Robust stability analysis of Takagi–Sugeno uncertain stochastic fuzzy recurrent neural networks with mixed time-varying delays

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In this paper, the global stability of Takagi–Sugeno (TS) uncertain stochastic fuzzy recurrent neural networks with discrete and distributed time-varying delays (TSUSFRNNs) is considered. A novel LMI-based stability criterion is obtained by using Lyapunov functional theory to guarantee the asymptotic stability of TSUSFRNNs. The proposed stability conditions are demonstrated through numerical examples. Furthermore, the supplementary requirement that the time derivative of time-varying delays must be smaller than one is removed. Comparison results are demonstrated to show that the proposed method is more able to guarantee the widest stability region than the other methods available in the existing literature.

Keywords: recurrent neural networks, linear matrix inequality, Lyapunov stability, time-varying delays, TS fuzzy model

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1. Introduction

Recurrent neural networks such as Hopfield neural networks (NNs) and cellular neural networks have been successfully applied to solve some previously unsolvable problems and improve system performance in many fields.[1] In the hardware implementation of recurrent neural networks, time delays occur due to the finite switching speed of the amplifiers and the communication time.[2] It has also been shown that the processing of moving images requires the introduction of delay in the signal transmitted through the networks. Recurrent neural networks with delays have many applications in fields such as signal processing, image processing, pattern recognition, associative memory and in optimization problems.[3−5] In the application of recurrent neural networks with delays, it is often required that the network model has a unique globally stable equilibrium point. In recent years, considerable effort has been devoted to the stability analysis of delayed recurrent neural networks via the Lyapunov functional method. Many conditions have been presented for the global stability of the equilibrium point, periodic attractor and synchronization of recurrent neural networks with delays (see for example, Refs. [6]–[9]).

Neural networks usually have a spatial extension due to the presence of a multitude of parallel pathways with a variety of axon sizes and length and hence there is a distribution of propagation delays over a period of time. It is worth noting that although the signal propagation is sometimes instantaneous and can be modelled with discrete delays, it may also be distributed during a certain time period, so the distributed delays should be incorporated in the model. In other words, it is often the case that the neural network model possesses both discrete and distributed delays. In recent years, the stability of Hopfield neural networks, cellular neural networks and bidirectional associative memory neural networks with distributed delays has been discussed in Refs. [10]–[16].

Fuzzy systems in the form of the Takagi–Sugeno (TS) model[17] have attracted a rapidly growing interest.[18,19] TS fuzzy systems are nonlinear systems described by a set of **IF−THEN** rules. It has been shown that the TS model can provide an effective way to represent complex nonlinear systems by some simple local linear dynamic systems with their linguistic description. Some nonlinear dynamic systems can be approximated by the overall fuzzy linear TS models for the purpose of stability analysis.[18,19] The authors[20,21] have proved that TS fuzzy systems can

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approximate any continuous function in a compact set of R^n at any level of precision. This allows designers to take advantage of conventional linear systems to analyse and design nonlinear systems. Based on the above discussions, the ordinary fuzzy models have been extended to describe delayed neural networks (see Refs. [22]–[28]).

In addition, parameter uncertainties can often be encountered in real systems, as well as neural networks, due to modelling inaccuracies and/or changes in the environment of the model. Over the past few years, to solve the problems caused by parameter uncertainty, robustness analysis for different uncertain systems has received considerable attention (see for example Refs. [29], [30], and the references therein). Motivated by this, we will generalize the ordinary TS fuzzy models to express a class of uncertain stochastic recurrent neural networks with discrete and distributed time-varying delays. The uncertainties are norm bounded and possibly time-varying as well. Based on the Lyapunov technique and the stochastic analysis approach, the global stability conditions for uncertain stochastic fuzzy recurrent neural networks are given in terms of linear matrix inequalities (LMIs), which can be easily solved by the MATLAB LMI toolbox.[31] Our results generalize many existing results in the literature. Moreover, the constraint $\dot{\tau}(t) < 1$ on time-varying delay imposed in Refs. [32] and [33] has been removed and a more general result obtained. We also provide numerical examples to demonstrate the effectiveness of the proposed stability results.

2. System description and preliminaries

Consider the following stochastic recurrent neural networks with both discrete and distributed delays described by

$$\begin{aligned}\dot{u}(t) = &-Au(t) + Bg(u(t)) + Cg(u(t-h(t))) \\ &+ D\int_{t-\tau(t)}^{t} g(u(s))\mathrm{d}s + I, \end{aligned} \tag{1}$$

where $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}} \in \mathbf{R}^n$ is the neuron state vector, the matrix $A = \mathrm{diag}\{a_1, a_2, \ldots, a_n\}$ is a diagonal matrix and $a_i > 0, i = 1, \ldots, n$. $B \in \mathbf{R}^{n\times n}$, $C \in \mathbf{R}^{n\times n}$ and $D \in \mathbf{R}^{n\times n}$ are the discretely delayed connection weight matrices and the distributively delayed connection weight matrix, respectively. $g(u(t)) = [g_1(u_1(t)),\ g_2(u_2(t)), \ldots, g_n(u_n(t))]^{\mathrm{T}} \in \mathbf{R}^n$ denotes the neuron activation function with $f(0) = 0$. $I = (I_1, I_2, \ldots, I_n) \in \mathbf{R}^n$ is a constant external input vector. $h(t) > 0$ and $\tau(t) > 0$ denote the discrete time-varying delay and the distributed time-varying delay, respectively, and are assumed to satisfy $0 \le h(t) \le \bar{h},\ 0 \le \tau(t) \le \bar{\tau},\ \dot{h}(t) \le d$ where $\bar{h},\ \bar{\tau}$ and d are known constants. The initial condition associated with model (1) is

$$x_i(s) = \phi_i(s),\ i = 1, 2, \ldots, n,$$

where $\phi_i(s)$ is bounded and continuously differential on $[-\rho, 0],\ \rho = \max\{h, \tau\}$.

It is well known that bounded activation functions always guarantee the existence of an equilibrium point for recurrent neural network (1). For convenience, we shift the equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^{\mathrm{T}}$ to the origin by translation $x(t) = u(t) - u^*$, which yields the following system

$$\begin{aligned}\dot{x}(t) = &-Ax(t) + Bf(x(t)) + Cf(x(t-h(t))) \\ &+ D\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s,\end{aligned}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}} \in \mathbf{R}^n$ is the state vector of the transformed system and

$$\begin{aligned}&f(x(\cdot)) = (f_1(x_1(\cdot)), \ldots, f_n(x_n(\cdot)))^{\mathrm{T}}, \\ &f_j(x(\cdot)) = g_j(x(\cdot) + u^*) - g_j(u^*).\end{aligned}$$

Throughout this paper, we make the following assumption:

There exist positive numbers L_q such that

$$|f(x) - f(y)| \le |L_q(x - y)|,\quad q = 1, 2, \ldots, n,$$

for all $x,\ y \in \mathbf{R},\ x \ne y$ and denote $L = \mathrm{diag}\{L_1, L_2, \ldots, L_n\}$.

In this section, we will consider a stochastic recurrent neural network with discrete and distributed time-varying delays, which is represented by a TS fuzzy model composed of a set of fuzzy implications and each implication is expressed as a linear system model.

The continuous fuzzy system was proposed to represent a nonlinear system.[24] The system dynamics can be captured by a set of fuzzy rules which characterize local correlation in the state space. Each local dynamic described by the fuzzy **IF–THEN** rule has the property of a linear input–output relation. Based

on the TS fuzzy model concept, a general class of TS fuzzy recurrent neural networks with time-varying delays is considered here. The model of Takagi–Sugeno stochastic fuzzy recurrent NNs with discrete and distributed time-varying delays is described as follows.

Plant Rule k**:**

IF $\{\theta_1(t)$ is $M_{k1}\}$ and ... and $\{\theta_r(t)$ is $M_{kr}\}$

THEN

$$\begin{aligned}\mathrm{d}x(t) = \Big\{&[-A_k x(t) + B_k f(x(t)) + C_k f(x(t-h(t)))\\ &+ D_k \int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s]\mathrm{d}t\\ &+ \sigma_k(t, x(t), x(t-h(t))),\\ &\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s)\mathrm{d}\omega(t)\Big\},\end{aligned}$$

where $\theta_i(t)$, $(i = 1, 2, \ldots, r)$ are known variables. M_{kl} $(k \in \{1, 2, \ldots, m\}$, $l \in \{1, 2, \ldots, r\})$ is the fuzzy set and m is the number of model rules. $w(t) = (w_1(t), w_2(t), \ldots, w_m(t))^{\mathrm{T}}$ is an m-dimensional Brownian motion defined on $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathscr{P})$. Let $\sigma(t, x, y, z) : R^+ \times R^n \times R^n \times R^n \to R^{n\times m}$ is locally Lipchitz continuous and satisfies the linear growth condition as well. Moreover σ_k satisfies,

$$\begin{aligned}&\mathrm{trace}\Big[\sigma_k\big(t, x(t), x(t-h(t)), \int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\big)^{\mathrm{T}}\sigma_k\big(t, x(t), x(t-h(t)), \int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\big)\Big]\\ &\leq |K_1 x(t)|^2 + |K_2 x(t-h(t))|^2 + \Big|K_3 \int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big|^2,\end{aligned}$$

where K_1, K_2 and K_3 are known constant matrices with appropriate dimensions. By inferring from the fuzzy models, the final output of TSSFRNNs is obtained by

$$\begin{aligned}\mathrm{d}x(t) &= \sum_{k=1}^{m}\beta_k(\theta(t))\Big\{\Big[-A_k x(t) + B_k f(x(t)) + C_k f(x(t-h(t))) + D_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\mathrm{d}t\\ &\quad + \sigma_k(t, x(t), x(t-h(t)),\ \int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s)\mathrm{d}\omega(t)\Big\}\Big/\sum_{k=1}^{m}\beta_k(\theta(t))\\ &= \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{\Big[-A_k x(t) + B_k f(x(t)) + C_k f(x(t-h(t))) + D_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\mathrm{d}t\\ &\quad + \sigma_k(t, x(t), x(t-h(t)), \int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s)\mathrm{d}\omega(t)\Big\}. \qquad (2)\end{aligned}$$

The weight and averaged weight of each fuzzy rule are denoted by

$$\beta_k(\theta(t)) = \prod_{l=1}^{t} M_{kl}(\theta(t))$$

and

$$\eta_k(\theta(t)) = \beta_k(\theta(t))/\sum_{k=1}^{m}\beta_k(\theta(t)),$$

respectively. The term $M_{kl}(\theta(t))$ is the grade membership of $\theta_l(t)$ in M_{kl}.

