A new kind of nonlinear phenomenon in coupled fractional-order chaotic systems: coexistence of anti-phase and complete synchronization*

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In this paper, we have found a kind of interesting nonlinear phenomenon—hybrid synchronization in linearly coupled fractional-order chaotic systems. This new synchronization mechanism, i.e., part of state variables are anti-phase synchronized and part completely synchronized, can be achieved using a single linear controller with only one drive variable. Based on the stability theory of the fractional-order system, we investigated the possible existence of this new synchronization mechanism. Moreover, a helpful theorem, serving as a determinant for the gain of the controller, is also presented. Solutions of coupled systems are obtained numerically by an improved Adams–Bashforth–Moulton algorithm. To support our theoretical analysis, simulation results are given.

Keywords: fractional-order unified chaotic system, hybrid synchronization, linear controller, single drive variable

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1. Introduction

Although fractional-order differential calculus dates from the 17th century, the application of fractional calculus to physics and engineering has been a recent focus of interest. There are essential differences between ordinary differential equation systems and fractional-order differential systems. Most properties and conclusions of ordinary differential equation systems cannot be simply extended to that of fractional-order differential systems. Therefore, fractional-order systems have attracted more and more attention. Over the centuries, theories of fractional-order calculus (fractional-order derivatives and integrals) have undergone significant development, primarily contributed by pure, not applied, mathematicians. However, in the past two decades, many researchers have pointed out that fractional-order calculus is very suitable for describing the properties of various real materials. Compared with its classical integer-order counterpart, fractional-order derivatives can provide a better instrument for the description of the memory and hereditary properties of various materials and processes.[1−7]

Recently, there has been growing interest in the study of chaos synchronization due to its potential applications in secure communication and process control. Several typical kinds of synchronization have been identified, such as complete synchronization,[8−10] phase synchronization,[11,12] generalized synchronization,[13,14] etc. Many theoretical studies have been carried out for unidirectional coupled identical systems, usually by means of the feedback control approach. In this case, linear feedback via a single drive variable may be a better choice due to its simplicity of theoretical analysis and circuit implementation. In Ref. [15], Li *et al.* first investigated a new kind of interesting synchronization phenomenon existing in coupled integer-order Chen chaotic systems. That is to say, part states of the interactive systems are synchronized in terms of complete synchronization and partly synchronized in terms of anti-phase synchronization. Therefore, it is natural for us to ask whether there still exists a similar hybrid synchronization phenomenon in some fractional-order coupled chaotic systems. There is no doubt that this is an interesting problem. However, to the best of our knowledge, there are limited (if

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any) investigation results reported in literature. This is mainly because fractional-order stability theory is immature and the corresponding stability theory of an integer-order system cannot be extended to a fractional order.

In this paper, motivated by the above discussions, an interesting nonlinear phenomenon—hybrid synchronization in linearly coupled fractional-order chaotic systems is investigated. It is worthwhile mentioning that complete synchronization and anti-phase synchronization coexist in interactive fractional-order unified chaotic systems by a single controller with one drive variable. A theorem used to determine the gain of the linear controller is also proposed based on the stability theory of fractional-order system. We apply the improved version of the Adams–Bashforth–Moulton algorithm to compute the solutions of coupled systems numerically. Finally, numerical simulations show that part of the states in the coupled systems is anti-phase synchronized and part completely synchronized under a single linear controller with one drive variable. This also demonstrates that an interesting nonlinear phenomenon—hybrid synchronization does exist in fractional-order coupled chaotic systems.

This paper is orgnized as follows: in Section 2, the model of the fractional-order unified chaotic system is briefly reviewed and a theorem is proposed for designing the gain of the controller. The improved version of the Adams–Bashforth–Moulton algorithm is applied to solve the coupled drive and response fractional-order systems numerically in Section 3. Some numerical simulation results supporting our proposed theorem from Section 2 are given in Section 4. Finally, we end our investigation in Section 5 with brief conclusions.

