Controlling the collision between two solitons in the condensates by a double-barrier potential*

Li Zhi-Jian(李志坚)[a] and Li Jin-Hui(李锦茴)[b]†

[a] Information Science and Engineering Department, Hunan First Normal University, Changsha 410205, China
[b] Mathematics and Science Department, Hunan First Normal University, Changsha 410205, China

(Received 18 December 2010; revised manuscript received 4 March 2011)

We present an analytical solution of two solitons of Bose–Einstein condensates trapped in a double-barrier potential by using a multiple-scale method. In the linear case, we find that the stable spots of the soliton formation are at the top of the barrier potential and at the region of barrier potential absence. For weak nonlinearity, it is shown that the height of the barrier potential has an important effect on the dark soliton dynamical properties. Especially, in the case of regarding a double-barrier potential as the output source of the solitons, the collision spots between two dark solitons can be controlled by the height of the barrier potential.

Keywords: Bose–Einstein condensates, solitons, double-barrier potential

PACS: 05.30.Jp, 02.90.+p, 11.10.Lm **DOI:** 10.1088/1674-1056/20/8/080502

1. Introduction

The realization of Bose–Einstein condensates (BEC) in weakly interacting gases has opened up a new avenue for the investigation of nonlinear phenomena in matter waves.[1,2] The identification of nonlinear excitations like dark and bright solitons has proved to be a catalyst leading to a flurry of further activity in this area.[3−26] Such matter wave solitons are of fundamental interest, not only conceptually but also practically as BECs have virtually become a natural experimental playground for nonlinear waves and robust structures, which are being coherently manipulated in matter waves analogous to the optical solitons in fibres.[25,26]

A relevant interesting issue in this field is to learn how to control the motion of the matter wave solitons. One of the possibilities of controlling the dynamical properties of the soliton is to vary the trapped external potential.[3−7] The other is to adjust the inter-atomic interaction strength via the process of Feshbach resonance.[10−13] To date, the effect of the interatomic interaction on the nonlinear dynamics of the BECs has been intensively investigated.[10−21] Meanwhile, physicists have also investigated some types of external potentials influencing a number of the BECs properties, including static and dynamical properties.[3−7,22−26]

Most related theoretical studies were focused on the adiabatic propagation of a BEC in the presence of obstacles.[27] Particularly interesting in this context is the propagation of a BEC through a double-barrier potential, which can be viewed as a Fabry–Perot interferometer for matter waves.[27] In the field of atom chips, a bosonic quantum dot could be realized by suitable geometries of microfabricated wires.[28,29] In this paper, we regard a double-barrier potential as the output source of the solitons and study the effect of a barrier potential on the collision between two solitons in the condensates. It is shown that the collision spots between two dark solitons can be controlled by the height of the barrier potential.

2. The double-barrier potential model and asymptotic expansion

In the presence of a strong cylindrical confinement with trapping frequency $\omega_\perp$, the mean-field dynamics of the condensate can be described by the effective one-dimensional (1D) Gross–Pitaevskii equation in dimensionless variables[1,2]

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi + g_{1\mathrm{D}}\,|\psi|^2\,\psi, \qquad (1)$$

with $g_{1\mathrm{D}} = 2\,|a_\mathrm{s}|\,/a_0$. Here a_s is the s-wave scattering length and $a_0 = \sqrt{\hbar/m\omega_\perp}$ (transverse harmonic-

*Project supported by the Science Research Foundation of the Education Bureau of Hunan Province of China (Grant No. 09C227).
†Corresponding author. E-mail: lijinhui1188@163.com
© 2011 Chinese Physical Society and IOP Publishing Ltd

oscillator length) with the atomic mass m. For the sake of definiteness, a double-barrier potential can be given by[27]

$$V(x) = V_1 \exp\left[-(x+L/2)^2/\sigma^2\right] + V_2 \exp\left[-(x-L/2)^2/\sigma^2\right], \quad (2)$$

