Time evolution of distribution functions in dissipative environments*

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By introducing the thermal entangled state representation, we investigate the time evolution of distribution functions in the dissipative channels by bridging the relation between the initial distribution function and the any time distribution function. We find that most of them are expressed as such integrations over the Laguerre–Gaussian function. Furthermore, as applications, we derive the time evolution of photon-counting distribution by bridging the relation between the initial distribution function and the any time photon-counting distribution, and the time evolution of R-function characteristic of nonclassicality depth.

Keywords: entangled state representation, dissipative environment, photon-counting distribution, R-function

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1. Introduction

Decoherence is an important topic in the fields of quantum optics, quantum computing and quantum information processing. It usually happens when a system is immersed in a thermal environment, or a signal (a quantum state) through a quantum channel, and is described by a master equation (ME) which is set up for a better understanding of how quantum information is processed.

Usually, an ME is solved by transforming it into a partial differential equation for either the characteristic function or a quasi-probability distribution (Fokker–Planck equation or Langevin equation), which is not suitable for any initial density operator. For different quantum optical systems, Moya-Cessa solved the above problem by using superoperator techniques,[1] which bring the ambiguity as regarding the explicit operator-form. By introducing a fictitious mode as a counterpart mode of the system mode under review,[2−5] we construct the thermal entangled state representation (TESR) $|\eta\rangle$, which can arrange MEs of density operators in quantum statistics as state-vector evolution equations due to the elegant properties of $|\eta\rangle$. In our approach, therefore, many MEs can be concisely solved. Thus the TESR is beneficial to quantum decoherence theory.

On the other hand, phase-space representations are of considerable theoretical interest because they constitute a framework within which the formal differences between quantum and classical mechanics are minimized and they provide an important framework within which the transition from quantum to classical physics can be understood.[6,7] In addition, distribution functions (such as Wigner-, Q- and P-functions) in phase space have been a major topic in studying quantum mechanics and quantum statistics. For example, the P-function is used to define the R-function which is continuous interpolation between Q and P functions, and can be used as a measure of nonclassicality of quantum states.[8] To the best of our knowledge, however, the time evolution of distribution functions (DFs) in dissipative environments, especially for the time evolution of photon-counting distribution and R-function, has not been derived analytically in literature up to now.

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In this paper, we will investigate the time evolution of DFs, including that of photon-counting distribution and R-function, in the dissipative channels by using the TESR. Our paper is arranged as follows. In Section 2, we briefly review the characteristic function (CF) formula in the TESR. In Sections 3 and 4, we derive the evolution formulas of Wigner-, Q- and P-functions for amplitude damping channel and thermal channel, respectively, i.e., bridge the relation between the initial DF and the any time DF. It is shown that most of them are expressed as such integrations over Laguerre–Gaussian function. Section 5 is devoted to applications of these evolution formulas, involving the time evolution of photon-counting distribution and R-function. Conclusions are drawn in the last section.

2. Characteristic function formula in thermal entangled state representation

Recall that the TESR is constructed in the doubled Fock space[9,10] based on Umezawa–Takahash thermo field dynamics (TFD),[11−13] i.e.,

$$\begin{aligned}|\eta\rangle &= \exp\left(-\frac{1}{2}|\eta|^2+\eta a^\dagger-\eta^*\tilde{a}^\dagger+a^\dagger\tilde{a}^\dagger\right)\left|0,\tilde{0}\right\rangle \\ &= D(\eta)\,|\eta=0\rangle\,, \end{aligned} \tag{1}$$

where $D(\eta)=\mathrm{e}^{\eta a^\dagger-\eta^* a}$ is the displacement operator, and $\tilde{a}^\dagger$ is a fictitious mode accompanying the real photon creation operator $a^\dagger$, $\left|0,\tilde{0}\right\rangle=|0\rangle\left|\tilde{0}\right\rangle$, and $\left|\tilde{0}\right\rangle$ is annihilated by $\tilde{a}$ with the relations $\left[\tilde{a},\tilde{a}^\dagger\right]=1$ and $\left[a,\tilde{a}^\dagger\right]=0$. It is easily seen that $|\eta=0\rangle$ has the properties of

$$|I\rangle\equiv|\eta=0\rangle=\mathrm{e}^{a^\dagger\tilde{a}^\dagger}\left|0,\tilde{0}\right\rangle=\sum_{n=0}^{\infty}|n,\tilde{n}\rangle\,, \tag{2}$$

where $\tilde{n}=n$, and $\tilde{n}$ denotes the number in the fictitious Hilbert space, and

$$\begin{cases} a\,|\eta=0\rangle=\tilde{a}^\dagger\,|\eta=0\rangle\,, \\ a^\dagger\,|\eta=0\rangle=\tilde{a}\,|\eta=0\rangle\,, \\ \left(a^\dagger a\right)^n|\eta=0\rangle=\left(\tilde{a}^\dagger\tilde{a}\right)^n|\eta=0\rangle\,. \end{cases} \tag{3}$$

Note that density operators $\rho(a^\dagger,a)$ are defined in the real space which are commutative with operators $(\tilde{a}^\dagger,\tilde{a})$ in the tilde space.