2. Fractional-order derivative and its approximation

There are several definitions of fractional-order derivatives[16] where the Caputo derivative[17] is a time domain computation method. In practical applications, the Caputo derivative is more popular than others since un-homogenous initial conditions are permitted if such conditions are necessary. Furthermore, these initial values are easy to measure since they all have idiographic meanings.[18] The Caputo definition of a fractional-order derivative is as follows:

$$\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = J^{n-\alpha}\frac{\mathrm{d}^{n} f(t)}{\mathrm{d}t^{n}}, \tag{1}$$

where n is the first integer which is not less than α, J^{θ} is the θ-order Riemann–Liouville integral operator. It can be described as[19−25]

$$J^{\theta}\phi(t) = \frac{1}{\Gamma(\theta)}\int_0^t \frac{\phi(\tau)}{(t-\tau)^{(1-\theta)}}\,\mathrm{d}\tau, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function with $0 < \theta \leq 1$.

In order to present our theoretical results on the unified chaotic system in the following section, we first make a brief review of this model. The unified chaotic system is introduced by Lü *et al.*[20] with a single parameter α

$$\begin{cases} \dot{x}_1 = (25\alpha + 10)(x_2 - x_1), \\ \dot{x}_2 = (28 - 35\alpha)x_1 - x_1x_3 + (29\alpha - 1)x_2, \\ \dot{x}_3 = x_1x_2 - (8 + \alpha)x_3/3, \end{cases} \tag{3}$$

where $\alpha \in [0, 1]$. When $\alpha \in [0, 0.8)$, system (3) is called the generalized Lorenz system. When $\alpha = 0.8$, system (3) is called the Lü system and when $\alpha = (0.8, 1]$, system (3) is called the generalized Chen system.

Now, consider the following fractional-order unified system as a drive system,

$$\begin{cases} \dfrac{\mathrm{d}^{q} x_1}{\mathrm{d}t^{q}} = (25\alpha + 10)(x_2 - x_1), \\ \dfrac{\mathrm{d}^{q} x_2}{\mathrm{d}t^{q}} = (28 - 35\alpha)x_1 - x_1x_3 + (29\alpha - 1)x_2, \\ \dfrac{\mathrm{d}^{q} x_3}{\mathrm{d}t^{q}} = x_1x_2 - (8 + \alpha)x_3/3, \end{cases} \tag{4}$$

where q is the order of fractional derivative and satisfies $0 \leq q < 1$.

According to Refs. [15]–[17], the predictor-corrector scheme for system (4) can be derived. This scheme is the generalization of the Adams–Bashforth–Moulton algorithm. The following differential equation

$$\begin{aligned} &\frac{\mathrm{d}^{\alpha} x}{\mathrm{d}t^{\alpha}} = f(t, x), \quad 0 \leq t \leq T, \\ &x^{(k)}(0) = x_0^{(k)}, \quad k = 0, 1, 2, \ldots, n-1, \end{aligned}$$

is equivalent to the Volterra integral equation[17]

$$x(t) = \sum_{k=0}^{n-1} x_0^{(k)}\frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau, x)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau.$$

Let $h = T/N$, $t_n = nh$, $(n = 0, 1, 2, \ldots, N)$. Then the above equation can be discretized as follows:

$$x_h(t_{n+1}) = \sum_{k=0}^{n-1} x_0^{(k)}\frac{t_{n+1}^k}{k!} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{n+1}, x_h^{\rho}(t_{n+1}))$$

$$+\frac{h^\alpha}{\Gamma(\alpha+2)}\sum a_{j,n+1}f(t_j,x_h(t_j)),$$

where

$$a_{j,n+1}=\begin{cases}n^{\alpha+1}-(n-\alpha)(n+1)^{\alpha+1}, & j=0,\\ (n-j+2)^{\alpha+1}+(n-j)^{\alpha+1} & \\ \qquad -2(n-j+1)^{\alpha+1}, & 1\le j\le n,\\ 1, & j=n+1,\end{cases}$$

$$x_h^\rho(t_{n+1})=\sum_{k=0}^{n-1}x_0^{(k)}\frac{t_{n+1}^k}{k!}+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}b_{j,n+1}f(t_j,x_h(t_j)),$$

$$b_{j,n+1}=\frac{h^\alpha}{\alpha}((n+1-j)^\alpha-(n-j)^\alpha).$$

The error estimate is $e=\mathrm{Max}|x(t_j)-x_h(t_j)|=O(h^\rho)$, $(j=0,1,\ldots,N)$, where $\rho=\mathrm{Min}(2,1+\alpha)$.