We assume that

$$\beta_k(\theta(t)) \geq 0,\ k \in \{1, 2, \ldots, m\}$$

and

$$\sum_{k=1}^{m}\eta_k(\theta(t)) = 1,\ \text{for all}\ \ t \geq 0.$$

Lemma 1 (Schur complement.[34]) The linear matrix inequality

$$\begin{bmatrix} Q(y) & S(y)\\ S(y)^{\mathrm{T}} & R(y)\end{bmatrix} < 0$$

is equivalent to

$$R(y) < 0,\ Q(y) - S(y)R(y)^{-1}S(y)^{\mathrm{T}} < 0,$$

where $Q(y) = Q(y)^{\mathrm{T}}$, $R(y) = R(y)^{\mathrm{T}}$ and $S(y)$ depend affinely on y.

Lemma 2[35] Given any real matrices K_1, K_2, Q of appropriate dimensions and a number $\nu > 0$ such that $0 < Q = Q^{\mathrm{T}}$, then the following inequality holds:

$$K_1^{\mathrm{T}}K_2 + K_2^{\mathrm{T}}K_1 \leq \nu K_1^{\mathrm{T}}QK_1 + \nu^{-1}K_2^{\mathrm{T}}Q^{-1}K_2.$$

Lemma 3[36] For any vectors $x,\ y \in R^n$, with

$P > 0$, the following inequality holds

$$2x^{\mathrm{T}}y \leq x^{\mathrm{T}}P^{-1}x + y^{\mathrm{T}}Py.$$

Lemma 4[37] For matrices $P \in \mathbf{R}^{n\times n}, M \in \mathbf{R}^{n\times k}$, $N \in \mathbf{R}^{l\times n}$ and $F \in \mathbf{R}^{k\times l}$ with $P > 0, \|F\| \leq 1$ and scalar $\epsilon > 0$, one has the following,

(i)

$$(MFN)^{\mathrm{T}}P + P(MFN) \leq \epsilon PMM^{\mathrm{T}}P + \epsilon^{-1}N^{\mathrm{T}}N.$$

(ii) If $P - \epsilon MM^{\mathrm{T}} > 0$, then

$$\begin{aligned}&(A + MFN)^{\mathrm{T}}P^{-1}(A + MFN)\\ \leq\ & A^{\mathrm{T}}(P - \epsilon MM^{\mathrm{T}})^{-1}A + \epsilon^{-1}N^{\mathrm{T}}N.\end{aligned}$$

3. Global stability results

In this section, some sufficient conditions of the stability criterion for system (2) are obtained. First, we introduce the free weighting matrices used in the following proofs. For convenience, we set

$$\begin{aligned}y(t) =& \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{ - A_kx(t) + B_kf(x(t))\\ &+ C_kf(x(t-h(t))) + D_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big\},\\ g(t) =& \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{\sigma_k(t,x(t),x(t-h(t)),\\ &\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s)\mathrm{d}\omega(t)\Big\},\end{aligned}$$

then system (2) becomes

$$\mathrm{d}x(t) = y(t)\mathrm{d}t + g(t)\mathrm{d}\omega(t).$$

Then, for any appropriately dimensioned matrices N_i, M_i $(i = 1,\ldots,6)$, the following equation holds

$$\begin{aligned}&2\Big[x^{\mathrm{T}}(t)\ M_1 + x^{\mathrm{T}}(t-h(t))\ M_2\\ &+ f^{\mathrm{T}}(x(t))\ M_3 + f^{\mathrm{T}}(x(t-h(t)))\ M_4\\ &+ \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}s\ M_5 + y^{\mathrm{T}}(t)M_6\Big]\\ &\times\Big[- y(t) + \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{ - A_kx(t) + B_kf(x(t))\\ &+ C_kf(x(t-h(t))) + D_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big\}\Big] = 0, \quad (3)\end{aligned}$$

and

$$\begin{aligned}&2\Big[x^{\mathrm{T}}(t)\ N_1 + x^{\mathrm{T}}(t-h(t))\ N_2\\ &+ f^{\mathrm{T}}(x(t))\ N_3 + f^{\mathrm{T}}(x(t-h(t)))\ N_4\\ &+ \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}s\ N_5 + y^{\mathrm{T}}(t)\ N_6\Big]\\ &\times\Big[x(t) - x(t-h(t)) - \int_{t-h(t)}^{t}\mathrm{d}x(s)\Big] = 0. \quad (4)\end{aligned}$$

Noting that X, Y, and V are positive-definite matrices and using the above assumption, we can have

$$x^{\mathrm{T}}(t)Yf(x(t)) \leq x^{\mathrm{T}}(t)YLx(t), \quad (5)$$

$$f^{\mathrm{T}}(x(t))Xf(x(t)) \leq x^{\mathrm{T}}(t)L^{\mathrm{T}}XLx(t), \quad (6)$$

$$\begin{aligned}&x^{\mathrm{T}}(t-h(t)))L^{\mathrm{T}}VLx(t-h(t))\\ &\quad - f^{\mathrm{T}}(x(t-h(t)))Vf(x(t-h(t))) > 0. \quad (7)\end{aligned}$$

Theorem 1 Given scalars $\bar{\tau}$, $\bar{h}$, and d, the equilibrium point of system (2) is globally asymptotically stable if there are symmetric positive-definite matrices $P > 0$, $Q > 0$, $R > 0$, $S > 0$, $T > 0$, $U > 0$, $X > 0$, $Y > 0$, $V > 0$, positive scalars ρ, μ and matrices N_i, M_i $(i = 1,\ldots,6)$, such that the following conditions hold for $k = 1, 2, \ldots, m$,

$$P < \rho\ I, \quad (8)$$

$$U < \mu\ I, \quad (9)$$

$$\Psi_k = \begin{bmatrix}\Psi_{k11} & \Psi_{k12} & \Psi_{k13} & \Psi_{k14} & \Psi_{k15} & \Psi_{k16} & N_1 & N_1\\ * & \Psi_{k22} & \Psi_{k23} & \Psi_{k24} & \Psi_{k25} & \Psi_{k26} & N_2 & N_2\\ * & * & \Psi_{k33} & \Psi_{k34} & \Psi_{k35} & \Psi_{k36} & N_3 & N_3\\ * & * & * & \Psi_{k44} & \Psi_{k45} & \Psi_{k46} & N_4 & N_4\\ * & * & * & * & \Psi_{k55} & \Psi_{k56} & N_5 & N_5\\ * & * & * & * & * & \Psi_{k66} & N_6 & N_6\\ * & * & * & * & * & * & -T & 0\\ * & * & * & * & * & * & * & -U\end{bmatrix} < 0, \quad (10)$$

where

$$\begin{aligned}
\Psi_{k11} &= -PA_k - A_k^{\mathrm{T}}P + Q + (\rho + h\ \mu)K_1 + N_1 \\
&\quad + N_1^{\mathrm{T}} - A_k^{\mathrm{T}}M_1 - M_1A_k + 2L^{\mathrm{T}}XL + 2YL, \\
\Psi_{k12} &= -A_k^{\mathrm{T}}M_2^{\mathrm{T}} - N_1 + N_2^{\mathrm{T}}, \\
\Psi_{k13} &= PB_k + N_3^{\mathrm{T}} + M_1B_k - A_k^{\mathrm{T}}M_3^{\mathrm{T}} - 2Y, \\
\Psi_{k14} &= PC_k + N_4^{\mathrm{T}} + M_1C_k - A_k^{\mathrm{T}}M_4^{\mathrm{T}}, \\
\Psi_{k15} &= PD_k + N_5^{\mathrm{T}} + M_1D_k - A_k^{\mathrm{T}}M_5^{\mathrm{T}}, \\
\Psi_{k16} &= N_6^{\mathrm{T}} - M_1 - A_k^{\mathrm{T}}M_6^{\mathrm{T}}, \\
\Psi_{k22} &= -(1-d)Q + (\rho + h\ \mu)K_2 \\
&\quad - N_2 - N_2^{\mathrm{T}} + 2L^{\mathrm{T}}VL, \\
\Psi_{k23} &= -N_3^{\mathrm{T}} + M_2B_k,\ \ \Psi_{k24} = -N_4^{\mathrm{T}} + M_2C_k, \\
\Psi_{k25} &= -N_5^{\mathrm{T}} + M_2D_k,\ \ \Psi_{k26} = -N_6^{\mathrm{T}} - M_2, \\
\Psi_{k33} &= R + \tau^2\ S + M_3B_k + B_k^{\mathrm{T}}M_3^{\mathrm{T}} - 2X, \\
\Psi_{k34} &= M_3C_k + B_k^{\mathrm{T}}M_4^{\mathrm{T}}, \quad \Psi_{k35} = M_3D_k + B_k^{\mathrm{T}}M_5^{\mathrm{T}}, \\
\Psi_{k36} &= -M_3 + B_k^{\mathrm{T}}M_6^{\mathrm{T}}, \\
\Psi_{k44} &= -(1-d)R + M_4C_k + C_k^{\mathrm{T}}M_4^{\mathrm{T}} - 2V, \\
\Psi_{k45} &= M_4D_k + C_k^{\mathrm{T}}M_5^{\mathrm{T}}, \quad \Psi_{k46} = -M_4 + C_k^{\mathrm{T}}M_6^{\mathrm{T}}, \\
\Psi_{k55} &= M_5D_k + D_k^{\mathrm{T}}M_5^{\mathrm{T}} + (\rho + h\ \mu)K_3 - S, \\
\Psi_{k56} &= D_k^{\mathrm{T}}M_6^{\mathrm{T}} - M_5,\ \ \Psi_{k66} = h^2T - M_6^{\mathrm{T}} - M_6.
\end{aligned}$$

