag{10}$$

By inserting sub-controller (10) into system (7), we can rewrite the response system as follows:

$$D^{\boldsymbol{\alpha}}\boldsymbol{y} = D^{-(1-\boldsymbol{\alpha})}g(\boldsymbol{y}) + U_{\rm II}(\boldsymbol{x}, \boldsymbol{y}). \tag{11}$$

Applying a Laplace transform to the above system and letting $\boldsymbol{Y}(s) = L\{\boldsymbol{y}(t)\}$, we obtain

$$\begin{aligned} &s^{\boldsymbol{\alpha}}\boldsymbol{Y}(s) - s^{\boldsymbol{\alpha}-1}\boldsymbol{y}(0) \\ &= s^{-(1-\boldsymbol{\alpha})}L\{g(\boldsymbol{y})\} + L\{U_{\rm II}(\boldsymbol{x}, \boldsymbol{y})\}. \end{aligned} \tag{12}$$

Multiplying both the left-hand and right-hand sides of Eq. (12) by $s^{1-\boldsymbol{\alpha}}$ and again applying the inverse Laplace transform to the result, we obtain the following system, which is equivalent to systems (9) and (7),

$$D^1\boldsymbol{y} = g(\boldsymbol{y}) + D^{1-\boldsymbol{\alpha}}[U_{\rm II}(\boldsymbol{x}, \boldsymbol{y})]. \tag{13}$$

By introducing the sub-controller $U_{\rm I}(\boldsymbol{x}, \boldsymbol{y})$, we then reduce the problem to the synchronization between integer-order systems. The synchronization problem between integer-order chaotic systems with identical or non-identical structures has been extensively studied in the literature and there are many available methods to achieve it. In this paper we propose the nonlinear observer of the following form as the sub-controller $U_{\rm II}(\boldsymbol{x}, \boldsymbol{y})$

$$\begin{aligned} U_{\rm II}(\boldsymbol{x}, \boldsymbol{y}) \;=\; & D^{-(1-\boldsymbol{\alpha})}[(\boldsymbol{A} - \boldsymbol{B})x - G(\boldsymbol{y}) \\ & + F(\boldsymbol{x}) - \boldsymbol{K}\boldsymbol{e}], \end{aligned} \tag{14}$$

where $\boldsymbol{K} \in \mathbf{R}^{n\times n}$ is a gain matrix. Now we have the following proposition.

Proposition 1 Given integer-order system (6) and fractional-order system (7), there exists a suitable gain matrix $\boldsymbol{K}$ to realize the synchronization between systems (6) and (7) when the controller $U(\boldsymbol{x}, \boldsymbol{y})$ designed above is adopted.

Proof Since system (13) is an equivalence of response system (7), substituting Eq. (14) into system (13) yields the response system in the following form

$$\begin{aligned} D^1\boldsymbol{y} \;=\; & \boldsymbol{B}\boldsymbol{y} + G(\boldsymbol{y}) + D^{1-\boldsymbol{\alpha}}D^{-(1-\boldsymbol{\alpha})} \\ & \times [(\boldsymbol{A} - \boldsymbol{B})\boldsymbol{x} - G(\boldsymbol{y}) + F(\boldsymbol{x}) - \boldsymbol{K}\boldsymbol{e}]. \end{aligned} \tag{15}$$

Recalling the properties of fractional calculus discussed in Section 2, we obtain the error dynamic system expressed as

$$\begin{aligned} \dot{\boldsymbol{e}} = \dot{\boldsymbol{y}} - \dot{\boldsymbol{x}} \;=\; & D^1\boldsymbol{y} - \dot{\boldsymbol{x}} \\ =\; & \boldsymbol{B}\boldsymbol{y} + G(\boldsymbol{y}) - [\boldsymbol{A}\boldsymbol{x} + F(\boldsymbol{x})] \\ & + [(\boldsymbol{A} - \boldsymbol{B})\boldsymbol{x} - G(\boldsymbol{y}) + F(\boldsymbol{x}) - \boldsymbol{K}\boldsymbol{e}] \\ =\; & (\boldsymbol{B} - \boldsymbol{K})\boldsymbol{e}. \end{aligned} \tag{16}$$

For the linear error system $\dot{\boldsymbol{e}}$, there indeed exist some matrices $\boldsymbol{K}$ to make all the eigenvalues λ_i $(i = 1, 2, \ldots, n)$ of $(\boldsymbol{B} - \boldsymbol{K})$ stay in the left-half plane (i.e., the eigenvalues are negative real numbers or complex numbers with negative real parts), which ensures that error system (16) is asymptotically stable. Hence the synchronization between integer-order system (6) and fractional-order system (7) is guaranteed. This completes the proof.