The response system with a single linear controller, which is added on the right-hand side of the second equation, has the following form

$$\begin{cases}\dfrac{\mathrm{d}^q y_1}{\mathrm{d}t^q}=(25\alpha+10)(y_2-y_1),\\ \dfrac{\mathrm{d}^q y_2}{\mathrm{d}t^q}=(28-35\alpha)y_1-y_1y_3+(29\alpha-1)y_2+u,\\ \dfrac{\mathrm{d}^q y_3}{\mathrm{d}t^q}=y_1y_2-(8+\alpha)y_3/3,\end{cases}\tag{5}$$

where u is the linear controller with single drive variable that comes from drive system (4).

Note that many methods and techniques have been developed to synchronize fractional-order chaotic systems, but the linear feedback control method is the most popular one due to its simplicity for theoretical analysis and circuit implementation. In the following, we will study the hybrid synchronization phenomenon stated above.

Here, our specific target is to investigate a theorem that can determine the range of the linear controller. For detailed demonstration, we suppose the first two variables in Eq. (4) are anti-phase synchronized with those in Eq. (5) and the third variables in them are completely synchronized. To this end, we define the error variables as follows:

$$e_1=x_1+y_1,\quad e_2=x_2+y_2,\quad e_3=x_3-y_3,\tag{6}$$

and set the form of linear controller as

$$u=-ke_2.\tag{7}$$

It follows from systems (4) and (5) that the error variables (6) are governed by the following dynamical system

$$\begin{cases}\dfrac{\mathrm{d}^q e_1}{\mathrm{d}t^q}=(25\alpha+10)(e_2-e_1),\\ \dfrac{\mathrm{d}^q e_2}{\mathrm{d}t^q}=(28-35\alpha-x_3)e_1+(29\alpha-1-k)e_2\\ \qquad\qquad +e_1e_3-x_1e_3,\\ \dfrac{\mathrm{d}^q e_3}{\mathrm{d}t^q}=x_2e_1+x_1e_2-e_1e_2-(8+\alpha)e_3/3.\end{cases}\tag{8}$$

Before giving our main results, we introduce the following fractional-order stability theorem. It is very important to bring out our results.

Lemma 1[17] For n-dimensional fractional-order system, if all the eigenvalues $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ of the Jacobian matrix of some equilibrium point satisfy

$$|\arg(\lambda_i)|>\alpha\pi/2,$$
$$\alpha=\max(q_1,q_2,\ldots,q_n),\quad (i=1,2,\ldots,n),$$

then the fractional-order system is asymptotically stable at equilibrium.

Based on Lemma 1, we give the following theorem to achieve hybrid synchronization.

Theorem 1 The anti-phase synchronization and complete synchronization coexist in the coupled fractional-order systems (4) and (5) if the control gain k satisfies

$$k>\max\{k_1^*,k_2^*\},\tag{9}$$

where

$$k_1^*=29\alpha-1+\frac{(25\alpha p+10p+28-35\alpha+N_2)^2}{4(25\alpha+10)p},\tag{10}$$

$$k_2^*=\frac{(8+\alpha)(25\alpha p+10p+28-35\alpha+N_3)^2-4(8+\alpha)(25\alpha+10)p(1-29\alpha)+3N_2^2(1-29\alpha)}{4(8+\alpha)(25\alpha+10)p-3N_2^2},\tag{11}$$

and

$$p>\frac{3N_2^2}{4(8+\alpha)(25\alpha+10)},\tag{12}$$

where $|x_2(t)| \leq N_2, |x_3(t)| \leq N_3$.

Proof Firstly, according to the characteristics of a chaotic system, we know that all the state variables are bounded. Equivalently, there are three positive, numbers N_1, N_2, and N_3, such that $|x_1(t)| \leq N_1$, $|x_2(t)| \leq N_2$ and $|x_3(t)| \leq N_3$. In order to better determine the control gain range, we first introduce a positive relaxed variable p to the first equation in Eq. (8), then the error dynamic equation becomes

$$\begin{pmatrix} p\dfrac{\mathrm{d}^q e_1}{\mathrm{d}t^q} \\ \dfrac{\mathrm{d}^q e_2}{\mathrm{d}t^q} \\ \dfrac{\mathrm{d}^q e_3}{\mathrm{d}t^q} \end{pmatrix} = \begin{bmatrix} -25\alpha p - 10p & 25\alpha p + 10p & 0 \\ 28 - 35\alpha - x_3 & 29\alpha - 1 - k & -x_1 + e_1 \\ x_2 & x_1 - e_1 & -(8+\alpha)/3 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{13}$$