Proof We use the following Lyapunov functional to derive the stability result,

$$\begin{aligned}
V(t,x(t)) &= V_1(t,x(t)) + V_2(t,x(t)) + V_3(t,x(t)) \\
&\quad + V_4(t,x(t)) + V_5(t,x(t)),
\end{aligned}$$

where

$$\begin{aligned}
V_1(t,x(t)) &= x^{\mathrm{T}}(t)Px(t), \\
V_2(t,x(t)) &= \int_{t-h(t)}^{t} [x^{\mathrm{T}}(s)Qx(s) \\
&\quad + f^{\mathrm{T}}(x(s))Rf(x(s))]\mathrm{d}s, \\
V_3(t,x(t)) &= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} f^{\mathrm{T}}(x(s))Sf(x(s))\mathrm{d}s\mathrm{d}\theta, \\
V_4(t,x(t)) &= h\int_{-h}^{0}\int_{t+\theta}^{t} y^{\mathrm{T}}(s)Ty(s)\mathrm{d}s\mathrm{d}\theta, \\
V_5(t,x(t)) &= \int_{-h}^{0}\int_{t+\theta}^{t} \mathrm{trace}(g^{\mathrm{T}}(s)Ug(s))\mathrm{d}s\mathrm{d}\theta,
\end{aligned}$$

By Itô's formula, the stochastic derivative of $V(t,x(t))$ along the trajectories of system (2) can be obtained as

$$\begin{aligned}
\mathscr{L}V(t,x(t)) \le \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{ & 2x^{\mathrm{T}}(t)P\Big[-A_kx(t) + B_kf(x(t)) + C_kf(x(t-h(t))) + D_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big] \\
& + \mathrm{trace}[g^{\mathrm{T}}(t)\ P\ g(t)] + x^{\mathrm{T}}(t)\ Qx(t) - (1-d)x^{\mathrm{T}}(t-h(t))\ Q\ x(t-h(t)) + f^{\mathrm{T}}(x(t))\ R \\
& \times f(x(t)) - (1-d)f^{\mathrm{T}}(x(t-h(t)))\ R\ f(x(t-h(t))) + \tau^2 f^{\mathrm{T}}(x(t))\ S\ f(x(t)) + h^2y^{\mathrm{T}}(t) \\
& \times Ty(t) - \tau\int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\ S\ f(x(s))\mathrm{d}s - h\int_{t-h(t)}^{t} y^{\mathrm{T}}(s)\ T\ y(s)\mathrm{d}s + h\ \mathrm{trace}(g^{\mathrm{T}}(t)U \\
& \times g(t)) - \int_{t-h(t)}^{t}\mathrm{trace}(g^{\mathrm{T}}(s)\ U\ g(s))\mathrm{d}s + 2\Big[x^{\mathrm{T}}(t)N_1 + x^{\mathrm{T}}(t-h(t))N_2 + f^{\mathrm{T}}(x(t))N_3 \\
& + f^{\mathrm{T}}(x(t-h(t)))N_4 + \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}s\ N_5 + y^{\mathrm{T}}(t)N_6\Big]\Big[x(t) - x(t-h(t)) \\
& - \int_{t-h(t)}^{t}[y(s)\mathrm{d}s + g(s)\mathrm{d}w(s)]\Big] + 2\Big[x^{\mathrm{T}}(t)M_1 + x^{\mathrm{T}}(t-h(t))M_2 + f^{\mathrm{T}}(x(t))M_3 \\
& + f^{\mathrm{T}}(x(t-h(t)))M_4 + \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}sM_5 + y^{\mathrm{T}}(t)M_6\Big]\Big[-A_kx(t) + B_kf(x(t)) \\
& + C_kf(x(t-h(t))) + D_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s - y(t)\Big] + 2x^{\mathrm{T}}(t)Yf(x(t)) - 2x^{\mathrm{T}}(t)Yf(x(t)) \\
& + 2f^{\mathrm{T}}(x(t))Xf(x(t)) - 2f^{\mathrm{T}}(x(t))Xf(x(t)) + 2x^{\mathrm{T}}(t-h(t))L^{\mathrm{T}}VLx(t-h(t)) \\
& - 2f^{\mathrm{T}}(x(t-h(t)))Vf(x(t-h(t)))\Big\}. \qquad (11)
\end{aligned}$$

By using Jensen's inequality[38] we have

$$-\tau\int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))Sf(x(s))\mathrm{d}s \le -\Big[\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]^{\mathrm{T}}S\Big[\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big], \qquad (12)$$

$$-h\int_{t-h(t)}^{t} y^{\mathrm{T}}(s)Ty(s)\mathrm{d}s \le -\Big[\int_{t-h(t)}^{t} y(s)\mathrm{d}s\Big]^{\mathrm{T}} T\Big[\int_{t-h(t)}^{t} y(s)\mathrm{d}s\Big]. \tag{13}$$

It follows from Lemma 3 that

$$2\Big[x^{\mathrm{T}}(t)\ N_1 + x^{\mathrm{T}}(t-h(t))\ N_2 + f^{\mathrm{T}}(x(t))\ N_3 + f^{\mathrm{T}}(x(t-h(t)))\ N_4 + \int_{t-h(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}s\ N_5 + y^{\mathrm{T}}(t)\ N_6\Big] \times \int_{t-h(t)}^{t} y(s)\mathrm{d}s \le \xi^{\mathrm{T}}(t)\ N\ T^{-1}\ N^{\mathrm{T}}\ \xi(t) + \Big(\int_{t-h(t)}^{t} y(s)\mathrm{d}s\Big)^{\mathrm{T}}\ T\ \Big(\int_{t-h(t)}^{t} y(s)\mathrm{d}s\Big), \tag{14}$$

$$2\Big[x^{\mathrm{T}}(t)\ N_1 + x^{\mathrm{T}}(t-h(t))\ N_2 + f^{\mathrm{T}}(x(t))\ N_3 + f^{\mathrm{T}}(x(t-h(t)))\ N_4 + \int_{t-h(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}sN_5 + y^{\mathrm{T}}(t)N_6\Big] \times \int_{t-h(t)}^{t} g(s)\mathrm{d}\omega(s) \le \xi^{\mathrm{T}}(t)\ N\ U^{-1}\ N^{\mathrm{T}}\ \xi(t) + \Big(\int_{t-h(t)}^{t} g(s)\mathrm{d}w(s)\Big)^{\mathrm{T}}\ U\ \Big(\int_{t-h(t)}^{t} g(s)\mathrm{d}w(s)\Big), \tag{15}$$

where

$$\xi^{\mathrm{T}}(t) = \Big[x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-h(t))\ f^{\mathrm{T}}(x(t))\ f^{\mathrm{T}}(x(t-h(t)))\ \Big(\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big)^{\mathrm{T}}\ y^{\mathrm{T}}(t)\Big],$$
$$N^{\mathrm{T}} = [N_1^{\mathrm{T}}\ N_2^{\mathrm{T}}\ N_3^{\mathrm{T}}\ N_4^{\mathrm{T}}\ N_5^{\mathrm{T}}\ N_6^{\mathrm{T}}].$$

Thus from expressions (3)–(7), (11)–(15) and by using Schur complement Lemma 1 we have

$$\mathscr{L}V(t,x(t)) \le \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{\xi^{\mathrm{T}}(t)\ \Psi_k\ \xi(t) - \int_{t-h(t)}^{t}\mathrm{trace}\big(g^{\mathrm{T}}(s)\ U\ g(s)\big)\mathrm{d}s + \Big(\int_{t-h(t)}^{t} g(s)\mathrm{d}w(s)\Big)^{\mathrm{T}}\ U\ \Big(\int_{t-h(t)}^{t} g(s)\mathrm{d}w(s)\Big)\Big\},$$

Therefore, it follows from the facts of $\lambda_{\max}(\Psi_k) < 0$ and $|x(t)| \le |\xi(t)|$ that

$$\mathscr{L}V(t,x(t)) \le \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{\lambda_{\max}(\Psi_k)|x(t)|^2 - \int_{t-h(t)}^{t}\mathrm{trace}\big(g^{\mathrm{T}}(s)\ U\ g(s)\big)\mathrm{d}s + \Big(\int_{t-h(t)}^{t} g(s)\mathrm{d}w(s)\Big)^{\mathrm{T}}\ U\ \Big(\int_{t-h(t)}^{t} g(s)\mathrm{d}w(s)\Big)\Big\}. \tag{16}$$

According to the arguments,[39] we have

$$E\Big\{\Big(\int_{t-h(t)}^{t} g(s)\mathrm{d}w(s)\Big)^{\mathrm{T}}\ U\ \Big(\int_{t-h(t)}^{t} g(s)\mathrm{d}w(s)\Big)\Big\} = E\Big\{\int_{t-h(t)}^{t}\mathrm{trace}(g^{\mathrm{T}}(s)\ U\ g(s))\mathrm{d}s\Big\}.$$

Taking the mathematical expectation on both sides of expression (16), we have

$$\mathscr{L}EV(t,x(t)) \le \sum_{k=1}^{m}\eta_k(\theta(t))\{\lambda_{\max}(\Psi_k)E|x(t)|^2\}.$$

It can now be concluded from Lyapunov stability theory that the dynamics of neural network (2) is globally, asymptotically stable in the mean square. This completes the proof of Theorem 1.

In the previous part, we developed a stability analysis approach for a stochastic fuzzy recurrent neural network with discrete and distributed delays based on LMI. Now we consider the robust stability in the mean square of the stochastic uncertain fuzzy recurrent neural network with discrete and distributed delays based on LMI.

