Far-field divergence of a vectorial plane wave diffracted by a circular aperture from the vectorial structure*

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Based on the vectorial structure of an electromagnetic wave, the analytical and concise expressions for the TE and TM terms of a vectorial plane wave diffracted by a circular aperture are derived in the far-field. The expressions of the energy flux distributions of the TE term, the TM term and the diffracted plane wave are also presented. The ratios of the power of the TE and TM terms to that of the diffracted plane wave are examined in the far-field. In addition, the far-field divergence angles of the TE term, the TM term and the diffracted plane wave, which are related to the energy flux distribution, are investigated. The different energy flux distributions of the TE and TM terms result in the discrepancy of their divergence angles. The influences of the linearly polarized angle and the radius of the circular aperture on the far-field divergence angles of the TE term, the TM term and the diffracted plane wave are discussed in detail. This research may promote the recognition of the optical propagation through a circular aperture.

Keywords: divergence, plane wave, vectorial structure, diffraction

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1. Introduction

A plane wave diffracted by a circular aperture is a typical diffraction problem, which has been treated by many different methods.[1−5] Only when the radius of aperture is of the order of the wavelength would the vector diffraction theory be used to treat the optical propagation through an aperture. Otherwise, the scalar diffraction theory is usually adopted. Here, the description of a plane wave diffracted by a circular aperture is directly started from Maxwell's equations and the method of vector angular spectrum is used to resolve Maxwell's equations. As the vector angular spectrum can be uniquely decomposed into the two terms in the frequency domain,[6] the diffracted plane wave is essentially composed of the TE and TM terms. The TE term denotes the electric field transverse to the propagation axis and the TM term means the associated magnetic field transverse to the propagation axis.[7−13] In the far-field, the TE and TM terms are orthogonal to each other. The analytical expressions of the TE and TM terms of a diffracted plane wave have been derived in the far-field and the sum of the light intensities of the TE and TM terms just constitutes the Airy disc pattern.[14] Because the influence of the radius of a circular aperture on the divergence of the diffracted plane wave is critical to the practical applications, the divergence of the diffracted plane wave will be investigated in this paper. Based on the second-order moments and the accurate expression of light intensity, the divergence angle of a scalar plane wave diffracted by a circular aperture will not exceed 64.3°.[15] By using the vectorial Rayleigh diffraction integrals, non-paraxial diffraction of vectorial plane waves at a small circular aperture has been examined.[16] The maximum divergence angles defined by the FWHM (the full width between half maximum power) in the two transversal directions have been shown to be 45° and 32.8°, respectively.[16] In the present paper, the vectorial case is considered. Moreover, the plane wave is treated to be linearly polarized, which is the familiar case in the theoretical researches and the practical applications. Secondly, the far-field divergence angle is defined by the second-order moments. Finally, the far-field divergence angle of the diffracted plane wave, which is related to the energy flux distribution, is investigated from the vectorial structure of an electromagnetic wave.

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2. Far-field divergence angles of the diffracted plane wave and its TE and TM terms

A linearly polarized plane wave described by

$$\begin{bmatrix} E_x(\boldsymbol{r}) \\ E_y(\boldsymbol{r}) \end{bmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \exp(\mathrm{i}kz) \tag{1}$$

passes through a co-axial circular aperture with radius R. Here, $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $k = 2\pi/\lambda$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are three unit vectors in the x-, y- and z-directions, respectively and λ is the wavelength of the incident plane wave. Jones vector $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ describes the linearly polarized state, where α is the linearly polarized angle and ranges from 0 to 360 degrees. The time dependent factor $\exp(-\mathrm{i}\omega t)$ is omitted in Eq. (1), and ω is the circular frequency. A suitable Cartesian coordinate system is constructed as follows. The circular aperture plane is selected as the x–y plane and the centre of the circular aperture is the origin. The z-axis is taken to be the propagation axis. The optical field just behind the aperture is

$$\begin{bmatrix} E_x(x_0, y_0, 0) \\ E_y(x_0, y_0, 0) \end{bmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \mathrm{circ}(\zeta), \tag{2}$$

where $\zeta = \rho_0/R$, $\rho_0 = (x_0^2 + y_0^2)^{1/2}$ and the aperture function $\mathrm{circ}(\zeta)$ is given by

$$\mathrm{circ}(\zeta) = \begin{cases} 1, & 0 \le \zeta < 1, \\ 0, & \zeta > 1. \end{cases} \tag{3}$$

According to the vector angular spectrum representation of Maxwell's equations, the diffracted plane wave propagating toward the half free space $z \ge 0$ is obtained by