Assume that λ is one of the eigenvalues of the real matrix A, and the corresponding non-zero eigenvector is $\beta = (\eta_1, \eta_2, \eta_3)^{\mathrm{T}}$, i.e.,

$$A\beta = \lambda\beta. \tag{14}$$

Transposing on both sides of Eq. (14) yields

$$\beta^H A^{\mathrm{T}} = \bar{\lambda}\beta^{\mathrm{H}}, \tag{15}$$

where the superscript H stands for the conjugate transpose of a matrix. Multiplying Eq. (14) left by $\frac{1}{2}\beta^{\mathrm{H}}$, and Eq. (15) right by $\frac{1}{2}\beta$, we obtain

$$\frac{1}{2}\beta^{\mathrm{H}}(A + A^{\mathrm{T}})\beta = \frac{1}{2}(\lambda + \bar{\lambda})\beta^{\mathrm{H}}\beta. \tag{16}$$

Therefore,

$$\frac{1}{2}(\lambda + \bar{\lambda})\beta^{\mathrm{H}}\beta = \frac{1}{2}\beta^{\mathrm{H}}(A + A^{\mathrm{T}})\beta. \tag{17}$$

So,

$$\begin{aligned}
&\frac{1}{2}\beta^{\mathrm{H}}(A + A^{\mathrm{T}})\beta \\
&= (\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3) \begin{bmatrix} -25\alpha p - 10p & \dfrac{25\alpha p + 10p + 28 - 35\alpha - x_3}{2} & \dfrac{x_2}{2} \\ \dfrac{25\alpha p + 10p + 28 - 35\alpha - x_3}{2} & 29\alpha - 1 - k & 0 \\ \dfrac{x_2}{2} & 0 & -(8+\alpha)/3 \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \\
&\leq (|\bar{\eta}_1|, |\bar{\eta}_2|, |\bar{\eta}_3|) \begin{bmatrix} -25\alpha p - 10p & \dfrac{25\alpha p + 10p + 28 - 35\alpha + N_3}{2} & \dfrac{N_2}{2} \\ \dfrac{25\alpha p + 10p + 28 - 35\alpha + N_3}{2} & 29\alpha - 1 - k & 0 \\ \dfrac{N_2}{2} & 0 & -(8+\alpha)/3 \end{bmatrix} \begin{pmatrix} |\eta_1| \\ |\eta_2| \\ |\eta_3| \end{pmatrix} \\
&= -(|\eta_1|, |\eta_2|, |\eta_3|) \begin{bmatrix} 25\alpha p + 10p & -\dfrac{25\alpha p + 10p + 28 - 35\alpha + N_3}{2} & -\dfrac{N_2}{2} \\ -\dfrac{25\alpha p + 10p + 28 - 35\alpha + N_3}{2} & -29\alpha + 1 + k & 0 \\ -\dfrac{N_2}{2} & 0 & (8+\alpha)/3 \end{bmatrix} \begin{pmatrix} |\eta_1| \\ |\eta_2| \\ |\eta_3| \end{pmatrix} \\
&= -(|\eta_1|, |\eta_2|, |\eta_3|) Q \begin{pmatrix} |\eta_1| \\ |\eta_2| \\ |\eta_3| \end{pmatrix}.
\end{aligned} \tag{18}$$

That is to say, if the control gain $k > \max\{k_1^*, k_2^*\}$, where

$$k_1^* = 29\alpha - 1 + \frac{(25\alpha p + 10p + 28 - 35\alpha + N_2)^2}{4(25\alpha + 10)p},$$

$$k_2^* = \frac{(8+\alpha)(25\alpha p + 10p + 28 - 35\alpha + N_3)^2 - 4(8+\alpha)(25\alpha + 10)p(1 - 29\alpha) + 3N_2^2(1 - 29\alpha)}{4(8+\alpha)(25\alpha + 10)p - 3N_2^2},$$

and

$$p > \frac{3N_2^2}{4(8+\alpha)(25\alpha+10)}.$$

Then the matrix Q is positive-definite. Thus, we obtain

$$\mathrm{Re}(\lambda) \leq -\frac{1}{\beta^{\mathrm{H}}\beta}(|\eta_1|,|\eta_2|,|\eta_3|)Q\begin{pmatrix}|\eta_1|\\|\eta_2|\\|\eta_3|\end{pmatrix} \leq 0. \quad (19)$$

From Eq. (19) we have $|\arg\lambda| > 0.5\pi$, therefore, $|\arg\lambda| > 0.5q\pi$.