Next we first bridge the relation between the CF and the entangled state representation $\langle\eta|$. According to the TFD theory, the CF of density operator ρ, $\chi_{\mathrm{S}}(\lambda,\lambda^*)=\mathtt{tr}(\rho\,\mathrm{e}^{\lambda a^\dagger-\lambda^* a})$, can be calculated as

$$\begin{aligned}\chi_{\mathrm{S}}(\lambda) &= \sum_{m,n}^{\infty}\langle n,\tilde{n}|\,\mathrm{e}^{\lambda a^\dagger-\lambda^* a}\rho\,|m,\tilde{m}\rangle \\ &= \langle\eta=0|\,D(\lambda)\,|\rho\rangle \\ &= \langle\eta=-\lambda|\,\rho\rangle\,, \end{aligned} \tag{4}$$

or

$$\begin{aligned}\chi_{\mathrm{S}}(\lambda) &= \sum_{m,n}^{\infty}\langle n,\tilde{n}|\,\rho\,\mathrm{e}^{\lambda a^\dagger-\lambda^* a}\,|m,\tilde{m}\rangle \\ &= \langle\rho|\,\eta=\lambda\rangle\,, \end{aligned} \tag{5}$$

where $|\rho\rangle\equiv\rho\,|I\rangle$. Equation (4) or (5) is the CF formula in TESR, with which the CF of density operator is simplified as an overlap between two pure states in enlarged Fock space, rather than using ensemble average in the system-mode space. Thus we can then simplify the calculation of $\chi_{\mathrm{S}}(\lambda)$ by virtue of some important properties of the entangled state representation $\langle\eta|$. For example, for the coherent state $\langle\alpha|$, $|\rho\rangle\equiv|\alpha\rangle\langle\alpha|\,I\rangle=|\alpha,\tilde{\alpha}^*\rangle=D(\alpha)\tilde{D}(\alpha^*)\left|0,\tilde{0}\right\rangle$, thus the CF of $\langle\alpha|$ is given by $\chi_{\mathrm{S}}(\lambda)=\langle\eta=-\lambda|\,\alpha,\tilde{\alpha}^*\rangle=\exp(-|\lambda|^2/2+\lambda\alpha^*-\lambda^*\alpha)$. In addition, further using the (inverse-) Fourier transformations relation between the CF and the quasi-probability functions (QPFs), such as W-, Q- and P-functions, one can obtain these QPFs.

3. Evolution formulas of Wigner-, Q- and P-functions for amplitude damping channel

In this section, we consider the time evolution of Wigner-, Q- and P-functions for amplitude damping channel, (dissipation in a lossy cavity) described by the following ME:[14]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=\kappa\left(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a\right), \tag{6}$$

where κ is the rate of decay. Operating Eq. (6) on the vector $|I\rangle$ and using Eq. (3), we can reform Eq. (6) as[5]

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho\rangle=\kappa\left(2a\tilde{a}-a^\dagger a-\tilde{a}^\dagger\tilde{a}\right)|\rho\rangle\,, \tag{7}$$

thus the formal solution of Eq. (7) is

$$\begin{aligned}&|\rho(t)\rangle \\ &= \mathrm{e}^{\kappa t\left(a\tilde{a}-\tilde{a}^\dagger a^\dagger+1\right)}\,\mathrm{e}^{\left(1-\mathrm{e}^{2\kappa t}\right)\left(a^\dagger-\tilde{a}\right)\left(a-\tilde{a}^\dagger\right)/2}\,|\rho_0\rangle\,, \end{aligned} \tag{8}$$

where we have used the operator identity[15]

$$\mathrm{e}^{\lambda(A+\sigma B)} = \mathrm{e}^{\lambda A} \exp\left[\sigma\left(1 - \mathrm{e}^{-\lambda\tau}\right) B/\tau\right], \quad (9)$$

which is valid for $[A, B] = \tau B$. Then projecting Eq. (8) on $\langle\eta|$, and noticing $\exp\left[\kappa t\left(a\tilde{a} - \tilde{a}^{\dagger}a^{\dagger}\right)\right]$ as the two-mode squeezing operator,[16]

$$\langle\eta| \exp\left[\kappa t\left(a\tilde{a} - \tilde{a}^{\dagger}a^{\dagger}\right)\right] = \mathrm{e}^{-\kappa t}\left\langle\eta\,\mathrm{e}^{-\kappa t}\right|, \quad (10)$$

as well as the eigen-equations of $|\eta\rangle$, i.e.,

$$\begin{aligned} &(a - \tilde{a}^{\dagger})\,|\eta\rangle = \eta\,|\eta\rangle\,,\ (a^{\dagger} - \tilde{a})\,|\eta\rangle = \eta^{*}\,|\eta\rangle\,, \\ &\langle\eta|\,(a^{\dagger} - \tilde{a}) = \eta^{*}\,\langle\eta|\,,\ \langle\eta|\,(a - \tilde{a}^{\dagger}) = \eta\,\langle\eta|\,, \end{aligned} \quad (11)$$

we can obtain

$$\begin{aligned} &\langle\eta\,|\rho\,(t)\rangle = \mathrm{e}^{-(1/2)T|\eta|^{2}}\left\langle\eta\,\mathrm{e}^{-\kappa t}\right|\rho_{0}\Big\rangle, \\ &T = 1 - \mathrm{e}^{-2\kappa t}. \end{aligned} \quad (12)$$

Note the relation between the CF and the TESR in Eq. (4), we can recast Eq. (12) into

$$\chi_{\mathrm{S}}\,(\lambda, t) = \mathrm{e}^{-(1/2)T|\lambda|^{2}}\chi_{\mathrm{S}}\left(\lambda\,\mathrm{e}^{-\kappa t}, 0\right), \quad (13)$$

which is just the relation between the any time CF and the initial CF in amplitude damping channel.