4. Robust stability analysis

In this section, the global robust asymptotic stability in the mean square for stochastic uncertain fuzzy recurrent neural networks with discrete and distributed delays is derived.

Now, we consider a case where the system has uncertain and stochastic perturbations. The neural

network can be described as follows:

$$\begin{aligned}\mathrm{d}x(t) = \Big[& -(A+\Delta A(t))x(t)+(B+\Delta B(t))f(x(t)) \\ & +(C+\Delta C(t))f(x(t-h(t))) \\ & +(D+\Delta D(t))\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\mathrm{d}t \\ & +\Big[(H+\Delta H(t))x(t) \\ & +(W+\Delta W(t))x(t-h(t)) \\ & +(Z+\Delta Z(t))\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\mathrm{d}\omega(t), \end{aligned} \tag{17}$$

The k-th rule of the TS fuzzy model is of the following form,

Plant Rule k:

IF $\{\theta_1(t)$ is $M_{k1}\}$ and $\ldots$ and $\{\theta_r(t)$ is $M_{kl}\}$

THEN

$$\begin{aligned}\mathrm{d}x(t) = \Big[& -(A_k+\Delta A_k(t))x(t) \\ & +(B_k+\Delta B_k(t))f(x(t)) \\ & +(C_k+\Delta C_k(t))f(x(t-h(t))) \\ & +(D_k+\Delta D_k(t))\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\mathrm{d}t \\ & +\Big[(H_k+\Delta H_k(t))x(t)) \\ & +(W_k+\Delta W_k(t))x(t-h(t)) \\ & +(Z_k+\Delta Z_k(t))\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\mathrm{d}\omega(t), \end{aligned}$$

where M_{kl} $(l = 1,2,\ldots,r,\ k = 1,2,\ldots,m)$ are fuzzy sets, $(\theta_1(t),\theta_2(t),\ldots,\theta_r(t))^{\mathrm{T}}$ is the premise variable vectors. $x(t)$ is the state variable, m is the number of **IF–THEN** rules and $w(t) = (w_1(t),w_2(t),\ldots,w_m(t))^{\mathrm{T}}$ is an m-dimensional Brownian motion defined on $(\Omega,\mathfrak{F},\{\mathfrak{F}_t\}_{t\geq 0},\mathscr{P})$.

The parameter uncertainties $\Delta A_k(t)$, $\Delta B_k(t)$, $\Delta C_k(t)$, $\Delta D_k(t)$, $\Delta H_k(t)$, $\Delta W_k(t)$, $\Delta Z_k(t)$ are time-varying matrices with appropriate dimensions, which are defined as follows:

$$\begin{aligned}&[\Delta A_k(t),\quad \Delta B_k(t),\quad \Delta C_k(t),\quad \Delta D_k(t),\quad \Delta H_k(t),\quad \Delta W_k(t),\quad \Delta Z_k(t)] \\ &= E_k F_k(t)[G_{1k},\quad G_{2k},\quad G_{3k},\quad G_{4k},\quad G_{5k},\quad G_{6k},\quad G_{7k}],\end{aligned} \tag{18}$$

where G_{1k}, G_{2k}, G_{3k}, G_{4k}, G_{5k}, G_{6k}, G_{7k}, E_k are known constant matrices of appropriate dimensions and $F_k(t)$ is a known time-varying matrix with Lebegue measurable elements bounded by,

$$F_k^{\mathrm{T}}(t)F_k(t) \leq I,$$

where I is the identity matrix with appropriate dimension.

For convenience we define,

$$\begin{aligned}&\bar{A}_k = A_k+\Delta A_k(t),\quad \bar{B}_k = B_k+\Delta B_k(t), \\ &\bar{C}_k = C_k+\Delta C_k(t),\quad \bar{D}_k = D_k+\Delta D_k(t), \\ &\bar{H}_k = H_k+\Delta H_k(t),\quad \bar{W}_k = W_k+\Delta W_k(t), \\ &\bar{Z}_k = Z_k+\Delta Z_k(t).\end{aligned}$$

By inferring from the fuzzy models, the final output of Takagi–Sugeno uncertain stochastic fuzzy recurrent neural networks with discrete and distributed delays (TSUSFRNNs) is obtained by

$$\begin{aligned}\mathrm{d}x(t) = \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{\Big[& -\bar{A}_k x(t)+\bar{B}_k f(x(t)) \\ & +\bar{C}_k f(x(t-h(t)))+\bar{D}_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\mathrm{d}t \\ & +\Big[\bar{H}_k x(t)+\bar{W}_k x(t-h(t)) \\ & +\bar{Z}_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\mathrm{d}\omega(t)\Big\}. \end{aligned} \tag{19}$$

The weight and averaged weight of each fuzzy rule are denoted by

$$\beta_k(\theta(t)) = \prod_{l=1}^{r} M_{kl}(\theta(t))$$

and

$$\eta_k(\theta(t)) = \beta_k(\theta(t))/\sum_{k=1}^{m}\beta_k(\theta(t)),$$

respectively. The term $M_{kl}(\theta(t))$ is the grade membership of $\theta_l(t)$ in M_{kl}.

We assume that

$$\beta_k(\theta(t)) \geq 0,\ k \in \{1,2,\ldots,m\}$$

and

$$\sum_{k=1}^{m}\eta_k(\theta(t)) = 1,\ \text{for all}\ \ t \geq 0.$$

For convenience, we set

$$\begin{aligned}y(t) = \sum_{k=1}^{m}\eta_k(\theta(t))\Big\{ & -\bar{A}_k x(t)+\bar{B}_k f(x(t)) \\ & +\bar{C}_k f(x(t-h(t)))+\bar{D}_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big\},\end{aligned}$$

$$g(t) = \sum_{k=1}^{m} \eta_k(\theta(t))\Big\{\bar{H}_k x(t) + \bar{W}_k x(t-h(t)) + \bar{Z}_k \int_{t-\tau(t)}^{t} f(x(s))\,\mathrm{d}s\Big\},$$

then system (19) becomes

$$\mathrm{d}x(t) = y(t)\,\mathrm{d}t + g(t)\,\mathrm{d}\omega(t).$$

Theorem 2 Given scalars $\bar{\tau}$, $\bar{h}$ and d, the equilibrium point of system (19) is globally, robustly and asymptotically stable if there are symmetric positive-definite matrices $P > 0$, $Q > 0$, $R > 0$, $S > 0$, $T > 0$, $U > 0$, $V > 0$, $X > 0$, $Y > 0$, matrices N_i, M_i $(i = 1, \ldots, 6)$ and positive scalars $\epsilon > 0$, $\epsilon_j > 0$, $(j = 0, 1, 2, \ldots, 7)$ such that the following conditions hold for $k = 1, 2, \ldots, m$

$$\Omega_k = \begin{bmatrix} \bar{\Omega}_k & PE_k & PE_k & 0 & 0 & 0 & 0 & 0 & H_k^{\mathrm{T}}(P + h\,U) & 0 \\ * & -\epsilon_1 I & 0 & PE_k & 0 & 0 & 0 & 0 & W_k^{\mathrm{T}}(P + h\,U) & 0 \\ * & * & -\epsilon_2 I & 0 & PE_k & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\epsilon_3 I & 0 & PE_k & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_4 I & 0 & PE_k & 0 & Z_k^{\mathrm{T}}(P + h\,U) & 0 \\ * & * & * & * & * & -\epsilon_5 I & 0 & PE_k & 0 & 0 \\ * & * & * & * & * & * & -\epsilon_6 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\epsilon_7 I & 0 & 0 \\ * & * & * & * & * & * & * & * & -(P + h\,U) & (P + h\,U)E_k \\ * & * & * & * & * & * & * & * & * & -\epsilon_0 I \end{bmatrix} < 0, \quad (20)$$

$$\bar{\Omega}_k = \begin{bmatrix} \Omega_{k11} & \Omega_{k12} & \Omega_{k13} & \Omega_{k14} & \Omega_{k15} & \Omega_{k16} & N_1 & N_1 \\ * & \Omega_{k22} & \Omega_{k23} & \Omega_{k24} & \Omega_{k25} & \Omega_{k26} & N_2 & N_2 \\ * & * & \Omega_{k33} & \Omega_{k34} & \Omega_{k35} & \Omega_{k36} & N_3 & N_3 \\ * & * & * & \Omega_{k44} & \Omega_{k45} & \Omega_{k46} & N_4 & N_4 \\ * & * & * & * & \Omega_{k55} & \Omega_{k56} & N_5 & N_5 \\ * & * & * & * & * & \Omega_{k66} & N_6 & N_6 \\ * & * & * & * & * & * & -T & 0 \\ * & * & * & * & * & * & * & -U \end{bmatrix}, \quad (21)$$

where

$$\begin{aligned}
\Omega_{k11} &= -PA_k - A_k^{\mathrm{T}}P + Q + N_1 + N_1^{\mathrm{T}} - A_k^{\mathrm{T}}M_1 - M_1A_k + 2L^{\mathrm{T}}XL + 2YL + (\epsilon_1 + \epsilon)G_{1k}^{\mathrm{T}}G_{1k} + \epsilon_0 G_{5k}^{\mathrm{T}}G_{5k}, \\
\Omega_{k12} &= \Psi_{k12} + \epsilon_0 G_{5k}^{\mathrm{T}}G_{6k}, \quad \Omega_{k13} = \Psi_{k13}, \quad \Omega_{k14} = \Psi_{k14}, \ \Omega_{k15} = \Psi_{k15} + \epsilon_0 G_{5k}^{\mathrm{T}}G_{7k}, \quad \Omega_{k16} = \Psi_{k16}, \\
\Omega_{k22} &= -(1-d)Q - N_2 - N_2^{\mathrm{T}} + \epsilon_0 G_{6k}^{\mathrm{T}}G_{6k}, \Omega_{k23} = \Psi_{k23}, \quad \Omega_{k24} = \Psi_{k24}, \quad \Omega_{k25} = \Psi_{k25} + \epsilon_0 G_{6k}^{\mathrm{T}}G_{7k}, \\
\Omega_{k26} &= \Psi_{k26}, \quad \Omega_{k33} = \Psi_{k33} + (\epsilon_1 + \epsilon)G_{2k}^{\mathrm{T}}G_{2k}, \quad \Omega_{k34} = \Psi_{k34}, \ \Omega_{k35} = \Psi_{k35}, \quad \Omega_{k36} = \Psi_{k36}, \\
\Omega_{k44} &= -(1-d_2)R + M_kC_k + C_k^{\mathrm{T}}M_k^{\mathrm{T}} + (\epsilon_1 + \epsilon)G_{3k}^{\mathrm{T}}G_{3k}, \quad \Omega_{k45} = \Psi_{k45}, \quad \Omega_{k46} = \Psi_{k46}, \\
\Omega_{k55} &= M_5D_k + D_k^{\mathrm{T}}M_5^{\mathrm{T}} - S + (\epsilon_1 + \epsilon)G_{4k}^{\mathrm{T}}G_{4k} + \epsilon_0 G_{7k}^{\mathrm{T}}G_{7k}, \quad \Omega_{k56} = \Psi_{k56}, \quad \Omega_{k66} = \Psi_{k66},
\end{aligned}$$