(see Fig. 1). As discussed in Ref. [27], our numerical calculations were performed for ^{87}Rb atoms ($a_s = 5.77$ nm) with $\omega_\perp = 2\pi\times10^3$ μm, $V_1 = V_2 = \hbar\omega_\perp$, and $L = 10\sigma = 5$ μm$\approx 14.7a_0$. So, the corresponding dimensionless parameters are $V_1 = V_2 = 1.0$, $L = 14.7$, and $\sigma = 1.47$. V_1 and V_2 denote the height of the left and right barriers, respectively. Expressing the wave function in terms of modulus and phase, we set $\psi(x,t) = A(x,t)\exp[-\mathrm{i}\mu t + \mathrm{i}\varphi(x,t)]$. Substituting it into Eq. (1) and then separating the real and imaginary parts, we obtain the following two equations

$$\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{1}{2}A\frac{\partial^2 \varphi}{\partial x^2} = 0, \quad (3)$$

$$-\frac{1}{2}\frac{\partial^2 A}{\partial x^2} + \left[\frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 - \mu + \frac{\partial \varphi}{\partial t} + V(x)\right]A + g_{1\mathrm{D}}A^3 = 0. \quad (4)$$

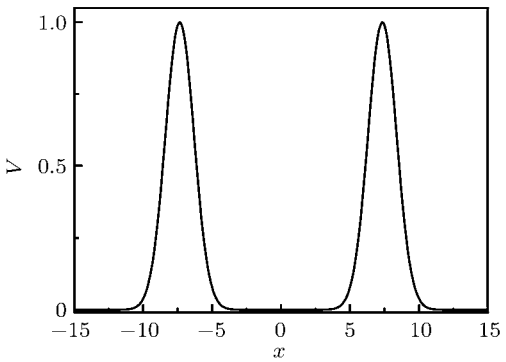


Fig. 1. A double-barrier potential. The parameters used are $V_1 = V_2 = 1$, $L = 14.7$, and $\sigma = 1.47$.

Subsequently, we study the linear stability condition of soliton formation. Due to the strong confinement in the transverse direction, the system is similar to a waveguide, in which the excitation from ground state propagates in the longitudinal direction.[7−11] The strong confinement also ensures the dynamical stability of a soliton excitation.[7−11] Therefore, we set $A = u_0(x) + \alpha(x,t)$ (without loss of generality, we assume that $u_0(x)$, which characterizes the condensate background, is positive) with $\alpha(x,t) = \alpha_0\exp(\mathrm{i}\beta)$ + c.c. and $\varphi = \varphi_0\exp(\mathrm{i}\beta)$ + c.c. Here, c.c. is complex conjugate and $\beta = kx - \omega t$, where k is wave number and ω indicates eigen-frequency. Considering that α_0 and φ_0 are small constants, we obtain

$$\mathrm{i}\omega\alpha_0 = \mathrm{i}k\varphi_0\frac{\partial u_0}{\partial x} - \frac{1}{2}k^2u_0\varphi_0, \quad (5)$$

$$\mu u_0 = -\frac{1}{2}\frac{\partial^2}{\partial x^2}u_0 + V(x)u_0 + g_{1\mathrm{D}}u_0^3, \quad (6)$$

$$\mu\alpha_0 = \frac{1}{2}k^2\alpha_0 - \mathrm{i}\omega\varphi_0u_0 + V(x)\alpha_0 + 3g_{1\mathrm{D}}u_0^2\alpha_0, \quad (7)$$

from the linearization of Eqs. (3) and (4). For a linear case, i.e., $g_{1\mathrm{D}} = 0$, equation (6) is turned into

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_0 + 2\Big[\mu - V_1\,\mathrm{e}^{-(x+L/2)^2/\sigma^2} + V_2\,\mathrm{e}^{-(x-L/2)^2/\sigma^2}\Big]u_0 = 0. \quad (8)$$

In the case of $g_{1\mathrm{D}} \neq 0$, and utilizing Eqs. (5), (6), and (7), the linear dispersion relation is given by

$$\omega^2 = \left(\frac{1}{2u_0}\frac{\partial^2 u_0}{\partial x^2} + \frac{1}{2}k^2 + 2g_{1\mathrm{D}}u_0^2\right) \times \left(\frac{1}{2}k^2 - \frac{\mathrm{i}k}{u_0}\frac{\partial u_0}{\partial x}\right). \quad (9)$$

We next discuss the stability problem of soliton formation. Setting $\omega = \omega_\mathrm{r} + \mathrm{i}\omega_\mathrm{i}$ (where the subscripts denote the real and imaginary parts),[5,30] we have $\omega_\mathrm{r}^2 = \left(\eta\chi + k^2\eta\right)/4$ and $\omega_\mathrm{i}^2 = \left(\eta\chi - k^2\eta\right)/4$, where

$$\eta = \frac{1}{2u_0}\frac{\partial^2 u_0}{\partial x^2} + k^2/2 + 2g_{1\mathrm{D}}u_0^2$$

and

$$\chi = \sqrt{k^4 + 4\frac{k^2}{u_0^2}\left(\frac{\partial u_0}{\partial x}\right)^2}.$$