4. Illustrative examples

In this section, we consider two examples to illustrate the effectiveness of the proposed method. One is for the synchronization between Lorenz systems with

an integer order and a commensurate fractional order. The other is concerned with the synchronization problem between different chaotic systems with non-identical orders. In the latter, the synchronization between an integer-order Chen system and an incommensurate fractional-order Liu system is investigated.

The numerical solution of a fractional-order differential equation is more difficult than that of an ordinary differential equation. Hartley[21] calculated the transfer function approximations for the fractional integrator of orders from 0.1 to 0.9 in steps of 0.1. This method has been widely used in the simulation of fractional-order systems. However, it suffers from the following defects.

(I) The number of fractional orders that can be numerically simulated is confined to 9, namely, from 0.1 to 0.9. In other words, we cannot simulate systems with other fractional orders using this method.

(II) It may demonstrate some incorrect numerical results, especially in the simulation of fractional-order chaotic systems. In Ref. [22] the author presented an example on the misapplication of this method.

There exist many other numerical methods. One can refer to Ref. [23] for a detailed survey on the comparison of different approximation methods. In this paper we adopt the so-called CRONE-Oustaloup method[24] to simulate the fractional-order systems and the fractional calculus operators. Besides, this method has been developed into a free toolbox named *ninteger* by Duarte Valerio,[25] which has facilitated our numerical simulations.

Note that for the simulations of the integer-order systems the commonly used fourth-order Runge–Kutta method is employed here. All the computations are done via *Matlab/Simulink*.

4.1. Synchronization between an integer-order Lorenz system and a commensurate fractional-order Lorenz system

Let the famous Lorenz system,[26] which is described as follows, be the drive system

$$\begin{cases} \dot{x}_1 = 10(x_2 - x_1), \\ \dot{x}_2 = 28x_1 - x_1x_3 - x_2, \\ \dot{x}_3 = x_1x_2 - 8x_3/3. \end{cases} \tag{17}$$

Consider its corresponding fractional-order system as the response system, which is given by

$$\begin{cases} D^\alpha y_1 = 10(y_2 - y_1) + u_1, \\ D^\alpha y_2 = 28y_1 - y_1y_3 - y_2 + u_2, \\ D^\alpha y_3 = y_1y_2 - 8y_3/3 + u_3, \end{cases} \tag{18}$$

where $U = [u_1, u_2, u_3]^{\mathrm{T}}$ is the controller. It is reported that the fractional-order Lorenz system (18) with fractional order of $\alpha = 0.99$ can behave chaotically.[2] The three-dimensional (3D) phase portraits of the Lorenz chaotic system with integer order and fractional order, respectively, are shown in Fig. 1.

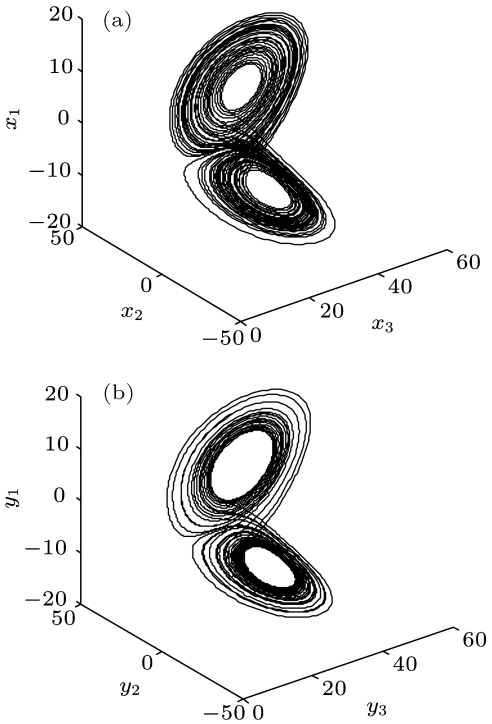


Fig. 1. The 3D phase portraits of the Lorenz chaotic system: (a) the integer-order Lorenz system (17); (b) the commensurate fractional-order Lorenz system (18) of order 0.99 (without control).