Ψ_{k12}, Ψ_{k13}, Ψ_{k14}, Ψ_{k15}, Ψ_{k16}, Ψ_{k23}, Ψ_{k24}, Ψ_{k25}, Ψ_{k26}, Ψ_{k33}, Ψ_{k34}, Ψ_{k35}, Ψ_{k36}, Ψ_{k45}, Ψ_{k46}, Ψ_{k55}, Ψ_{k56}, Ψ_{k66} are defined in Theorem 1.

Proof Pre, post multiplying Ω_k by $\mathrm{diag}(I, I, I, I, I, I, I, I, I, I, I, I, I, I, I, (P + h\,U)^{-1}, I)$ and by using the Schur complement (Lemma 1) $\Omega_k < 0$ implies that

$$\Xi_k < 0,$$

where

$$\Xi_k = \Pi + [H_k\ W_k\ 0\ 0\ Z_k\ 0\ 0\ 0]^{\mathrm{T}}((P + h\,U)^{-1} - \epsilon_0^{-1}E_kE_k^{\mathrm{T}})^{-1}[H_k\ W_k\ 0\ 0\ Z_k\ 0\ 0\ 0].$$

Here

$$\Pi = (\Pi_{i,j})_{8\times 8},\ i,j = 1,2,\ldots,8,$$

where

$$\begin{aligned}
&\Pi_{1,1} = \Omega_{k11} + (\epsilon_1^{-1} + \epsilon_2^{-1})PE_kE_k^{\mathrm{T}}P, \quad \Pi_{1,2} = \Omega_{k12}, \quad \Pi_{1,5} = \Omega_{k15},\\
&\Pi_{2,2} = \Omega_{k22} + \epsilon_3^{-1}PE_kE_k^{\mathrm{T}}P, \quad \Pi_{2,5} = \Omega_{k25}, \quad \Pi_{3,3} = \Omega_{k33} + \epsilon_4^{-1}PE_kE_k^{\mathrm{T}}P,\\
&\Pi_{4,4} = \Omega_{k44} + \epsilon_5^{-1}PE_kE_k^{\mathrm{T}}P, \quad \Pi_{5,5} = \Omega_{k55} + \epsilon_6^{-1}PE_kE_k^{\mathrm{T}}P, \quad \Pi_{6,6} = \Omega_{k66} + \epsilon_7^{-1}PE_kE_k^{\mathrm{T}}P,\\
&\Pi_{1,3} = \Omega_{k13}, \quad \Pi_{1,4} = \Omega_{k14}, \quad \Pi_{1,6} = \Omega_{k16}, \quad \Pi_{1,7} = N_1, \quad \Pi_{1,8} = N_1, \quad \Pi_{2,3} = \Omega_{k23}, \Pi_{2,4} = \Omega_{k24},\\
&\Pi_{2,5} = \Omega_{k25}, \quad \Pi_{2,6} = \Omega_{k26}, \quad \Pi_{2,7} = N_2, \quad \Pi_{2,8} = N_2, \quad \Pi_{3,4} = \Omega_{k34}, \quad \Pi_{3,5} = \Omega_{k35}, \quad \Pi_{3,6} = \Omega_{k36},\\
&\Pi_{3,7} = N_3, \quad \Pi_{3,8} = N_3, \quad \Pi_{4,5} = \Omega_{k45}, \quad \Pi_{4,6} = \Omega_{k46}, \quad \Pi_{4,7} = N_4, \quad \Pi_{4,8} = N_4, \quad \Pi_{5,6} = \Omega_{k56},\\
&\Pi_{5,7} = N_5, \ \Pi_{5,8} = N_5, \quad \Pi_{6,7} = N_6, \quad \Pi_{6,8} = N_6, \quad \Pi_{7,7} = -T, \quad \Pi_{7,8} = 0, \quad \Pi_{8,8} = -U.
\end{aligned}$$

We use the Lyapunov functional used in the proof of Theorem 1 to derive the stability result. By Itô's formula, the stochastic derivative of $V(t,x(t))$ along the trajectories of system (19) can be obtained as,

$$\begin{aligned}
\mathscr{L}V(t,x(t)) \le \sum_{k=1}^{m}\lambda_k(\xi(t))\Big\{ & 2x^{\mathrm{T}}(t)P\Big[-\bar{A}_kx(t) + \bar{B}_kf(x(t)) + \bar{C}_kf(x(t-h(t))) + \bar{D}_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\\
& + \Big[\bar{H}_kx(t) + \bar{W}_kx(t-h(t)) + \bar{Z}_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]^{\mathrm{T}}(P + h\ U)\Big[\bar{H}_kx(t) + \bar{W}_kx(t-h(t))\\
& + \bar{Z}_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big] + x^{\mathrm{T}}(t)\ Qx(t) - (1-d)x^{\mathrm{T}}(t-h(t))\ Q\ x(t-h(t))\\
& + f^{\mathrm{T}}(x(t))\ Rf(x(t)) - (1-d)f^{\mathrm{T}}(x(t-h(t)))R\ f(x(t-h(t))) + \tau^2f^{\mathrm{T}}(x(t))\ S\ f(x(t))\\
& + h^2y^{\mathrm{T}}(t)Ty(t) - \tau\int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\ S\ f(x(s))\mathrm{d}s - h\int_{t-h(t)}^{t} y^{\mathrm{T}}(s)\ T\ y(s)\mathrm{d}s\\
& - \int_{t-h(t)}^{t}\mathrm{trace}\big(g^{\mathrm{T}}(s)\ U\ g(s)\big)\mathrm{d}s + 2\Big[x^{\mathrm{T}}(t)N_1 + x^{\mathrm{T}}(t-h(t))N_2 + f^{\mathrm{T}}(x(t))N_3\\
& + f^{\mathrm{T}}(x(t-h(t)))N_4 + \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}sN_5 + y^{\mathrm{T}}(t)N_6\Big]\Big[x(t) - x(t-h(t))\\
& - \int_{t-h(t)}^{t}[\,y(s)\mathrm{d}s + g(s)\mathrm{d}w(s)]\Big] + 2\Big[x^{\mathrm{T}}(t)M_1 + x^{\mathrm{T}}(t-h(t))M_2 + f^{\mathrm{T}}(x(t))M_3\\
& + f^{\mathrm{T}}(x(t-h(t)))M_4 + \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}sM_5 + y^{\mathrm{T}}(t)M_6\Big]\Big[-\bar{A}_kx(t) + \bar{B}_kf(x(t))\\
& + \bar{C}_kf(x(t-h(t))) + \bar{D}_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s - y(t)\Big] + 2x^{\mathrm{T}}(t)Yf(x(t)) - 2x^{\mathrm{T}}(t)Yf(x(t))\\
& + 2f^{\mathrm{T}}(x(t))Xf(x(t)) - 2f^{\mathrm{T}}(x(t))Xf(x(t)) + 2x^{\mathrm{T}}(t-h(t))L^{\mathrm{T}}VLx(t-h(t))\\
& - 2f^{\mathrm{T}}(x(t-h(t)))Vf(x(t-h(t)))\Big\},
\end{aligned} \tag{22}$$

It follows from Lemma 4 and the fact $F_k^{\mathrm{T}}F_k \le I$ that

$$\begin{aligned}
& 2x^{\mathrm{T}}(t)P\Big[-\Delta A_k(t)x(t) + \Delta B_k(t)f(x(t)) + \Delta C_k(t)f(x(t-h(t))) + \Delta D_k(t)\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]\\
\le\ & \epsilon_1^{-1}x^{\mathrm{T}}(t)PE_kE_k^{\mathrm{T}}Px(t) + \epsilon_1\Big[x^{\mathrm{T}}(t)G_{1k}^{\mathrm{T}}G_{1k}x(t) + f^{\mathrm{T}}(x(t))G_{2k}^{\mathrm{T}}G_{2k}f(x(t))\\
& + f^{\mathrm{T}}(x(t-h(t)))G_{3k}^{\mathrm{T}}G_{3k}f(x(t-h(t))) + \Big(\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big)^{\mathrm{T}}G_{4k}^{\mathrm{T}}G_{4k}\Big(\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big)\Big],
\end{aligned} \tag{23}$$

$$2\Big[x^{\mathrm{T}}(t)M_1 + x^{\mathrm{T}}(t-h(t))M_2 + f^{\mathrm{T}}(x(t))M_3 + f^{\mathrm{T}}(x(t-h(t)))M_4 + \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}sM_5 + y^{\mathrm{T}}(t)M_6\Big]$$