If the imaginary part of quasiparticle frequency is a nonzero value, the corresponding matter wave exhibits exponential growth and hence the state ψ is of dynamical instability.[5,30] If the frequency of the associated quasiparticle spectrum is real, the soliton would be stable. From the expression of the imaginary part of the frequency, one can find the dependence of the instability growth rate ω_i on k and $(1/u_0)(\partial u_0/\partial x)$. On the one hand, when $k = 0$, one finds $\omega = 0$, which is an inessential solution. On the other hand, we obtain $(1/u_0)(\partial u_0/\partial x) = 0$. Combining with Eq. (8) and Fig. 1, we find that at the top (maximum) of the barrier potential and the region of the absence of barrier potential, the wave function ψ possesses dynamical stability. We therefore conclude that there is a stable soliton at the top of the barrier potential and at the region of absence of barrier potential.

The stability of the propagating soliton suggests a possibility to consider collision between two solitons. To understand the effect of the barrier potential on soliton collisions in BECs, we here make

use of a multiple-scale method[31−33] to derive an explicit expression of the wave function of the BECs. First, we should introduce that two "slow" variables $\xi = \varepsilon(x - V_g t)$ and $\tau = \varepsilon^3 t$ characterize the slow variation of soliton dynamics, and a "fast" variable $x_0 = x$ represents the direction of soliton propagation. Moreover, we assume that $A(x,t)$ and $\varphi(x,t)$ can be expanded into a polynomial form in terms of ε in the forms of $A = u_0 + \varepsilon^2\left[a^{(0)}(\xi,\tau) + \varepsilon^2 a^{(1)}(\xi,\tau) + \cdots\right]$, and $\varphi = \varepsilon\left[\varphi^{(0)}(\xi,\tau) + \varepsilon^2\varphi^{(1)}(\xi,\tau) + \cdots\right]$, where ε is a smallness parameter characterizing the relative amplitude of the excitation. Inserting them into Eq. (3), and then separating each order in terms of the power of ε, respectively, from Eq. (3), we have

$$V_g \frac{\partial a^{(j)}}{\partial \xi} - \frac{1}{2}u_0 \frac{\partial^2 \varphi^{(j)}}{\partial \xi^2} = M^{(j)}, \quad j = 0, 1\ldots; \tag{10}$$

from Eq. (4), we have

$$-\mu u_0 + V(x_0)u_0 + g_{1D}u_0^3 = 0, \tag{11}$$

$$-\mu a^{(j)} + V(x_0)a^{(j)} + 3g_{1D}u_0^2 a^{(j)} = N^{(j)}, \quad j = 0, 1, \ldots. \tag{12}$$

Here $M^{(0)} = 0$,

$$M^{(1)} = \frac{\partial a^{(0)}}{\partial \xi}\frac{\partial \varphi^{(0)}}{\partial \xi} + \frac{1}{2}a^{(0)}\frac{\partial^2 \varphi^{(0)}}{\partial \xi^2} + \frac{\partial a^{(0)}}{\partial \tau},$$

$$N^{(0)} = V_g u_0 \frac{\partial \varphi^{(0)}}{\partial \xi},$$

and

$$N^{(1)} = \frac{1}{2}\frac{\partial^2 a^{(0)}}{\partial \xi^2} - \frac{1}{2}u_0\left(\frac{\partial \varphi^{(0)}}{\partial \xi}\right)^2 - 3g_{1D}u_0\left(a^{(0)}\right)^2 + V_g u_0 \frac{\partial \varphi^{(1)}}{\partial \xi} + V_g a^{(0)}\frac{\partial \varphi^{(0)}}{\partial \xi} - u_0\frac{\partial \varphi^{(0)}}{\partial \tau}.$$

From Eq. (11) we obtain that the chemical potential of the system is $\mu = V(x_0) + g_{1D}u_0^2$. Comparing the case of $j = 0$ of Eqs. (10) and (12), we obtain $V_g = \pm\sqrt{g_{1D}}u_0$. Subsequently, for the case of $j = 1$ in Eqs. (10) and (12), we obtain

$$\frac{\partial \varphi^{(0)}}{\partial \tau} + \frac{3}{4}\left(\frac{\partial \varphi^{(0)}}{\partial \xi}\right)^2 - \frac{1}{8V_g}\frac{\partial^3 \varphi^{(0)}}{\partial \xi^3} = 0. \tag{13}$$