3.1. The time evolution of Wigner function (WF)

Using the Fourier transformations of CF $\chi_{\mathrm{S}}\,(\lambda)$ and the WF, i.e.,[17]

$$W\,(\alpha) = \int \frac{\mathrm{d}^{2}\lambda}{\pi^{2}}\,\mathrm{e}^{\lambda^{*}\alpha - \lambda\alpha^{*}}\,\chi_{\mathrm{S}}\,(\lambda)\,, \quad (14)$$

so multiplying the item $\int(\mathrm{d}^{2}\lambda/\pi^{2})\,\mathrm{e}^{\lambda^{*}\alpha - \lambda\alpha^{*}}$ to the left side of Eq. (13), we have

$$\begin{aligned} &W\,(\alpha, t) \\ &= \int \frac{\mathrm{d}^{2}\lambda}{\pi^{2}}\,\mathrm{e}^{\lambda^{*}\alpha - \lambda\alpha^{*}}\,\mathrm{e}^{-(1/2)T|\lambda|^{2}}\chi_{\mathrm{S}}\left(\lambda\,\mathrm{e}^{-\kappa t}, 0\right) \\ &= \mathrm{e}^{2\kappa t}\int \frac{\mathrm{d}^{2}\lambda}{\pi^{2}}\,\mathrm{e}^{-(1/2)T\,\mathrm{e}^{2\kappa t}|\lambda|^{2} + (\lambda^{*}\alpha - \lambda\alpha^{*})\,\mathrm{e}^{\kappa t}} \\ &\quad \times \chi_{\mathrm{S}}\,(\lambda, 0)\,. \end{aligned} \quad (15)$$

Thus we can obtain the evolution of WF by Eq. (15) once the initial CFs are known. For example, we still consider the coherent state $|\beta\rangle$, whose CF is shown above, after substituting it into Eq. (15), we have

$$\begin{aligned} W\,(\alpha, t) &= \mathrm{e}^{2\kappa t}\int \frac{\mathrm{d}^{2}\lambda}{\pi^{2}}\exp\Big[-\frac{1}{2}\,\mathrm{e}^{2\kappa t}\,|\lambda|^{2} \\ &\quad + \lambda\left(\beta^{*} - \mathrm{e}^{\kappa t}\alpha^{*}\right) + \lambda^{*}\left(\mathrm{e}^{\kappa t}\alpha - \beta\right)\Big] \\ &= \frac{2}{\pi}\exp\left[-2\left|\alpha - \beta\,\mathrm{e}^{-\kappa t}\right|^{2}\right], \end{aligned} \quad (16)$$

where we have used the following formula[18]

$$\int \frac{\mathrm{d}^{2}z}{\pi}\,\mathrm{e}^{\zeta|z|^{2} + \xi z + \eta z^{*}} = -\frac{1}{\zeta}\,\mathrm{e}^{-(\xi\eta/\zeta)}, \quad \mathrm{Re}\,\zeta < 0. \quad (17)$$

Equation (16) indicates that the coherent state $|\beta\rangle$ evolves to another coherent pure state $|\beta\,\mathrm{e}^{-\kappa t}\rangle$ through the amplitude damping channel, which is a known result.[1,3]

Furthermore, employing the inverse Fourier transformations,

$$\chi_{\mathrm{S}}\,(\lambda) = \int \mathrm{e}^{\lambda\alpha^{*} - \lambda^{*}\alpha}W\,(\alpha)\,\mathrm{d}^{2}\alpha, \quad (18)$$

for $\chi_{\mathrm{S}}\,(\lambda, 0)$ in Eq. (15), we can rewrite Eq. (15) as

$$\begin{aligned} &W\,(\alpha, t) \\ &= \mathrm{e}^{2\kappa t}\int \frac{\mathrm{d}^{2}\lambda}{\pi^{2}}\,\mathrm{e}^{-(1/2)T\,\mathrm{e}^{2\kappa t}|\lambda|^{2} + (\lambda^{*}\alpha - \lambda\alpha^{*})\,\mathrm{e}^{\kappa t}} \\ &\quad \times \int \mathrm{e}^{\lambda\beta^{*} - \lambda^{*}\beta}W\,(\beta, 0)\,\mathrm{d}^{2}\beta \\ &= \frac{2}{T}\int \frac{\mathrm{d}^{2}\beta}{\pi}\,\mathrm{e}^{-(2/T)\left|\alpha - \beta\,\mathrm{e}^{-\kappa t}\right|^{2}}W\,(\beta, 0)\,, \end{aligned} \quad (19)$$

where we have interchanged the order of integrations and used Eq. (17). Equation (19) is another new expression to obtain the evolution of WF when the initial WF is known, which is also obtained by using another method.[19]