$$\times\Big[-\Delta A_k(t)x(t)+\Delta B_k(t)f(x(t))+\Delta C_k(t)f(x(t-h(t)))+\Delta D_k(t)\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big]$$
$$\le \epsilon_2^{-1}x^{\mathrm{T}}(t)PE_kE_k^{\mathrm{T}}Px(t)+\epsilon_3^{-1}x^{\mathrm{T}}(t-h(t))PE_kE_k^{\mathrm{T}}Px(t-h(t))+\epsilon_4^{-1}f^{\mathrm{T}}(x(t))PE_k$$
$$\times E_k^{\mathrm{T}}Pf(x(t))+\epsilon_5^{-1}f^{\mathrm{T}}(x(t-h(t)))PE_kE_k^{\mathrm{T}}Pf(x(t-h(t)))+\epsilon_6^{-1}\Big(\int_{t-\tau(t)}^{t}f^{\mathrm{T}}(x(s))\mathrm{d}s\Big)^{\mathrm{T}}$$
$$\times PE_kE_k^{\mathrm{T}}P\Big(\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big)+\epsilon_7^{-1}y^{\mathrm{T}}(t)PE_kE_k^{\mathrm{T}}Py(t)+\epsilon\Big[x^{\mathrm{T}}(t)G_{1k}^{\mathrm{T}}G_{1k}x(t)$$
$$+f^{\mathrm{T}}(x(t))G_{2k}^{\mathrm{T}}G_{2k}f(x(t))+f^{\mathrm{T}}(x(t-h(t)))G_{3k}^{\mathrm{T}}G_{3k}f(x(t-h(t)))$$
$$+\int_{t-\tau(t)}^{t}f^{\mathrm{T}}(x(s))\mathrm{d}sG_{4k}^{\mathrm{T}}G_{4k}\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big], \tag{24}$$
$$\Big[\bar{C}_kx(t)+\bar{H}_kx(t-h(t))+\bar{Z}_k\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big]^{\mathrm{T}}(P+h\ U)$$
$$\times\Big[\bar{C}_kx(t)+\bar{H}_kx(t-h(t))+\bar{Z}_k\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big]$$
$$\le\Big[C_kx(t)+H_kx(t-h(t))+Z_k\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big]^{\mathrm{T}}((P+hU)^{-1}-\epsilon_0^{-1}M_kM_k^{\mathrm{T}})^{-1}$$
$$\times\Big[C_kx(t)+H_kx(t-h(t))+Z_k\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big]+\epsilon_0\Big[G_{5k}x(t)+G_{6k}x(t-h(t))$$
$$+G_{7k}\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big]^{\mathrm{T}}\Big[G_{5k}x(t)+G_{6k}x(t-\tau(t))+G_{7k}\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big]. \tag{25}$$

Thus from expressions (22)–(25), by using expressions (3)–(7), (12)–(15) and Schur complement Lemma 1 we have,

$$\mathscr{L}V(t,x(t))\le\sum_{k=1}^{m}\lambda_k(\xi(t))\Big\{\xi^{\mathrm{T}}(t)\ \ \Xi_k\ \ \xi(t)-\int_{t-h(t)}^{t}\mathrm{trace}(g^{\mathrm{T}}(s)\ U\ g(s))\mathrm{d}s$$
$$+\Big(\int_{t-h(t)}^{t}g(s)\mathrm{d}w(s)\Big)^{\mathrm{T}}\ U\ \Big(\int_{t-h(t)}^{t}g(s)\mathrm{d}w(s)\Big)\Big\},$$

where

$$\xi^{\mathrm{T}}(t)=\Big[x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-h(t))\ f^{\mathrm{T}}(x(t))\ f^{\mathrm{T}}(x(t-h(t)))\ \Big(\int_{t-\tau(t)}^{t}f(x(s))\mathrm{d}s\Big)^{\mathrm{T}}\ y^{\mathrm{T}}(t)\Big],$$

$\epsilon=\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7$. Therefore it follows from the facts of $\lambda_{\max}(\Xi_k)<0$ and $|x(t)|\le|\xi(t)|$ that

$$\mathscr{L}V(t,x(t))\le\sum_{k=1}^{m}\eta_k(\theta(t))\Big\{\lambda_{\max}(\Xi_k)|x(t)|^2-\int_{t-h(t)}^{t}\mathrm{trace}(g^{\mathrm{T}}(s)\ U\ g(s))\mathrm{d}s$$
$$+\Big(\int_{t-h(t)}^{t}g(s)\mathrm{d}w(s)\Big)^{\mathrm{T}}\ U\ \Big(\int_{t-h(t)}^{t}g(s)\mathrm{d}w(s)\Big)\Big\}. \tag{26}$$

According to the arguments,[39] we have

$$E\Big\{\Big(\int_{t-h(t)}^{t}g(s)\mathrm{d}w(s)\Big)^{\mathrm{T}}U\Big(\int_{t-h(t)}^{t}g(s)\mathrm{d}w(s)\Big)\Big\}$$
$$=E\Big\{\int_{t-h(t)}^{t}\mathrm{trace}(g^{\mathrm{T}}(s)\ U\ g(s))\mathrm{d}s\Big\}.$$

Taking the mathematical expectation on both sides of inequality (26), we have

$$\mathscr{L}EV(t,x(t))\le\sum_{k=1}^{m}\eta_k(\theta(t))\{\lambda_{\max}(\Xi_k)E|x(t)|^2\}.$$

It can now be concluded from Lyapunov stability theory that the dynamics of neural network (19) is globally, robustly and asymptotically stable in the mean square. This completes the proof of Theorem 2.

In what follows, we will show that our result can be specialized to several cases, including those studied extensively in the literature.

We first consider the following delayed neural network with discrete and distributed time-varying delays[32,33]

$$\dot{x}(t) = -Ax(t) + Bf(x(t)) + Cf(x(t-h(t))) + D\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s, \tag{27}$$

when $r = 1$ with $\sigma = 0$ it is a special form of system (2). For convenience, we deleted the subscript "1".

Theorem 3 Given scalars $\bar{\tau}$, $\bar{h}$, α and d, the equilibrium point of system (27) is globally exponentailly stable if there are symmetric positive-definite matrices $P > 0$, $Q > 0$, $R > 0$, $S > 0$, $T > 0$, $V > 0$, $X > 0$, $Y > 0$, and matrices N_i, M_i $(i = 1, \ldots, 6)$, such that the following conditions hold for,

$$\Gamma_k = \begin{bmatrix} \Gamma_{k11} & \Gamma_{k12} & \Gamma_{k13} & \Gamma_{k14} & \Gamma_{k15} & \Gamma_{k16} & \Gamma_{k17} \\ * & \Gamma_{k22} & \Gamma_{k23} & \Gamma_{k24} & \Gamma_{k25} & \Gamma_{k26} & \Gamma_{k27} \\ * & * & \Gamma_{k33} & \Gamma_{k34} & \Gamma_{k35} & \Gamma_{k36} & \Gamma_{k37} \\ * & * & * & \Gamma_{k44} & \Gamma_{k45} & \Gamma_{k46} & \Gamma_{k47} \\ * & * & * & * & \Gamma_{k55} & \Gamma_{k56} & \Gamma_{k57} \\ * & * & * & * & * & \Gamma_{k66} & \Gamma_{k67} \\ * & * & * & * & * & * & \Gamma_{k77} \end{bmatrix} < 0, \tag{28}$$

where

$$\begin{aligned}
&\Gamma_{k11} = 2\alpha P - PA_k - A_k^{\mathrm{T}}P + Q + N_1 + N_1^{\mathrm{T}} - A_k^{\mathrm{T}}M_1 - M_1A_k + 2L^{\mathrm{T}}XL + 2YL,\\
&\Gamma_{k12} = -A_k^{\mathrm{T}}M_2^{\mathrm{T}} - N_1 + N_2^{\mathrm{T}},\ \ \Gamma_{k13} = PB_k + N_3^{\mathrm{T}} + M_1B_k - A_k^{\mathrm{T}}M_3^{\mathrm{T}} - 2Y,\\
&\Gamma_{k14} = PC_k + N_4^{\mathrm{T}} + M_1C_k - A_k^{\mathrm{T}}M_4^{\mathrm{T}},\ \Gamma_{k15} = PD_k + N_5^{\mathrm{T}} + M_1D_k - A_k^{\mathrm{T}}M_5^{\mathrm{T}},\\
&\Gamma_{k16} = N_6^{\mathrm{T}} - M_1 - A_k^{\mathrm{T}}M_6^{\mathrm{T}},\ \ \Gamma_{k17} = \mathrm{e}^{\alpha h(t)}N_1,\ \ \Gamma_{k22} = -\mathrm{e}^{-2\alpha h(t)}(1-d)Q - N_2 - N_2^{\mathrm{T}} + L^{\mathrm{T}}VL,\\
&\Gamma_{k23} = -N_3^{\mathrm{T}} + M_2B_k,\ \ \Gamma_{k24} = -N_4^{\mathrm{T}} + M_2C_k,\ \ \Gamma_{k25} = -N_5^{\mathrm{T}} + M_2D_k,\\
&\Gamma_{k26} = -N_6^{\mathrm{T}} - M_2,\ \ \Gamma_{k27} = \mathrm{e}^{\alpha h(t)}N_2\ \Gamma_{k33} = R + \tau^2\ S + M_3B_k + B_k^{\mathrm{T}}M_3^{\mathrm{T}} - 2X,\\
&\Gamma_{k34} = M_3C_k + B_k^{\mathrm{T}}M_4^{\mathrm{T}},\ \ \Gamma_{k35} = M_3D_k + B_k^{\mathrm{T}}M_5^{\mathrm{T}},\ \ \Gamma_{k36} = -M_3 + B_k^{\mathrm{T}}M_6^{\mathrm{T}},\ \ \Gamma_{k37} = \mathrm{e}^{\alpha h(t)}N_3,\\
&\Gamma_{k44} = -\mathrm{e}^{-2\alpha h(t)}(1-d)R + M_4C_k + C_k^{\mathrm{T}}M_4^{\mathrm{T}} - 2V,\ \ \Gamma_{k45} = M_4D_k + C_k^{\mathrm{T}}M_5^{\mathrm{T}},\\
&\Gamma_{k46} = -M_4 + C_k^{\mathrm{T}}M_6^{\mathrm{T}},\ \ \Gamma_{k47} = \mathrm{e}^{\alpha h(t)}N_4\ \Gamma_{k55} = M_5D_k + D_k^{\mathrm{T}}M_5^{\mathrm{T}} - \mathrm{e}^{-2\alpha\tau(t)}S,\\
&\Gamma_{k56} = D_k^{\mathrm{T}}M_6^{\mathrm{T}} - M_5,\ \ \Gamma_{k57} = \mathrm{e}^{\alpha h(t)}N_5,\ \ \Gamma_{k66} = h^2T - M_6^{\mathrm{T}} - M_6,\ \ \Gamma_{k67} = \mathrm{e}^{\alpha h(t)}N_6,\ \Gamma_{k77} = -T.
\end{aligned}$$