According to the stability theory of a fractional-order system, the equilibrium point in Eq. (11) is asymptotically stable and so we have $\lim_{t\to\infty}|e_i(t)| = 0$, $(i = 1, 2, 3)$. That is all for the proof.

3. Numerical algorithm for fractional-order coupled unified chaotic systems

The initial conditions for drive system (4) and response system (5) are selected as (x_0^1, x_0^2, x_0^3) and (y_0^1, y_0^2, y_0^3), respectively. Set $h = T/N$, $N \in Z^+$, and $t_n = nh$, $(n = 0, 1, \ldots, N)$, where T is total simulation time and h is step size, respectively. Applying the improved version of the Adams–Bashforth–Moulton algorithm,[3] the numerical solutions to coupled fractional order systems (4) and (5) can be computed as follows:

$$\begin{cases}
(x_1)_{n+1} = x_0^1 + \dfrac{h^q}{\Gamma(q+2)}(25\alpha+10)[(x_2)_{n+1}^p - (x_1)_{n+1}^p] + \dfrac{h^q}{\Gamma(q+2)}\sum\limits_{j=0}^{n}\alpha_{1,j,n+1}(25\alpha+10)[(x_2)_j - (x_1)_j],\\
(x_2)_{n+1} = x_0^2 + \dfrac{h^q}{\Gamma(q+2)}[(28-35\alpha)(x_1)_{n+1}^p - (x_1)_{n+1}^p(x_3)_{n+1}^p + (29\alpha-1)(x_2)_{n+1}^p]\\
\qquad + \dfrac{h^q}{\Gamma(q+2)}\sum\limits_{j=0}^{n}\alpha_{2,j,n+1}[(28-35\alpha)(x_1)_j - (x_1)_j(x_3)_j + (29\alpha-1)(x_2)_j],\\
(x_3)_{n+1} = x_0^3 + \dfrac{h^q}{\Gamma(q+2)}[(x_1)_{n+1}^p(x_2)_{n+1}^p - (8+\alpha)(x_3)_{n+1}^p/3]\\
\qquad + \dfrac{h^q}{\Gamma(q+2)}\sum\limits_{j=0}^{n}\alpha_{3,j,n+1}[(x_1)_j(x_2)_j - (8+\alpha)(x_3)_j/3],
\end{cases} \quad (20)$$

$$\begin{cases}
(y_1)_{n+1} = y_0^1 + \dfrac{h^q}{\Gamma(q+2)}(25\alpha+10)[(y_2)_{n+1}^p - (y_1)_{n+1}^p] + \dfrac{h^q}{\Gamma(q+2)}\sum\limits_{j=0}^{n}\alpha_{1,j,n+1}(25\alpha+10)[(y_2)_j - (y_1)_j],\\
(y_2)_{n+1} = y_0^2 + \dfrac{h^q}{\Gamma(q+2)}\{(28-35\alpha)(y_1)_{n+1}^p - (y_1)_{n+1}^p(y_3)_{n+1}^p + (29\alpha-1)(y_2)_{n+1}^p - k[(y_2)_{n+1}^p + (x_2)_{n+1}^p]\}\\
\qquad + \dfrac{h^q}{\Gamma(q+2)}\sum\limits_{j=0}^{n}\alpha_{2,j,n+1}\{(28-35\alpha)(y_1)_j - (y_1)_j(y_3)_j + (29\alpha-1)(y_2)_j - k[(y_2)_j + (x_2)_j]\},\\
(y_3)_{n+1} = y_0^3 + \dfrac{h^q}{\Gamma(q+2)}[(y_1)_{n+1}^p(y_2)_{n+1}^p - (8+\alpha)(y_3)_{n+1}^p/3]\\
\qquad + \dfrac{h^q}{\Gamma(q+2)}\sum\limits_{j=0}^{n}\alpha_{3,j,n+1}[(y_1)_j(y_2)_j - (8+\alpha)(y_3)_j/3],
\end{cases} \quad (21)$$