3.2. The time evolutions of Q- and P-functions

In a similar way to derive Eqs. (15) and (19), multiplying $\int(\mathrm{d}^{2}\lambda/\pi^{2})\,\mathrm{e}^{\alpha\lambda^{*} - \alpha^{*}\lambda}\,\mathrm{e}^{-|\lambda|^{2}/2}$, $\int(\mathrm{d}^{2}\lambda/\pi^{2})\,\mathrm{e}^{\alpha\lambda^{*} - \alpha^{*}\lambda}\,\mathrm{e}^{|\lambda|^{2}/2}$ to the left side of Eq. (13), respectively, and noticing

$$Q\,(\alpha) = \int \frac{\mathrm{d}^{2}\lambda}{\pi^{2}}\,\mathrm{e}^{\alpha\lambda^{*} - \alpha^{*}\lambda}\chi_{\mathrm{A}}\,(\lambda)\,, \quad (20)$$

$$P\,(\alpha) = \int \frac{\mathrm{d}^{2}\lambda}{\pi^{2}}\,\mathrm{e}^{\alpha\lambda^{*} - \alpha^{*}\lambda}\chi_{\mathrm{N}}\,(\lambda)\,, \quad (21)$$

and the corresponding inverse transforms with

$$\chi_{\mathrm{S}}\,(\lambda) = \chi_{\mathrm{A}}\,(\lambda)\,\mathrm{e}^{|\lambda|^{2}/2} = \chi_{\mathrm{N}}\,(\lambda)\,\mathrm{e}^{-|\lambda|^{2}/2}, \quad (22)$$

we have

$$\begin{aligned} &Q\,(\alpha, t) \\ &= \int \frac{\mathrm{d}^{2}\lambda}{\pi^{2}}\,\mathrm{e}^{\alpha\lambda^{*} - \alpha^{*}\lambda}\,\mathrm{e}^{-(1/2)(2 - \mathrm{e}^{-2\kappa t})|\lambda|^{2}}\chi_{\mathrm{S}}\left(\lambda\,\mathrm{e}^{-\kappa t}, 0\right) \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{2\kappa t}\int\frac{\mathrm{d}^2\lambda}{\pi^2}\,\mathrm{e}^{-(\mathrm{e}^{2\kappa t}-1)|\lambda|^2+\alpha\,\mathrm{e}^{\kappa t}\lambda^*-\alpha^*\lambda\,\mathrm{e}^{\kappa t}}\chi_{\mathrm{A}}(\lambda,0)\\
&= \mathrm{e}^{2\kappa t}\int\frac{\mathrm{d}^2\lambda}{\pi^2}\,\mathrm{e}^{-(\mathrm{e}^{2\kappa t}-1)|\lambda|^2+\alpha\,\mathrm{e}^{\kappa t}\lambda^*-\alpha^*\lambda\,\mathrm{e}^{\kappa t}}\\
&\quad\times\int\mathrm{e}^{\lambda\beta^*-\lambda^*\beta}Q(\beta,0)\,\mathrm{d}^2\beta\\
&= \int\frac{\mathrm{d}^2\beta}{\pi T}\exp\left(-\frac{1}{T}\left|\alpha-\beta\,\mathrm{e}^{-\kappa t}\right|^2\right)Q(\beta,0)\,, \qquad (23)
\end{aligned}
$$

and

$$
\begin{aligned}
&P(\alpha,t)\\
&= \int\frac{\mathrm{d}^2\lambda}{\pi^2}\,\mathrm{e}^{\alpha\lambda^*-\alpha^*\lambda}\,\mathrm{e}^{-(1/2)(T-1)|\lambda|^2}\chi_{\mathrm{S}}\left(\lambda\,\mathrm{e}^{-\kappa t},0\right)\\
&= \mathrm{e}^{2\kappa t}\int\frac{\mathrm{d}^2\lambda}{\pi^2}\,\mathrm{e}^{\alpha\,\mathrm{e}^{\kappa t}\lambda^*-\alpha^*\lambda\,\mathrm{e}^{\kappa t}}\chi_{\mathrm{N}}(\lambda,0)\\
&= \mathrm{e}^{2\kappa t}\int\frac{\mathrm{d}^2\lambda}{\pi^2}\,\mathrm{e}^{\alpha\,\mathrm{e}^{\kappa t}\lambda^*-\alpha^*\lambda\,\mathrm{e}^{\kappa t}}\\
&\quad\times\int\mathrm{e}^{\lambda\beta^*-\lambda^*\beta}P(\beta,0)\,\mathrm{d}^2\beta\\
&= \mathrm{e}^{2\kappa t}\int\mathrm{d}^2\beta P(\beta,0)\\
&\quad\times\int\frac{\mathrm{d}^2\lambda}{\pi^2}\,\mathrm{e}^{\left(\alpha\,\mathrm{e}^{\kappa t}-\beta\right)\lambda^*-\left(\alpha^*\,\mathrm{e}^{\kappa t}-\beta^*\right)\lambda}\\
&= \mathrm{e}^{2\kappa t}P\left(\alpha\,\mathrm{e}^{\kappa t},0\right), \qquad (24)
\end{aligned}
$$

where in the last step, we have used the formula

$$
\int\frac{\mathrm{d}^2z}{\pi^2}\exp\left(\alpha z^*-\alpha'^*z\right)=\delta(\alpha)\,\delta\left(\alpha'^*\right). \qquad (25)
$$

Equations (23) and (24) are the time evolution of Q- and P-functions, respectively.

For the coherent state $|\beta\rangle$, as an example, the corresponding Q- and P-functions are $Q(\alpha,0)=(1/\pi)\exp(-|\alpha-\beta|^2)$ and $P(\alpha,0)=\delta(\beta-\alpha)\,\delta(\beta^*-\alpha^*)$, respectively, then

$$
\begin{aligned}
Q_\beta(\alpha,t) &= Q_{\beta\,\mathrm{e}^{-\kappa t}}(\alpha,0)\,, \qquad (26)\\
P_\beta(\alpha,t) &= \delta\left(\beta\,\mathrm{e}^{-\kappa t}-\alpha\right)\delta\left(\beta^*\,\mathrm{e}^{-\kappa t}-\alpha^*\right)\\
&= P_{\beta\,\mathrm{e}^{-\kappa t}}(\alpha,0)\,, \qquad (27)
\end{aligned}
$$

which are in agreement with the result of Eq. (16), i.e., the output state is still a pure coherent state with the amplitude $\beta\,\mathrm{e}^{-\kappa t}$.