Proof We use the following Lyapunov functional to derive the stability result,

$$V(t, x(t)) = V_1(t, x(t)) + V_2(t, x(t)) + V_3(t, x(t)) + V_4(t, x(t)) + V_5(t, x(t)),$$

where

$$\begin{aligned}
&V_1(t, x(t)) = \mathrm{e}^{2\alpha t}x^{\mathrm{T}}(t)Px(t),\quad V_2(t, x(t)) = \int_{t-h(t)}^{t} \mathrm{e}^{2\alpha s}x^{\mathrm{T}}(s)Qx(s)\mathrm{d}s,\\
&V_3(t, x(t)) = \int_{t-h(t)}^{t} \mathrm{e}^{2\alpha s}f^{\mathrm{T}}(x(s))Rf(x(s))\mathrm{d}s,\ \ V_4(t, x(t)) = \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} \mathrm{e}^{2\alpha s}f^{\mathrm{T}}(x(s))Sf(x(s))\mathrm{d}s\mathrm{d}\theta,\\
&V_5(t, x(t)) = h\int_{-h}^{0}\int_{t+\theta}^{t} \mathrm{e}^{2\alpha s}\dot{x}^{\mathrm{T}}(s)T\dot{x}(s)\mathrm{d}s\mathrm{d}\theta.
\end{aligned}$$

The time derivative of $V(t, x(t))$ along the trajectories of system (27) can be obtained as,

$$\begin{aligned}
\dot{V}(t, x(t)) \le\ & \mathrm{e}^{2\alpha t}\Big\{2x^{\mathrm{T}}(t)\alpha Px(t) + 2x^{\mathrm{T}}(t)P\Big[-A_kx(t) + B_kf(x(t)) + C_kf(x(t-h(t)))\\
&+ D_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big] + x^{\mathrm{T}}(t)\ Qx(t) - \mathrm{e}^{-2\alpha h(t)}(1-d)x^{\mathrm{T}}(t-h(t))\ Q\ x(t-h(t))\\
&+ f^{\mathrm{T}}(x(t))\ Rf(x(t)) - \mathrm{e}^{-2\alpha h(t)}(1-d)f^{\mathrm{T}}(x(t-h(t)))\ R\ f(x(t-h(t))) + \tau^2 f^{\mathrm{T}}(x(t))
\end{aligned}$$

$$\times S\ f(x(t)) + h^2\dot{x}^{\mathrm{T}}(t)T\dot{x}(t) - \tau\,\mathrm{e}^{-2\alpha\tau(t)}\int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\ S\ f(x(s))\mathrm{d}s$$
$$\times\ -h\,\mathrm{e}^{-2\alpha h(t)}\int_{t-h(t)}^{t}\dot{x}^{\mathrm{T}}(s)\ T\ \dot{x}(s)\mathrm{d}s + 2\Big[x^{\mathrm{T}}(t)N_1 + x^{\mathrm{T}}(t-h(t))N_2 + f^{\mathrm{T}}(x(t))N_3$$
$$+ f^{\mathrm{T}}(x(t-h(t)))N_4 + \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}s\ N_5 + \dot{x}^{\mathrm{T}}(t)N_6\Big]\Big[x(t) - x(t-h(t))$$
$$- \int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\Big] + 2\Big[x^{\mathrm{T}}(t)M_1 + x^{\mathrm{T}}(t-h(t))M_2 + f^{\mathrm{T}}(x(t))M_3 + f^{\mathrm{T}}(x(t-h(t)))M_4$$
$$+ \int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}sM_5 + \dot{x}^{\mathrm{T}}(t)M_6\Big]\Big[- A_kx(t) + B_kf(x(t)) + C_kf(x(t-h(t)))$$
$$+ D_k\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s - \dot{x}(t)\Big] + 2x^{\mathrm{T}}(t)Yf(x(t)) - 2x^{\mathrm{T}}(t)Yf(x(t)) + 2f^{\mathrm{T}}(x(t))Xf(x(t))$$
$$- 2f^{\mathrm{T}}(x(t))Xf(x(t)) + 2x^{\mathrm{T}}(t-h(t))L^{\mathrm{T}}VLx(t-h(t)) - 2f^{\mathrm{T}}(x(t-h(t)))Vf(x(t-h(t)))\Big\}. \quad (29)$$

By using Jensen's inequality[38] we have

$$-\tau\,\mathrm{e}^{-2\alpha\tau(t)}\int_{t-\tau(t)}^{t} f^{\mathrm{T}}(x(s))Sf(x(s))\mathrm{d}s \le -\,\mathrm{e}^{-2\alpha\tau(t)}\Big[\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big]^{\mathrm{T}}S\Big[\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big], \quad (30)$$

$$-h\,\mathrm{e}^{-2\alpha h(t)}\int_{t-h(t)}^{t}\dot{x}^{\mathrm{T}}(s)T\dot{x}(s)\mathrm{d}s \le -\,\mathrm{e}^{-2\alpha h(t)}\Big[\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\Big]^{\mathrm{T}}T\Big[\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\Big]. \quad (31)$$

It follows from Lemma 2 that

$$2\Big[x^{\mathrm{T}}(t)\ N_1 + x^{\mathrm{T}}(t-h(t))\ N_2 + f^{\mathrm{T}}(x(t))\ N_3 + f^{\mathrm{T}}(x(t-h(t)))\ N_4$$
$$+ \int_{t-h(t)}^{t} f^{\mathrm{T}}(x(s))\mathrm{d}s\ N_5 + y^{\mathrm{T}}(t)\ N_6\Big]\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s$$
$$\le\ \mathrm{e}^{2\alpha h(t)}\xi^{\mathrm{T}}(t)\ N\ T^{-1}\ N^{\mathrm{T}}\ \xi(t) + \mathrm{e}^{-2\alpha h(t)}\Big(\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\Big)^{\mathrm{T}}\ T\ \Big(\int_{t-h(t)}^{t}\dot{x}(s)\mathrm{d}s\Big), \quad (32)$$

where

$$\xi^{\mathrm{T}}(t) = \Big[x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-h(t))\ f^{\mathrm{T}}(x(t))\ f^{\mathrm{T}}(x(t-h(t)))\ \Big(\int_{t-\tau(t)}^{t} f(x(s))\mathrm{d}s\Big)^{\mathrm{T}}\ \dot{x}^{\mathrm{T}}(t)\Big],$$
$$N^{\mathrm{T}} = [N_1^{\mathrm{T}}\ N_2^{\mathrm{T}}\ N_3^{\mathrm{T}}\ N_4^{\mathrm{T}}\ N_5^{\mathrm{T}}\ N_6^{\mathrm{T}}].$$

From expressions (3)–(7), (12)–(15) and by using Schur complement Lemma 1 we have

$$\dot{V}(t,x(t))\ \le\ \mathrm{e}^{2\alpha t}\Big\{\xi^{\mathrm{T}}(t)\ \Gamma_k\ \xi(t)\Big\}.$$

Thus, if the LMI (28) holds system (27) is globally, exponentially stable. This completes the proof.

Remark 1 Especially, when $B = 0$, $D = 0$, $K_3 = 0$, the system discussed in Theorem 1 reduces the one proposed in Ref. [24]. In our paper, the discrete, distributed delays and uncertainty structure are considered as additional features to those in Ref. [24]. Hence our results provide the more general case of the existing results.

Remark 2 In this paper, by constructing a new Lyapunov–Krasovskii functional and using LMI method, sufficient stability conditions for delayed stochastic fuzzy recurrent neural networks to guarantee the stability of a trivial solution are derived. In order to derive the stability conditions, free-weighting matrices method has been introduced in this paper, which may develop less conservative results. In addition, the approaches discussed in this paper can also be applied to stability analysis for other stochastic fuzzy neural networks including stochastic Hopfield neural networks and stochastic BAM neural networks, etc.

5. Numerical example

Example 1 Consider the TSSFRNNs (2) with $m = 1$ and the following parameters

$$A_1 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.1 & 0.9 \\ 0.8 & 1.2 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.5 & -0.5 \\ 0.2 & 0.7 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2.4 & 0.2 \\ 0 & 2.4 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.2 & 0.5 \\ 0.4 & 0.3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0.07 & 0 \\ 0 & 0.07 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.09 \end{bmatrix}$$

with $f(x) = \tanh(x)$, we select $\bar{h} = 2.65$, $\bar{\tau} = 0.5$, and $d = 1.5$. The membership functions for Rule 1 and Rule 2 are $\eta^1 = 1/\,\mathrm{e}^{-2u_1(t)}$, $\eta^2 = 1 - \eta^1$, by using the Matlab LMI toolbox, we solve the LMIs (8)–(10), the feasible solutions are

$$P = \begin{bmatrix} 20.9602 & 0.1442 \\ 0.1442 & 20.8135 \end{bmatrix},$$
$$Q = \begin{bmatrix} 4.1584 & 0.1578 \\ 0.1578 & 4.8890 \end{bmatrix},$$
$$R = \begin{bmatrix} 8.6792 & -0.0913 \\ -0.0913 & 9.4299 \end{bmatrix},$$
$$S = \begin{bmatrix} 20.6951 & 0.0939 \\ 0.0939 & 21.4944 \end{bmatrix},$$
$$T = \begin{bmatrix} 1.3971 & -0.0333 \\ -0.0333 & 1.3339 \end{bmatrix},$$
$$U = \begin{bmatrix} 14.1815 & 0.0001 \\ 0.0001 & 14.1827 \end{bmatrix},$$
$$V = \begin{bmatrix} 9.9985 & 1.2196 \\ 1.2196 & 12.5484 \end{bmatrix},$$
$$X = \begin{bmatrix} 29.0179 & 1.1341 \\ 1.1341 & 25.5488 \end{bmatrix},$$
$$Y = \begin{bmatrix} 11.0381 & 1.5663 \\ 1.5663 & 8.5599 \end{bmatrix}.$$

Therefore, the concerned neural network with time-varying delays is globally asymptotically stable, as shown in Figs. 1 and 2.