where

$$\begin{cases}
(x_1)_{n+1}^p = x_0^1 + \dfrac{1}{\Gamma(q)}\sum\limits_{j=0}^{n}\beta_{1,j,n+1}(25\alpha+10)[(x_2)_j - (x_1)_j],\\
(x_2)_{n+1}^p = x_0^2 + \dfrac{1}{\Gamma(q)}\sum\limits_{j=0}^{n}\beta_{2,j,n+1}[(28-35\alpha)(x_1)_j - (x_1)_j(x_3)_j + (29\alpha-1)(x_2)_j],\\
(x_3)_{n+1}^p = x_0^3 + \dfrac{1}{\Gamma(q)}\sum\limits_{j=0}^{n}\beta_{3,j,n+1}[(x_1)_j(x_2)_j - (8+\alpha)(x_3)_j/3],
\end{cases} \quad (22)$$

$$\begin{cases}
(y_1)_{n+1}^p = y_0^1 + \dfrac{1}{\Gamma(q)}\sum\limits_{j=0}^{n}\beta_{1,j,n+1}(25\alpha+10)[(y_2)_j - (y_1)_j],\\
(y_2)_{n+1}^p = y_0^2 + \dfrac{1}{\Gamma(q)}\sum\limits_{j=0}^{n}\beta_{2,j,n+1}\{(28-35\alpha)(y_1)_j - (y_1)_j(y_3)_j + (29\alpha-1)(y_2)_j - k[(y_2)_j + (x_2)_j]\},\\
(y_3)_{n+1}^p = y_0^3 + \dfrac{1}{\Gamma(q)}\sum\limits_{j=0}^{n}\beta_{3,j,n+1}[(y_1)_j(y_2)_j - (8+\alpha)(y_3)_j/3],
\end{cases} \quad (23)$$

$$\alpha_{i,j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & \text{if } j = 0, \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j = n+1, \end{cases} \tag{24}$$

and

$$\beta_{i,j,n+1} = \frac{h^q}{q}((n+1-j)^q - (n-j)^q), \quad 0 \le j \le n, \quad i = 1, 2, 3. \tag{25}$$

4. Simulation results

In this section, the system parameters are chosen as: fractional order $q = 0.96$, $\alpha = 0.5$, total simulation time $T = 50$ s, step size $h = 0.005$ and the initial conditions for system (4) are selected as $(x_0^1,\ x_0^2,\ x_0^3) = (0.12,\ 0.2,\ 0.94)$. By solving Eqs. (20), (22), (24) and (25), the chaotic attractor in a fractional-order generalized Lorenz system is shown in Figs. 1(a)–1(d). Next, we will determine the gain of a linear controller with a single drive state variable.

By calculation, we obtain $|x_1(t)| \le 25.1371$, $|x_2(t)| \le 32.4742$, and $|x_3(t)| \le 49.5924$. Setting

$$p = \frac{3N_2^2}{4(8+\alpha)(25\alpha+10)} + 5 = 9.4048$$

and computing Eqs. (10) and (11) yields $k_1^* = 97.7182$ and $k_2^* = 173.1718$. So, we select the control gain $k = 175$. The initial conditions for response system (5) are chosen to be $(y_0^1,\ y_0^2,\ y_0^3) = (0.13,\ 0.65,\ 0.98)$. Figure 2(a) shows the convergence dynamics of error system (8) when the control gain $k = 175$. From Fig. 2(a), we can see $x_1(t) + y_1(t) \to 0$, $x_2(t) + y_2(t) \to 0$, and $x_3(t) - y_3(t) \to 0$ as time t approaches infinity.