4. Evolution formulas for thermal channel

When a quantum state evolves in the thermal channel, the evolution of the density matrix can be described by ME,[14]

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \kappa(\bar{n}+1)\left(2a\rho a^\dagger-a^\dagger a\rho-\rho a^\dagger a\right)\\
&\quad+\kappa\bar{n}\left(2a^\dagger\rho a-aa^\dagger\rho-\rho aa^\dagger\right), \qquad (28)
\end{aligned}
$$

where κ represents the dissipative coefficient and $\bar{n}$ denotes the average thermal photon number of the environment. When $\bar{n}=0$, Eq. (28) reduces to the ME describing the photon-loss channel (amplitude damping).

Similarly to Eq. (7), we can obtain[2,15]

$$
\begin{aligned}
&|\rho(t)\rangle\\
&= \exp\left[\left(a\tilde{a}-\tilde{a}^\dagger a^\dagger+1\right)\kappa t\right]\\
&\quad\times\exp\left[\frac{(2\bar{n}+1)\left(1-\mathrm{e}^{2\kappa t}\right)}{2}\left(a^\dagger-\tilde{a}\right)\left(a-\tilde{a}^\dagger\right)\right]\\
&\quad\times|\rho_0\rangle\,. \qquad (29)
\end{aligned}
$$

Projecting it on $\langle\eta|$ and using Eq. (11), we have

$$
\begin{aligned}
\langle\eta|\,\rho(t)\rangle &= \exp\left(-\frac{A}{2}|\eta|^2\right)\langle\eta\,\mathrm{e}^{-\kappa t}|\,\rho_0\rangle\,, \qquad (30)\\
&\quad(A=(2\bar{n}+1)\,T),
\end{aligned}
$$

or

$$
\chi_{\mathrm{S}}(\lambda,t)=\mathrm{e}^{-(1/2)A|\lambda|^2}\chi_{\mathrm{S}}\left(\lambda\,\mathrm{e}^{-\kappa t},0\right). \qquad (31)
$$

In comparison with Eq. (13), we can directly obtain the evolution formulas about Wigner- and Q-functions,

$$
W(\alpha,t)=\frac{2}{A}\int\frac{\mathrm{d}^2\beta}{\pi}\,\mathrm{e}^{-(2/A)\left|\alpha-\beta\,\mathrm{e}^{-\kappa t}\right|^2}W(\beta,0)\,, \qquad (32)
$$

$$
\begin{aligned}
Q(\alpha,t) &= \frac{1}{(\bar{n}+1)\,T}\int\frac{\mathrm{d}^2\beta}{\pi}Q(\beta,0)\\
&\quad\times\mathrm{e}^{-(|\alpha-\beta\,\mathrm{e}^{-\kappa t}|^2/(\bar{n}+1)T)}, \qquad (33)
\end{aligned}
$$

and using a similar way to derive Eq. (24), we also have

$$
\begin{aligned}
&P(\alpha,t)\\
&= \frac{1}{\bar{n}T}\int\frac{\mathrm{d}^2\beta}{\pi}\,\mathrm{e}^{-(1/\bar{n}T)\left|\alpha-\beta\,\mathrm{e}^{-\kappa t}\right|^2}P(\beta,0)\,. \qquad (34)
\end{aligned}
$$

For the coherent state $|\beta\rangle$, the W-, Q-, P-functions of the output state are

$$
\begin{aligned}
W(\alpha,t) &= \frac{2/\pi}{2\bar{n}T+1}\,\mathrm{e}^{-2|\alpha-\beta\,\mathrm{e}^{-\kappa t}|^2/(2\bar{n}T+1)},\\
Q(\alpha,t) &= \frac{1/\pi}{\bar{n}T+1}\,\mathrm{e}^{-|\alpha-\beta\,\mathrm{e}^{-\kappa t}|^2/(\bar{n}T+1)},\\
P(\alpha,t) &= \frac{1/\pi}{\bar{n}T}\,\mathrm{e}^{-|\alpha-\beta\,\mathrm{e}^{-\kappa t}|^2/\bar{n}T},
\end{aligned}
$$

respectively, which imply that in the thermal channel, the output state is no longer the pure coherent state but a mixed state.

5. Applications of time evolution of distribution functions

5.1. Evolution of the photon-counting distribution

As an important application of the above evolution formulas, we consider the evolution of photon-counting distribution in thermal channel. In quantum optics photon counting is important for judging the nonclassical features of light field, most measurements of the electromagnetic field are based on the absorption of photons via the photoelectric effect. This is true not only in so far as photodiodes, photomultipliers, being used, but also for such homely devices as the photographic plate and the eye. So the problem of photo-electric detection attracts the attention of many physicists and scientists.