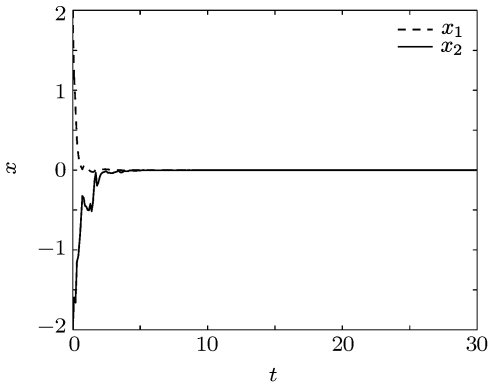


Fig. 1. State variations of x_1 and x_2 in Example 1 when $m = 1$.

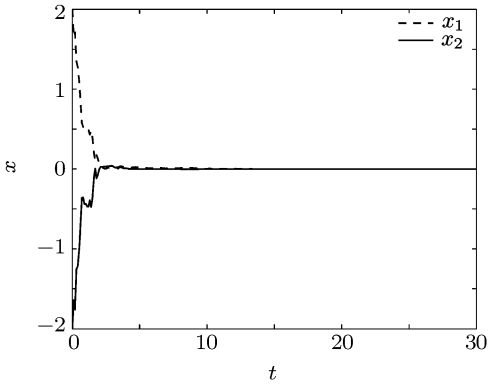


Fig. 2. State variations of x_1 and x_2 in Example 1 when $m = 2$.

Example 2 Consider the TSUSFRNNs (19) with $m = 2$ and the following parameters

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0.2 \\ 0 & -0.2 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} -0.3 & -0.2 \\ 0 & 0.1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 & -4 \\ 0.1 & 0.3 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.7 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$H_1 = \begin{bmatrix} 0 & 0.2 \\ 0 & -0.2 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -0.3 & -0.2 \\ 0 & 0.1 \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0.5 & -0.1 \\ -0.5 & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad G_i = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad i = 1, 2, \ldots, 7$$

with $f(x) = \tanh(x)$, we select $\bar{h} = 1.5$, $\bar{\tau} = 0.5$, and $d = 1.5$. The membership functions for Rule 1 and Rule 2 are $\eta^1 = 1/\mathrm{e}^{-2u_1(t)}$, $\eta^2 = 1 - \eta^1$, by using the Matlab LMI toolbox, we solve the LMIs (20)–(21), the feasible solutions are

$$P = \begin{bmatrix} 73.2403 & -4.4483 \\ -4.4483 & 107.9676 \end{bmatrix},$$

$$Q = \begin{bmatrix} 2.8037 & 0.4363 \\ 0.4363 & 7.6283 \end{bmatrix}, \quad R = \begin{bmatrix} 9.6781 & 1.9430 \\ 1.9430 & 31.8615 \end{bmatrix},$$

$$S = \begin{bmatrix} 637.6186 & -80.8633 \\ -80.8633 & 764.8084 \end{bmatrix},$$

$$T = \begin{bmatrix} 2.1334 & 0.2098 \\ 0.2098 & 1.4486 \end{bmatrix}, \quad U = \begin{bmatrix} 7.1377 & 8.7694 \\ 8.7694 & 52.8938 \end{bmatrix},$$

$$V = \begin{bmatrix} 28.9804 & 6.5576 \\ 6.5576 & 60.2304 \end{bmatrix},$$

$$X = \begin{bmatrix} 119.4362 & -10.1179 \\ -10.1179 & 415.3133 \end{bmatrix},$$

$$Y = \begin{bmatrix} 4.8796 & -0.6616 \\ -0.6616 & 23.7834 \end{bmatrix}.$$

Therefore, the concerned neural network with time-varying delays is globally, robustly and asymptotically stable, as shown in Figs. 3 and 4.

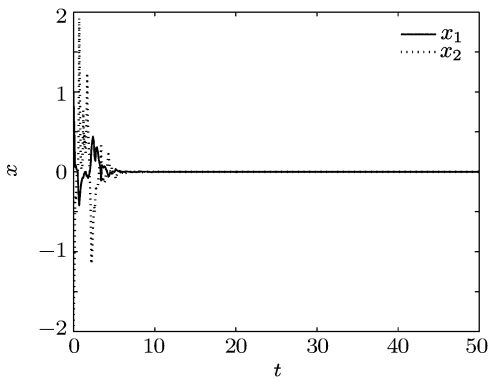


Fig. 3. State variations of x_1 and x_2 in Example 2 when $m = 1$.

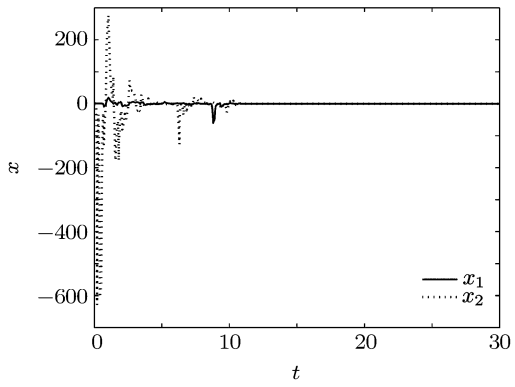


Fig. 4. State variations of x_1 and x_2 in Example 2 when $m = 2$.

Example 3 Consider a cellular neural network with discrete and distributed time-varying delays[32,33]

$$A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1.2 & -0.8 & 0.6 \\ 0.5 & -0.5 & 0.7 \\ -0.8 & -1.2 & -1.4 \end{bmatrix},$$

$$C = \begin{bmatrix} -1.4 & 0.9 & 0.5 \\ -0.6 & 1.2 & 0.8 \\ 0.5 & -1.2 & -1.4 \end{bmatrix},$$

$$D = \begin{bmatrix} 1.8 & 0.7 & -0.8 \\ 0.6 & 1.4 & -0.8 \\ -0.4 & -0.6 & 1.2 \end{bmatrix}, \quad L = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.4 & 0 \\ 0 & 0 & 2.4 \end{bmatrix},$$

we select $\bar{\tau} = 0.2$, $\bar{h} = 0.5$, $\alpha = 0.649$, and $d = 1.5$. By using the Matlab LMI toolbox, we solve the LMI (28), the feasible solutions are

$$P = \begin{bmatrix} 11.7012 & -4.8564 & 3.9604 \\ -4.8564 & 8.3984 & 3.2064 \\ 3.9604 & 3.2064 & 7.8746 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.0440 & -0.0218 & 0.0105 \\ -0.0218 & 0.0124 & -0.0038 \\ 0.0105 & -0.0038 & 0.0043 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.0307 & -0.0130 & 0.0037 \\ -0.0130 & 0.0063 & -0.0011 \\ 0.0037 & -0.0011 & 0.0008 \end{bmatrix},$$

$$S = \begin{bmatrix} 188.7019 & -54.4810 & 28.1938 \\ -54.4810 & 51.1291 & -1.7506 \\ 28.1938 & -1.7506 & 18.4845 \end{bmatrix},$$

$$T = \begin{bmatrix} 0.0335 & -0.0178 & 0.0065 \\ -0.0178 & 0.0105 & -0.0026 \\ 0.0065 & -0.0026 & 0.0025 \end{bmatrix},$$

$$V = \begin{bmatrix} 4.5004 & -2.2598 & 0.2100 \\ -2.2598 & 2.0694 & 0.6002 \\ 0.2100 & 0.6002 & 0.8558 \end{bmatrix},$$

$$X = \begin{bmatrix} 4.2236 & -0.3702 & 1.0487 \\ -0.3702 & 4.9685 & 2.0109 \\ 1.0487 & 2.0109 & 1.9124 \end{bmatrix},$$

$$Y = \begin{bmatrix} 11.4269 & -3.7649 & 3.1214 \\ -3.7649 & 1.2407 & -1.0282 \\ 3.1214 & -1.0282 & 0.8528 \end{bmatrix}.$$

Therefore, the concerned neural network with time-varying delays is globally asymptotically stable. Some comparisons of the upper bounds of time delay guaranteeing the global exponential stability of these system are made in Table 1.

Table 1. The achieved maximum convergence rate k corresponding to different values of $\bar{h}$ and d.

$\bar{h}$, $\bar{\tau}$, d	Ref. [32]	Ref. [33]	Theorem 3
$\bar{h} = 0.6$, $\bar{\tau} = 0.2$, $d = 0.5$	0.44	0.56	0.60
$\bar{h} = 0.8$, $\bar{\tau} = 0.2$, $d = 0.5$	0.22	0.43	0.54
$\bar{h} = 1.0$, $\bar{\tau} = 0.2$, $d = 0.5$	0.11	0.34	0.49
$\bar{h} = 0.5$, $\bar{\tau} = 0.2$, $d = 1.0$	Fail	Fail	0.63
$\bar{h} = 0.5$, $\bar{\tau} = 0.2$, $d = 1.5$	Fail	Fail	0.63

6. Conclusion

A new sufficient condition is derived to guarantee the stability of the equilibrium point for uncertain stochastic fuzzy recurrent neural networks with discrete and distributed time-varying delays. A linear matrix inequality approach has been developed to solve the problem. It also generalizes many existing results in the literature. Our results can be easily verified and are also less conservative than in previously known criteria.

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