Equation (13) can also be called variable coefficient Korteweg–de Vries (VCKdV) equation. Making transformation $\eta = \varepsilon\varphi^{(0)}$, $\xi = \varepsilon X = \varepsilon(x - V_g t)$, and $\tau = \varepsilon^3 t$, equation (13) becomes

$$\frac{\partial \eta}{\partial t} + \frac{3}{4}\left(\frac{\partial \eta}{\partial X}\right)^2 - \frac{1}{8V_g}\frac{\partial^3 \eta}{\partial X^3} = 0. \tag{14}$$

Furthermore, we have the two soliton solutions by solving the VCKdV equation (14). The corresponding wave function of the condensates is

$$\psi = u_0\left\{1 - V_g(A_2 - A_1)\frac{A_2\,\mathrm{csch}^2\theta_2 + A_1\,\mathrm{sech}^2\theta_1}{\left(\sqrt{A_2}\coth\theta_2 - \sqrt{A_1}\tanh\theta_1\right)^2}\right\} \times \exp\left\{-\mathrm{i}\mu t - \mathrm{i}\sqrt{2}\left[\left(\sqrt{A_1} + \sqrt{A_2}\right) + \frac{A_2 - A_1}{\sqrt{A_2}\coth\theta_2 - \sqrt{A_1}\tanh\theta_1}\right]\right\}, \tag{15}$$

where $\theta_1 = V_g\sqrt{2A_1}\,[x - V_g t + A_1 V_g t - x_{01}]$ and $\theta_2 = V_g\sqrt{2A_2}\,[x - V_g t + A_2 V_g t - x_{02}]$; A_1 and A_2 are positive integrating constants, x_{01} and x_{02} are constants denoting the initial position of soliton on the pedestal background.

3. Numerical results and discussions

As discussed above, there is a stable soliton at the top of the barrier potential. So, we regard the double-barrier potential as the output source of the solitons and give a numerical calculation to study the effect of the barrier potential on the collision between two solitons in the condensates.

Figure 2 presents the space-time distribution of the density of the condensates with different heights of double-barrier potential. The source positions of the output of two solitons are chosen on the spots $x_{01} = -L/2$ and $x_{02} = L/2$, which are at the top of the double-barrier potential. When the integrating constants are $A_1 = 0.5$ and $A_2 = 2$, the space-time evolution of the density of the condensates in the absence of the double-barrier potential ($V_1 = V_2 = 0$) is shown in Fig. 2(a). We see that at the initial stage a larger amplitude and narrower dark soliton is on the right-hand side $x = L/2$, while a smaller amplitude and wider dark soliton is on the left-hand side $x = -L/2$. With the time going on, the rightward soliton propagates leftward, while the leftward soliton propagates rightward. That is to say, the two dark solitons