In Refs. [20] and [21], the TESR was used so that some new formulas of photon-counting distribution are derived. Three of those are

$$p(m) = \frac{2(-\zeta)^m}{(2-\zeta)^{m+1}} \int \mathrm{d}^2\alpha\, \mathrm{e}^{-2\zeta|\alpha|^2/(2-\zeta)} \times L_m\left\{\frac{4|\alpha|^2}{2-\zeta}\right\} W(\alpha), \tag{35}$$

$$p(m) = \frac{(-\zeta)^m}{(1-\zeta)^{m+1}} \int \mathrm{d}^2\alpha\, \mathrm{e}^{-\zeta|\alpha|^2/(1-\zeta)} \times L_m\left\{\frac{|\alpha|^2}{1-\zeta}\right\} Q(\alpha), \tag{36}$$

$$p(m) = \frac{\zeta^m}{m!} \int \mathrm{d}^2\alpha\, |\alpha|^{2m}\, \mathrm{e}^{-\zeta|\alpha|^2} P(\alpha), \tag{37}$$

where ζ is called the quantum efficiency (a measure) of the detector, m is the photon-number detected, and $L_m\{x\}$ is the Laguerre polynomials. Thus substituting Eqs. (32)–(34) into Eqs. (35)–(37) and using the following integration formula:[21]

$$\int \frac{\mathrm{d}^2\alpha}{\pi}\, \mathrm{e}^{-B|\alpha|^2+C\alpha+C^*\alpha^*} L_m\left\{\bar{A}|\alpha|^2\right\} = \frac{(B-\bar{A})^m}{B^{m+1}}\, \mathrm{e}^{CC^*/B} L_m\left\{\frac{\bar{A}CC^*/B}{B-\bar{A}}\right\}, \tag{38}$$

we can obtain

$$p(m,t) = \frac{2[(A-1)\zeta]^m}{[(A-1)\zeta+2]^{m+1}} \int \mathrm{d}^2\beta W(\beta,0) \times \mathrm{e}^{-2\zeta\,\mathrm{e}^{-2\kappa t}|\beta|^2/[(A-1)\zeta+2]} \times L_m\left\{\frac{-4\,\mathrm{e}^{-2\kappa t}/(A-1)}{(A-1)\zeta+2}|\beta|^2\right\}, \tag{39}$$

$$p(m,t) = \frac{(N\zeta)^m}{(N\zeta+1)^{m+1}} \int \mathrm{d}^2\beta Q(\beta,0) \times \mathrm{e}^{-\zeta\,\mathrm{e}^{-2t\kappa}|\beta|^2/(N\zeta+1)} \times L_m\left\{-\frac{\mathrm{e}^{-2t\kappa}/N}{N\zeta+1}|\beta|^2\right\}, \quad (N = T + T\bar{n} - 1), \tag{40}$$

and

$$p(m,t) = \frac{(\bar{n}T\zeta)^m}{(\bar{n}T\zeta+1)^{m+1}} \int \mathrm{d}^2\beta P(\beta,0) \times \mathrm{e}^{-\zeta\,\mathrm{e}^{-2\kappa t}|\beta|^2/(\bar{n}T\zeta+1)} \times L_m\left\{-\frac{\mathrm{e}^{-2\kappa t}/(T\bar{n})}{\bar{n}T\zeta+1}|\beta|^2\right\}, \tag{41}$$

where we have interchanged the order of integrations and in the last equation, we have used the following formula[21]

$$\frac{\partial^m}{\partial\tau^m}\frac{\partial^m}{\partial\upsilon^m}\, \mathrm{e}^{A\tau\upsilon+B\tau+C\upsilon}\big|_{\tau=\upsilon=0} = (A)^m m! L_m\left(\frac{BC}{-A}\right), \tag{42}$$

which corresponds to the case of amplitude damping when $\bar{n} = 0$. Under the limiting expression $\lim_{x\to 0} x^m L_m(-|\alpha|^2/x) = (1/m!)|\alpha|^{2m}$, Eqs. (39)–(41) are reduced to

$$p(m,t) = \frac{2(-\bar{\zeta})^m}{(2-\bar{\zeta})^{m+1}} \int \mathrm{d}^2\beta\, \mathrm{e}^{-2\bar{\zeta}|\beta|^2/(2-\bar{\zeta})} \times L_m\left\{\frac{4|\beta|^2}{2-\bar{\zeta}}\right\} W(\beta,0), \tag{43}$$

$$p(m,t) = \frac{(-\bar{\zeta})^m}{(1-\bar{\zeta})^{m+1}} \int \mathrm{d}^2\beta\, \mathrm{e}^{-\bar{\zeta}|\beta|^2/(1-\bar{\zeta})} \times L_m\left\{\frac{|\beta|^2}{1-\bar{\zeta}}\right\} Q(\beta,0), \tag{44}$$

$$p(m,t) = \frac{\bar{\zeta}^m}{m!} \int \mathrm{d}^2\alpha\, |\alpha|^{2m}\, \mathrm{e}^{-\bar{\zeta}|\alpha|^2} P(\alpha,0), \quad (\bar{\zeta} = \zeta\,\mathrm{e}^{-2\kappa t}), \tag{45}$$

respectively. For the amplitude damping channel, thus, one can see that the photon-counting distribution is related to the decay of quantum efficiency $\bar{\zeta} = \zeta\,\mathrm{e}^{-2\kappa t}$.