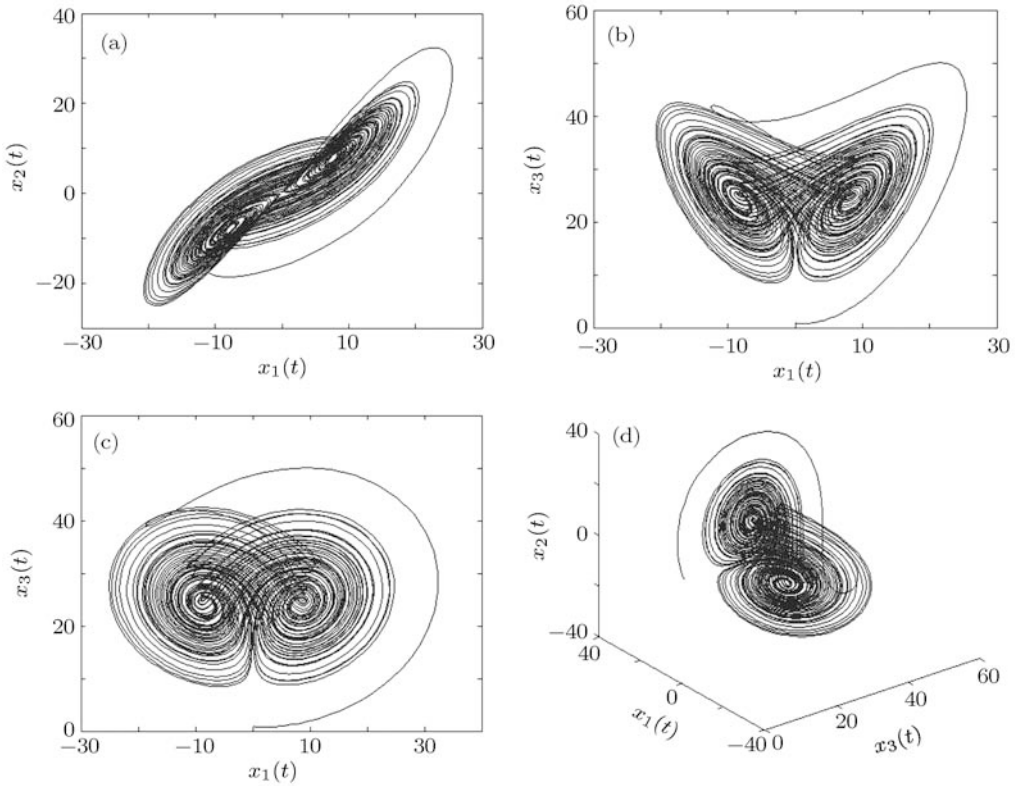


Fig. 1. Phase portrait of the chaotic attractor in fractional-order unified chaotic system (4): (a) $x_1(t)$–$x_2(t)$ plane, (b) $x_1(t)$–$x_3(t)$ plane, (c) $x_2(t)$–$x_3(t)$ plane, (d) $x_3(t)$–$x_1(t)$–$x_2(t)$ hyperplane.

It seems that the gain of linear controller obtained from our theorem is relatively large. The reason is that the conservative of the theorem is low after using the scaling law. In fact, the theorem is only a sufficient condition. Through simulations, we found that the controller is also very effective when the gain reduces to

$k = 5$ and the simulation results are shown in Fig. 2(b). This is helpful when it is used in practical engineering applications.

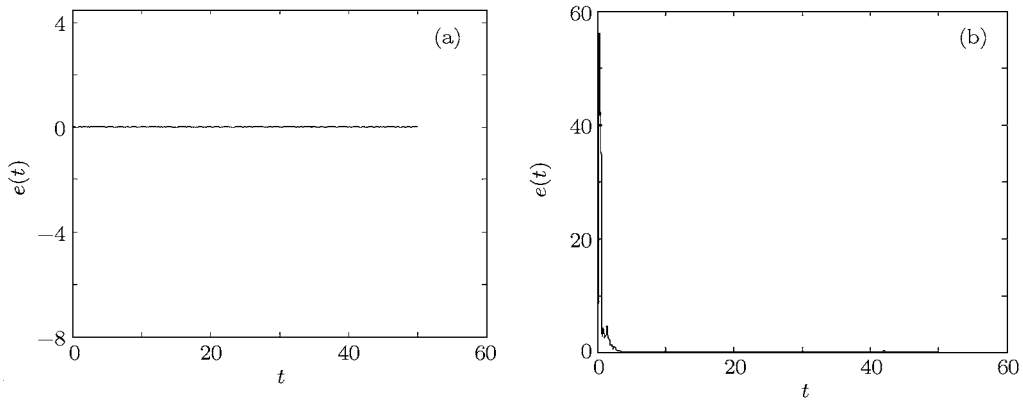


Fig. 2. The time response of synchronization error $e(t) = \sqrt{\sum_{i=1}^{3} e_i^2(t)}$ as time t varies for $k = 175$ (a) and $k = 5$ (b).

5. Conclusions

In this paper, a new kind of nonlinear phenomenon, hybrid synchronization, is studied by a single linear controller with only one drive state variable. Based on the stability of a fractional-order system, a theorem serving as a determinant for the gain of the linear controller is proposed. The simulation results show the effectiveness and feasibility of the proposed scheme. Through simulation, two pairs of states are anti-phase synchronized and the third pair is complete synchronized by a coupled fractional-order unified chaotic systems by linear control via a single drive state variable.

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