Based on the general method proposed above, the sub-controllers $U_{\mathrm{I}}(\boldsymbol{x}, \boldsymbol{y})$ and $U_{\mathrm{II}}(\boldsymbol{x}, \boldsymbol{y})$ can be derived directly from Eqs. (10) and (14), respectively. When we assemble the sub-controllers into the whole controller $U(\boldsymbol{x}, \boldsymbol{y})$, some reduplicate parts with opposite signs may be eliminated. Here we omit the expressions of the sub-controllers and give the simplified forms of the controller function $U(\boldsymbol{x}, \boldsymbol{y})$ as

$$\begin{cases} u_1 = (D^{-0.01} - 1)[10(y_2 - y_1)] \\ \qquad + D^{-0.01}(-\boldsymbol{k}_1\boldsymbol{e}), \\ u_2 = (D^{-0.01} - 1)(28y_1 - y_2) + y_1y_3 \\ \qquad + D^{-0.01}(-x_1x_3 - \boldsymbol{k}_2\boldsymbol{e}), \\ u_3 = (D^{-0.01} - 1)(-8y_3/3) - y_1y_2 \\ \qquad + D^{-0.01}(x_1x_2 - \boldsymbol{k}_3\boldsymbol{e}), \end{cases} \tag{19}$$

where

$$\begin{cases} \boldsymbol{e} = [e_1, e_2, e_3]^{\mathrm{T}} = [y_1 - x_1, y_2 - x_2, y_3 - x_3]^{\mathrm{T}}, \\ \boldsymbol{k}_i = [k_{i1}, k_{i2}, k_{i3}] \quad (i = 1, 2, 3), \\ \boldsymbol{K} = [\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3]^{\mathrm{T}}. \end{cases} \tag{20}$$

Thus, one can obtain the error system in the form as follows:

$$\dot{\boldsymbol{e}} = \begin{bmatrix} -10 - k_{11} & 10 - k_{12} & -k_{13} \\ 28 - k_{21} & -1 - k_{22} & -k_{23} \\ -k_{31} & -k_{32} & -8/3 - k_{33} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{21}$$

If the gain matrix is chosen as $\boldsymbol{K} = \mathrm{diag}(15, 15, 15)$, then the eigenvalues of the matrix $(\boldsymbol{B} - \boldsymbol{K})$ can be calculated to be $\lambda_1 = -37.8277$, $\lambda_2 = -17.6667$, $\lambda_3 = -3.1723$, which all satisfy λ_i $(i = 1, 2, 3) < 0$. Hence the error system is asymptotically stable and the synchronization is achieved.

Let the initial states of the drive system (17) and the response system (18) be $(x_1(0), x_2(0), x_3(0))^{\mathrm{T}} = (3, 4, 5)^{\mathrm{T}}$ and $(y_1(0), y_2(0), y_3(0))^{\mathrm{T}} = (-6, -10, 13.5)^{\mathrm{T}}$, respectively. Through numerical simulations, the time responses of state variables both in the integer-order Lorenz system and in the fractional-order Lorenz system are depicted in Fig. 2. The synchronization errors are shown in Fig. 3. We can see that the simulation results are in agreement with the theoretical analysis. The synchronization between integer-order and fractional-order Lorenz systems is observed.

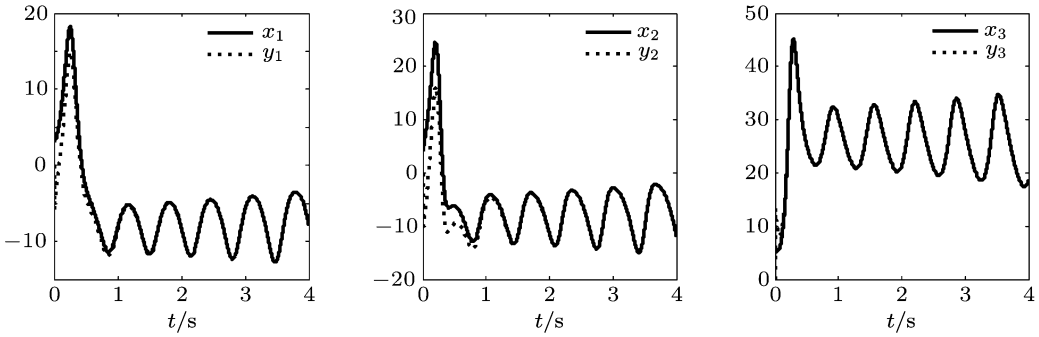


Fig. 2. Time responses of state variables in an integer-order drive system (solid line) and a fractional-order response system (dashed line).