As another example, we consider the coherent state, whose P-function is $P(\beta,0) =$

$\delta(\beta-\alpha)\,\delta(\beta^*-\alpha^*)$. Using Eq. (41) we immediately have

$$p_\alpha(m,t) = \frac{(\bar{n}T\zeta)^m}{(\bar{n}T\zeta+1)^{m+1}} \exp\left(-\frac{\zeta\,\mathrm{e}^{-2\kappa t}\,|\alpha|^2}{\bar{n}T\zeta+1}\right) \times L_m\left\{-\frac{\mathrm{e}^{-2\kappa t}/(T\bar{n})}{\bar{n}T\zeta+1}\,|\alpha|^2\right\}, \quad (46)$$

which is the photon-counting distribution of coherent state in thermal environment, a Laguerre–Gaussian function. Of course, equation (46) can be obtained by Eq. (39) or Eq. (40).

For the photon-added coherent state, its density operator is $\rho_0 = C_{\alpha,n}a^{\dagger n}\,|\alpha\rangle\,\langle\alpha|\,a^n$,[22] where $C_{\alpha,n} = [n!L_n(-|\alpha|^2)]^{-1}$ is the normalization factor, $|\alpha\rangle = \exp(-|\alpha|^2/2+\alpha a^\dagger)\,|0\rangle$ is the coherent state,[23,24] and $L_n(x)$ is the nth-order Laguerre polynomial. The Q-function of ρ_0 is $Q(\beta,0) = C_{\alpha,n}\,|\beta|^{2n}\exp[-|\beta|^2 - |\alpha|^2+\alpha\beta^*+\alpha^*\beta]/\pi$, thus substituting it into Eq. (40), we finally obtain

$$p(m,t) = C_{\alpha,n}\,\mathrm{e}^{-|\alpha|^2}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\left[\mathrm{e}^{|\alpha|^2}p_\alpha(m,t)\right]. \quad (47)$$

In particular, when $n=0$, i.e., the case of coherent state, equation (47) is just reduced to Eq. (46).

5.2. Evolution of the R-function

Let us use the new formula Eq. (34) to obtain the time evolution of R-function. The R-function is defined as[8]

$$R(z,\tau) = \frac{1}{\tau}\int_{-\infty}^{\infty}\frac{\mathrm{d}^2\alpha}{\pi}\exp\left(-\frac{1}{\tau}\,|\alpha-z|^2\right)P(\alpha), \quad (48)$$

where α and z are complex variables, $P(\alpha)$ is a P-function and τ is a continuous parameter. When we vary τ from 0 to 1, there exists a critical point τ_m such that $R(z,\tau)$ becomes non-negative and non-singular in the whole phase space when $\tau_m \leqslant \tau < 1$. The greatest lower bound of τ is called nonclassical depth.

Substituting Eq. (34) into Eq. (48) and interchanging the order of integrations, we have

$$R(z,\tau,t) = \frac{1}{\tau+\bar{n}T}\int_{-\infty}^{\infty}\frac{\mathrm{d}^2\beta}{\pi}\,\mathrm{e}^{-\left|z-\beta\,\mathrm{e}^{-\kappa t}\right|^2/(\tau+\bar{n}T)} \times P(\beta,0), \quad (49)$$

where we have used Eq. (17). Equation (49) seems to be a new time evolution formula for R-function and is reduced to Eq. (48) at the initial time, it can be used to investigate the nonclassicality of quantum states.[25,26] For example, we take the number state $|m\rangle\,\langle m|$ as an example. Using the following formal expression of P-function in terms of the density operator ρ,[27]

$$P(\beta,0) = \mathrm{e}^{|\beta|^2}\int\frac{\mathrm{d}^2\alpha}{\pi}\,\langle-\alpha|\,\rho\,|\alpha\rangle\,\mathrm{e}^{|\alpha|^2}\,\mathrm{e}^{\alpha^*\beta-\alpha\beta^*}, \quad (50)$$

where $\langle-\alpha|\,,|\alpha\rangle$ are the coherent states, the P-function of $|m\rangle\,\langle m|$ is

$$P(\beta,0) = \frac{(-1)^m}{m!}\,\mathrm{e}^{|\beta|^2}\int\frac{\mathrm{d}^2\alpha}{\pi}\,|\alpha|^{2m}\,\mathrm{e}^{\alpha^*\beta-\alpha\beta^*}. \quad (51)$$

Substituting it into Eq. (49) and interchanging the order of integrations, we have

$$R(z,\tau,t) = \frac{\left(\tau+\bar{n}T-\mathrm{e}^{-2\kappa t}\right)^m}{(\tau+\bar{n}T)^{m+1}}\,\mathrm{e}^{-|z|^2/(\tau+\bar{n}T)} \times L_m\left\{-\frac{\mathrm{e}^{-2\kappa t}\,|z|^2/(\tau+\bar{n}T)}{\tau+\bar{n}T-\mathrm{e}^{-2\kappa t}}\right\}, \quad (52)$$

where the integrable condition is $-\bar{n}T \leqslant \tau < \mathrm{e}^{-2\kappa t}-\bar{n}T$ and we have used the formula

$$L_m(\xi\eta) = \frac{\mathrm{e}^{\xi\eta}}{m!}\int\frac{\mathrm{d}^2z}{\pi}z^m z^{*m}\,\mathrm{e}^{-|z|^2+\xi z-\eta z^*}. \quad (53)$$