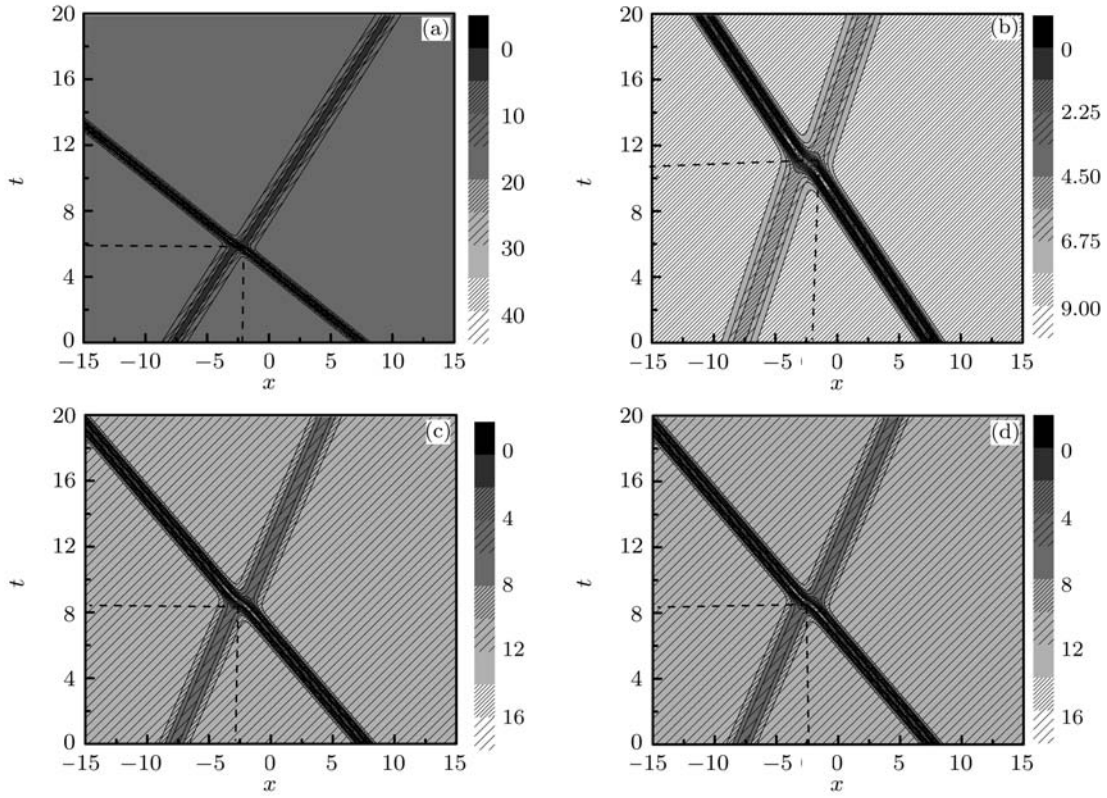


Fig. 2. The space-time distribution of the density of the condensates with different heights of the double-barrier potential, (a) $V_1 = V_2 = 0$, (b) $V_1 = V_2 = 1$, (c) $V_1 = 0.5$, $V_2 = 1$, and (d) $V_1 = 1$, $V_2 = 0.5$. The other parameters used are $\mu = 2.8$, $g_{1D} = 0.1$, $A_1 = 0.5$, $A_2 = 2$, $L = 14.7$, $\sigma = 1.47$, $x_{01} = -L/2$, and $x_{02} = L/2$.

propagate in opposite directions and approach each other. At position $x \approx -2.0$ and time $t \approx 6.0$, collisions of the two dark solitons take place. After the collision, the two solitons asymptotically separate away, and their propagating mode does not change. This obviously indicates that no energy exchange between the two dark solitons has occurred. So, we can conclude that the collision of two dark solitons is nearly elastic. To obtain the effect of the barrier potential on the dynamics of the solitons, we plot the space-time distribution of the condensates in the presence of a barrier potential with height $V_1 = V_2 = 1$ in Fig. 2(b). One sees that with the increasing height of the barrier potential, the amplitudes of the two dark solitons become smaller and their width wider. The two dark solitons propagate in opposite directions, approach each other, collide and asymptotically separate away. The position of the collision is still at $x \approx -2.0$, while the time is $t \approx 10.8$. It denotes that the two solitons propagate more slowly. Based on the above analyses, we think that the spots of the collisions between two dark solitons have changed, which may be caused by the double-barrier potentials. So we show the space–time evolution of the condensates trapped for heights $V_1 = 0.5$ and $V_2 = 1$ in Fig. 2(c). Compared with the case of the heights $V_1 = V_2 = 1$, although the collision between the two solitons is still elastic, the collision position for heights $V_1 = 0.5$ and $V_2 = 1$ is at $x \approx -3.0$ and the time is $t \approx 8.5$. Similarly, figure 2(d) presents the space-time distribution of the condensates for heights $V_1 = 1$ and $V_2 = 0.5$. We find that the collision position for heights $V_1 = 1$ and $V_2 = 0.5$ is at $x \approx -2.5$ and the time is $t \approx 8.5$. Figures 2(a) to 2(d) illustrate that when the condensates are confined in different heights of double-barrier potential, the spots of the collisions between two dark solitons are different. So, we conclude that the spots of the collisions between two dark solitons can be controlled by the height of the double-barrier potential.