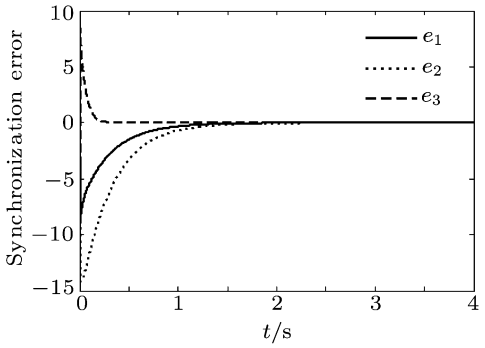


Fig. 3. Synchronization errors between the drive system and the response system.

4.2. Synchronization between an integer-order Chen system and an incommensurate fractional-order Liu system

Chen *et al.*[27] introduced a new chaotic system named the Chen system in 1999. The Chen system, which is similar but not topologically equivalent to Lorenz system, is described as

$$\begin{cases} \dot{x}_1 = a_1(x_2 - x_1), \\ \dot{x}_2 = (c_1 - a_1)x_1 - x_1 x_3 + c_1 x_2, \\ \dot{x}_3 = x_1 x_2 - b_1 x_3. \end{cases} \tag{22}$$

The default parameters for the Chen system are $a_1 = 35$, $b_1 = 3$, and $c_1 = 28$.

In Ref. [28], a new chaotic attractor called the Liu system was proposed. Here we consider the fractional-order version of the Liu system

$$\begin{cases} D^{\alpha_1} y_1 = a_2(y_2 - y_1), \\ D^{\alpha_2} y_2 = b_2 y_1 - k_2 y_1 y_3, \\ D^{\alpha_3} y_3 = -c_2 y_3 + h_2 y_1^2. \end{cases} \tag{23}$$

The commensurate fractional-order Liu system (i.e. system (23) with fractional order $\alpha_1 = \alpha_2 = \alpha_3$) has been studied in Refs. [5], [29], etc. However, there are few papers concerning the analysis and synchronization of an incommensurate fractional-order Liu system.

Via numerical method, we find that the incommensurate fractional-order Liu system of order $\alpha_1 =$

0.93, $\alpha_2 = 0.94$, $\alpha_3 = 0.95$ with the default parameters of $a_2 = 10$, $b_2 = 40$, $c_2 = 2.5$, $k_2 = 1$, and $h_2 = 4$ still exhibits chaotic dynamics. The chaotic attractor of the incommensurate fractional-order Liu system, together with that of the integer-order Chen system, is depicted in Fig. 4.

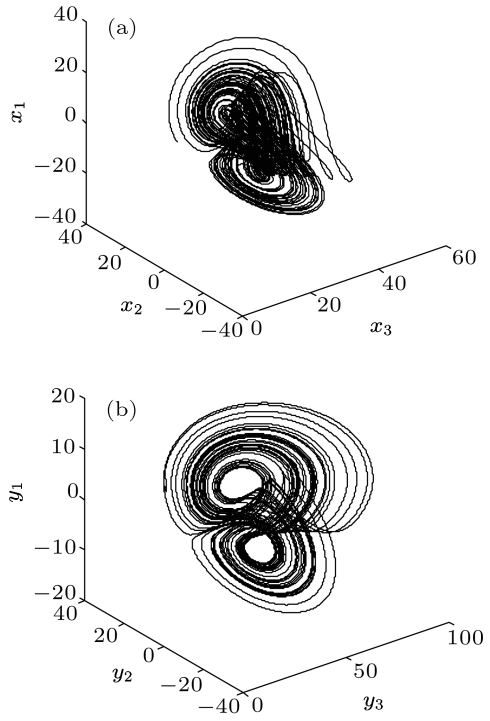


Fig. 4. The 3D phase portraits of the chaotic systems: (a) the integer-order Chen system (22) with parameters as $a_1 = 35$, $b_1 = 3$, $c_1 = 28$; (b) the incommensurate fractional-order Liu system (23) of order $\alpha_1 = 0.93$, $\alpha_2 = 0.94$, $\alpha_3 = 0.95$ with parameters $a_2 = 10$, $b_2 = 40$, $c_2 = 2.5$, $k_2 = 1$, and $h_2 = 4$.