It is clearly seen that Eq. (52) just becomes Eq. (10) in Ref. [8] when $t=0$, and that for any z, when $\tau+\bar{n}T-\mathrm{e}^{-2\kappa t} > 0$, i.e., $\tau > \mathrm{e}^{-2\kappa t}-\bar{n}T$ ($\leqslant 1$), $R(z,\tau,t)$ is always positive. In particular, if $0 \leq \mathrm{e}^{-2\kappa t}-\bar{n}T$, we have

$$\kappa t \leq \kappa_{\mathrm{c}}t = \frac{1}{2}\ln\frac{\bar{n}+1}{\bar{n}}, \quad (54)$$

which agrees with the result of keeping nonclassciality in Ref. [28], then $\tau_m = \mathrm{e}^{-2\kappa t}-\bar{n}T$. This indicates that the output state is nonclassical when equation (54) is satisfied. On the other hand, in comparison with the nonclassical depth ($\tau_m = 1$) of number state, we see that the nonclassical depth decreases due to the effect of environment, as expected. It is interesting to point out that the threshold value of the decay time keeping nonclassicality is larger than that ($(1/2)\ln[(2\bar{n}+2)/(2\bar{n}+1)]$) of keeping negativity of WF.[29,30] When $\bar{n}=0$ (amplitude damping), especially, $\kappa_{\mathrm{c}}t \to \infty$, i.e., under the amplitude damping case, the number state is always nonclassical. In this sense, the nonclassical depth is more insensitive to the environment than the negativity of WF. This property of nonclassical depth may be useful for characterizing nonclassicality in experiment.

6. Conclusion

In summary, by introducing the TESR, we converted the calculation of CF to an overlap between two "pure states" in enlarged Fock space. Further, we obtained the relation between the initial CF and the any time CF when a quantum state evolves in amplitude damping channel and thermal channel. Using the Fourier transform relation between the CF and the distribution functions, we further derived the time evolution of Wigner-, Q- and P-functions. The results show that most formulas are expressed as such integrations over Laguerre–Gaussian function. As some important applications of these evolution formulas, we considered the evolution of photon-counting distribution in thermal channel, i.e., we presented three new formulas to calculate the any time photon-counting distribution from the initial W-, Q-, and P-functions so that it would be convenient for investigating the evolution of quasi-probability functions in a dissipative channel by using these formulas. Finally, we should mention that our work can be further expanded to both the nonlinear case and the generalized parameter distribution.[31−33]

References

[1] Moya-Cessa H 2006 *Phys. Rep.* **432** 1

[2] Fan H Y and Hu L Y 2008 *Opt. Commun.* **281** 5571

[3] Fan H Y and Hu L Y 2008 *Mod. Phys. Lett.* B **22** 2435

[4] Fan H Y and Hu L Y 2009 *Commun. Theor. Phys.* **52** 729

[5] Fan H Y and Hu L Y 2009 *Opt. Commun.* **282** 932

[6] Jaffe C, Kanfer S and Brumer P 1985 *Phys. Rev. Lett.* **54** 8

[7] Wilkie J and Brumer P 2000 *Phys. Rev.* A **61** 064101

[8] Lee C T 1991 *Phys. Rev.* A **44** R2775

[9] Fan H Y and Fan Y 2001 *Phys. Lett.* A **282** 269

[10] Fan H Y and Fan Y 2002 *J. Phys.* A **35** 6873

[11] Takahashi Y and Umezawa H 1975 *Int. J. Mod. Phys.* B **2** 55

[12] Umezawa H 1993 *Advanced Field Theory – Micro, Macro, and Thermal Physics* (New York: AIP)

[13] Takahashi Y and Umezawa H 1975 *Collecive Phenomena* **2** 55

[14] Gardiner C and Zoller P 2000 *Quantum Noise* (Berlin: Springer)

[15] Fan H Y 1997 *Representation and Transformation Theory in Quantum Mechanics* (Shanghai: Shanghai Scientific & Technical Press) p. 114 (in Chinese)

[16] Fan H Y and Klauder J R 1994 *Phys. Rev.* A **49** 704

[17] Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge University Press)

[18] Puri R R 2001 *Mathematical Methods of Quantum Optics* (Berlin: Springer-Verlag) Appendix A

[19] Hu L Y and Fan H Y 2009 *Opt. Commun.* **282** 4379

[20] Fan H Y and Hu L Y 2008 *Opt. Lett.* **33** 443

[21] Hu L Y, Wang Z S, Kwek L C and Fan H Y 2010 **20** (in press)

[22] Agarwal G S and Tara K 1991 *Phys. Rev.* A **43** 492

[23] Glauber R J 1963 *Phys. Rev.* **130** 2529

[24] Klauder J R and Skargerstam B S 1985 *Coherent States* (Singapore: World Scientific)

[25] Hu L Y, Xu X X, Wang Z S and Xu X F 2010 *Phys. Rev.* A **82** 043842

[26] Yang Y and Li F L 2009 *J. Opt. Soc. Am.* B **26** 830

[27] Mehta C L 1967 *Phys. Rev. Lett.* **18** 752

[28] Carmichael H J 1999 *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations* (Berlin: Springer-Verlag)

[29] Hu L Y and Fan H Y 2008 *J. Opt. Soc. Am.* B **25** 1995

[30] Hu L Y, Xu X X and Fan H Y 2010 *J. Opt. Soc. Am.* B **27** 286

[31] Fan H Y, Ren G, Hu L Y and Jiang N Q 2010 *Chin. Phys.* B **19** 114206

[32] Fan H Y, Yuan H C and Hu L Y 2010 *Chin. Phys.* B **19** 104204

[33] Fan H Y 2010 *Chin. Phys.* B **19** 050303