4. Conclusions

In summary, we have used the multiple-scale method to analytically study the effect of the double-barrier potential on the dynamical characteristics in a BEC. In the linear case, we have obtained that there are stable solitons at the top of the barrier potential and in the region of an absence of barrier potential. For the nonlinear case, the amplitude and the wave function phase of the condensates was governed by the VCKdV equation. Our numerical calculation showed

that the height of the barrier potential had an important effect on the dark soliton dynamical characteristics. With increasing height of the barrier potential, the amplitude of the two dark soliton becomes smaller, their widths become wider and the solitons propagate more slowly. Especially, when we regard a double-barrier potential as the output source of the solitons, the collision spots between the two dark solitons can be controlled by the height of the barrier potential.

References

[1] Pethick C J and Smith H 2002 *Bose–Einstein Condensation in Dilute Gases* (Cambridge: Cambridge University Press)

[2] Pitaevskii L and Stringari S 2003 *Bose–Einstein Condensation* (Oxford: Clarendon)

[3] Wang D S, Hu X H, Hu J P and Liu W M 2010 *Phys. Rev.* A **81** 025604

[4] Ji A C, Sun Q, Xie X C and Liu W M 2009 *Phys. Rev. Lett.* **102** 023602

[5] Wang D L, Yan X H and Liu W M 2008 *Phys. Rev.* E **78** 026606

[6] Li Z D, Li Q Y, Li L and Liu W M 2007 *Phys. Rev.* E **76** 026605

[7] Zhang X F, Yang Q, Zhang J F, Chen X Z and Liu W M 2008 *Phys. Rev.* A **77** 023613

[8] Huang G X, Velarde M G and Makarov V A 2001 *Phys. Rev.* A **64** 013617

[9] Huang G X, Szeftel J and Zhu S H 2002 *Phys. Rev.* A **65** 053605

[10] Zhang W X, Wang D L, He Z M, Wang F J and Ding J W 2008 *Phys. Lett.* A **372** 4407

[11] Wang D L, Yan X H and Wang F J 2007 *Chin. Phys. Lett.* **24** 1817

[12] Zhang J M and Liu W M 2010 *Phys. Rev.* A **82** 025602

[13] Chen Y H, Wu W, Tao H S and Liu W M 2010 *Phys. Rev.* A **82** 043625

[14] Li Q Y, Li Z D, Yao S F, Li L and Fu G S 2010 *Chin. Phys.* B **19** 080501

[15] Song W W, Li Q Y, Li Z D and Fu G S 2010 *Chin. Phys.* B **19** 070503

[16] Li Z D, Li Q Y, He P B, Liang J Q, Liu W M and Fu G S 2010 *Phys. Rev.* A **81** 015602

[17] He Z M, Wang D L, Zhang W X, Wang F J and Ding J W 2008 *Chin. Phys.* B **17** 3640

[18] Zhang W X, Wang D L and Ding J W 2008 *Acta Phys. Sin.* **57** 6786 (in Chinese)

[19] Xi Y D, Wang D L, He Z M and Ding J W 2009 *Chin. Phys.* B **18** 0939

[20] He Z M and Wang D L 2007 *Acta Phys. Sin.* **56** 3088 (in Chinese)

[21] Li Q Y, Li Z D, Wang S X, Song W W and Fu G S 2009 *Opt. Commun.* **282** 1676

[22] Xi Y D, Wang D L, She Y C, Wang F J and Ding J W 2010 *Acta Phys. Sin.* **59** 3720 (in Chinese)

[23] She Y C, Wang D L and Ding J W 2009 *Acta Phys. Sin.* **58** 3198 (in Chinese)

[24] She Y C, Wang D L, Zhang W X, He Z M and Ding J W 2010 *J. Opt. Soc. Am.* B **27** 208

[25] Andersen J O 2004 *Rev. Mod. Phys.* **76** 599

[26] Burger S, Bongs K, Dettmer S, Ertmer W and Sengstock K 1999 *Phys. Rev. Lett.* **83** 5198

[27] Paul T, Richter K and Schlagheck P 2005 *Phys. Rev. Lett.* **94** 020404

[28] Carusotto I and La Rocca G C 1999 *Phys. Rev. Lett.* **84** 399

[29] Carusotto I 2001 *Phys. Rev.* A **63** 023610

[30] Johansson M and Kivshar Y S 1999 *Phys. Rev. Lett.* **82** 85

[31] Wang D L, Yan X H and Tang Y 2004 *J. Phys. Soc. Jpn.* **73** 123

[32] Wang D L, Yang R S and Yang Y Y 2007 *Commun. Theor. Phys.* (Beijing, China) **48** 917

[33] Wang D L and Yan X H 2011 *Int. J. Mod. Phys.* B **25** 781