Consider the integer-order Chen system (22) as the drive system and the incommensurate fractional-order Liu system as the response system. Rewrite the controlled fractional-order Liu system as follows:

$$\begin{cases} D^{0.93}y_1 = a_2(y_2 - y_1) + u_1, \\ D^{0.94}y_2 = b_2 y_1 - k_2 y_1 y_3 + u_2, \\ D^{0.95}y_3 = -c_2 y_3 + h_2 y_1^2 + u_3. \end{cases} \tag{24}$$

Combine the sub-controllers derived from Eq. (10) and (14) into a whole and we obtain the controller $U(\boldsymbol{x}, \boldsymbol{y})$ in the following form,

$$\begin{cases} u_1 = (D^{-0.07} - 1)[a_2(y_2 - y_1)] \\ \qquad + D^{-0.07}[(a_1 - a_2)(x_2 - x_1) - \boldsymbol{k}_1 \boldsymbol{e}], \\ u_1 = (D^{-0.06} - 1)(b_2 y_1) + k_2 y_1 y_3 \\ \qquad + D^{-0.06}[(c_1 - a_1 - b_2)x_1 \\ \qquad - x_1 x_3 + c_1 x_2 - \boldsymbol{k}_2 \boldsymbol{e}], \\ u_3 = (D^{-0.05} - 1)(-c_2 y_3) - h_2 y_1^2 \\ \qquad + D^{-0.05}[(c_2 - b_1)x_3 + x_1 x_2 - \boldsymbol{k}_3 \boldsymbol{e}), \end{cases} \tag{25}$$

where the definition of $\boldsymbol{k_i}$ $(i = 1, 2, 3)$ and $\boldsymbol{e}$ is the same as that in Eq. (20).

Then, the error system is calculated as

$$\dot{\boldsymbol{e}} = \begin{bmatrix} -a_2 - k_{11} & a_2 - k_{12} & -k_{13} \\ b_2 - k_{21} & -k_{22} & -k_{23} \\ -k_{31} & -k_{32} & -c_2 - k_{33} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{26}$$

Here we set the default parameters as described above. If one chooses the gain matrix as

$$\boldsymbol{K} = \begin{bmatrix} 0 & 10 & 0 \\ 10 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \tag{27}$$

then the eigenvalues of the matrix $(\boldsymbol{B} - \boldsymbol{K})$ are $\lambda_1 = -12.5 < 0$, $\lambda_2 = \lambda_3 = -10 < 0$, with default system parameters. Now we can conclude that, as the error system converges to the original point, the integer-order Chen system and the incommensurate fractional-order Liu system will be synchronized.

We also have performed some numerical computations to verify the analytical results. The initial conditions for the drive and response systems are taken as $(x_1(0), x_2(0), x_3(0))^{\mathrm{T}} = (5, 6, 7)^{\mathrm{T}}$ and $(y_1(0), y_2(0), y_3(0))^{\mathrm{T}} = (2.8, 4.7, 26)^{\mathrm{T}}$, respectively.

The results obtained from the numerical simulations are illustrated in Figs. 5 and 6. It can be seen that the state variables of the fractional-order response system successfully track the corresponding states of the integer-order drive system very quickly. The error system arrives at the stable point in a short time of less than one second. Hence, the synchronization between the integer-order Chen system and the incommensurate fractional-order Liu system is indeed achieved.

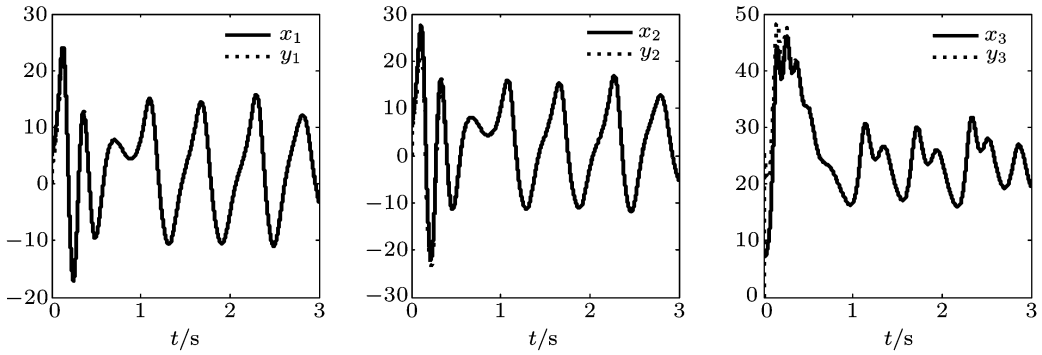


Fig. 5. Time responses of state variables in integer-order drive system (solid line) and fractional-order response system (dashed line). Note that the parameters of the drive and response systems are set at the default values in our simulation.

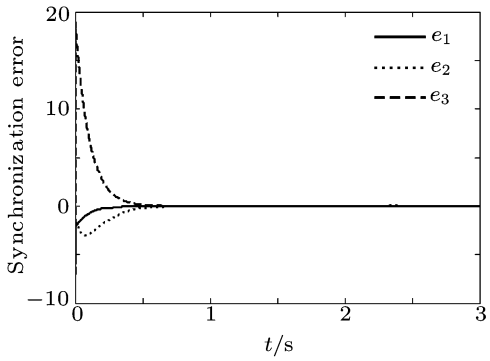


Fig. 6. Synchronization errors between the drive system and the response system.

5. Conclusions

In this paper we investigated the synchronization problem between an integer-order chaotic system and a fractional-order chaotic system. A general method is proposed to deal with this problem, which is theoretically rigorous and practically feasible. Two examples are given in the present paper, i.e., the synchronization between an integer-order Lorenz system and a fractional-order Lorenz system; and the synchronization between an integer-order Chen system and an incommensurate fractional-order Liu system. Based on the proposed method, the nonlinear controller functions are designed respectively to realize the synchronization of the two examples. At the same time, the numerical simulations are performed to verify the feasibility and effectiveness of the general method. The numerical results are in good accordance with the theoretical analyses.

References

[1] Diethelm K 2010 *The Analysis of Fractional Differential Equations* (Berlin: Springer)
[2] Grigorenko I and Grigorenko E 2003 *Phys. Rev. Lett.* **91** 34101
[3] Li C and Chen G 2004 *Physica* A **341** 55
[4] Li C and Chen G 2004 *Chaos, Solitons & Fractals* **22** 549
[5] Lu J J and Liu C X 2007 *Chin. Phys.* **16** 1586
[6] Boccaletti S, Kurths J, Osipov G, Valladares D and Zhou C 2002 *Phys. Rep.* **366** 1
[7] Wu Z M and Xie J Y 2007 *Chin. Phys.* **16** 1901
[8] Zhou P, Wei L J and Cheng X F 2009 *Chin. Phys.* B **18** 2674
[9] Zhang R X and Yang S P 2010 *Chin. Phys.* B **19** 020510
[10] Wang J W and Chen A M 2010 *Chin. Phys. Lett.* **27** 110501
[11] Wu D and Li J J 2010 *Chin. Phys.* B **19** 120505
[12] Zhang R and Xu Z Y 2010 *Chin. Phys.* B **19** 120511
[13] Lee S M, Kwon O M and Park J H 2011 *Chin. Phys.* B **20** 010506
[14] Li H Q, Liao X F and Huang H Y 2011 *Acta Phys. Sin.* **60** 020512 (in Chinese)
[15] Zhou P and Kuang F 2010 *Acta Phys. Sin.* **59** 6851 (in Chinese)
[16] Zhou P, Cheng Y M and Kuang F 2010 *Chin. Phys.* B **19** 090503
[17] Zhou P and Cao Y X 2010 *Chin. Phys.* B **19** 100507
[18] Odibat Z M 2010 *Nonlinear Dyn.* **60** 479
[19] Jia L X, Dai H and Hui M 2010 *Chin. Phys.* B **19** 110509
[20] Podlubny I 1999 *Fractional Differential Equations* (New York: Academic Press)
[21] Hartley T T, Lorenzo C F and Qammer K 1995 *IEEE Trans. Circuits Syst.* I **42** 485
[22] Tavazoei M S 2009 *IEEE Trans. Circuits Syst.* II **56** 519
[23] Vinagre B M, Podlubny I, Hernandez A and Feliu V 2000 *Fractional Calculus & Appl. Anal.* **3** 231
[24] Oustaloup A, Levron F, Mathieu B and Nanot F 2000 *IEEE Trans. Circuits Syst.* I **47** 25
[25] Valerio D and Sa da Costa J 2004 *Fractional Derivatives and Applications*, IFAC, Bordeaux
[26] Lorenz E N 1963 *J. Atmos. Sci.* **20** 130
[27] Chen G and Ueta T 1999 *Int. J. Bifur. Chaos* **9** 1465
[28] Liu C, Liu T, Liu L and Liu K 2004 *Chaos, Solitons & Fractals* **22** 1031
[29] Xu Z and Liu C X 2008 *Chin. Phys.* B **17** 4033