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{A}(p,q) \\ & \times \exp[\mathrm{i}k(px + qy + \gamma z)]\mathrm{d}p\mathrm{d}q, \end{aligned} \tag{4}$$

where $\gamma = (1 - p^2 - q^2)^{1/2}$ and the vector angular spectrum $\boldsymbol{A}(p,q)$ is expressed as[17]

$$\begin{aligned} \boldsymbol{A}(p,q) = & A_x(p,q)\left(\boldsymbol{i} - \frac{p}{\gamma}\boldsymbol{k}\right) \\ & + A_y(p,q)\left(\boldsymbol{j} - \frac{q}{\gamma}\boldsymbol{k}\right). \end{aligned} \tag{5}$$

Here, $A_x(p,q)$ and $A_y(p,q)$ are the transverse components of the vector angular spectrum and are found to be

$$\begin{aligned} A_x(p,q) = & \frac{1}{\lambda^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x(x_0, y_0, 0) \\ & \times \exp[-\mathrm{i}k(px_0 + qy_0)]\mathrm{d}x_0\mathrm{d}y_0 \\ = & \frac{R\cos\alpha J_1(kRb)}{\lambda b}, \end{aligned} \tag{6}$$

$$\begin{aligned} A_y(p,q) = & \frac{1}{\lambda^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y(x_0, y_0, 0) \\ & \times \exp[-\mathrm{i}k(px_0 + qy_0)]\mathrm{d}x_0\mathrm{d}y_0 \\ = & \frac{R\sin\alpha J_1(kRb)}{\lambda b}, \end{aligned} \tag{7}$$

where $b = (p^2 + q^2)^{1/2}$ and J_1 is the first order Bessel function of the first kind. The longitudinal component stems from the transversality condition of the optical field $\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}) = 0$. The dot denotes the scalar product. In the frequency domain, we can define three unit vectors $\boldsymbol{s}$, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ as

$$\begin{aligned} \boldsymbol{s} &= p\boldsymbol{i} + q\boldsymbol{j} + \gamma\boldsymbol{k}, \\ \boldsymbol{e}_1 &= (q\boldsymbol{i} - p\boldsymbol{j})/b, \\ \boldsymbol{e}_2 &= \frac{\gamma}{b}(p\boldsymbol{i} + q\boldsymbol{j}) - b\boldsymbol{k}. \end{aligned} \tag{8}$$

The above three unit vectors form a mutually perpendicular righthanded system as follows:

$$\begin{aligned} \boldsymbol{s} \times \boldsymbol{e}_1 &= \boldsymbol{e}_2, \\ \boldsymbol{e}_1 \times \boldsymbol{e}_2 &= \boldsymbol{s}, \\ \boldsymbol{e}_2 \times \boldsymbol{s} &= \boldsymbol{e}_1. \end{aligned} \tag{9}$$

In this system, the vector angular spectrum $\boldsymbol{A}(p,q)$ can be decomposed into two terms

$$\boldsymbol{A}(p,q) = [\boldsymbol{A}(p,q) \cdot \boldsymbol{e}_1]\boldsymbol{e}_1 + [\boldsymbol{A}(p,q) \cdot \boldsymbol{e}_2]\boldsymbol{e}_2. \tag{10}$$

Therefore, the diffracted plane wave can be decomposed into the TE and TM terms

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r}) + \boldsymbol{E}_{\mathrm{TM}}(\boldsymbol{r}), \tag{11}$$

where $\boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r})$ and $\boldsymbol{E}_{\mathrm{TM}}(\boldsymbol{r})$ are given by

$$\begin{aligned} & \boldsymbol{E}_{\mathrm{TE}}(\boldsymbol{r}) \\ = & \frac{R}{\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(q\cos\alpha - p\sin\alpha)J_1(kRb)}{b^3} \\ & \times (q\boldsymbol{i} - p\boldsymbol{j})\exp[\mathrm{i}k(px + qy + \gamma z)]\mathrm{d}p\mathrm{d}q, \end{aligned} \tag{12}$$

$$\begin{aligned} & \boldsymbol{E}_{\mathrm{TM}}(\boldsymbol{r}) \\ = & \frac{R}{\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(p\cos\alpha + q\sin\alpha)J_1(kRb)}{\gamma b^3} \\ & \times (p\gamma\boldsymbol{i} + q\gamma\boldsymbol{j} - b^2\boldsymbol{k}) \\ & \times \exp[\mathrm{i}k(px + qy + \gamma z)]\mathrm{d}p\mathrm{d}q. \end{aligned} \tag{13}$$

By taking the curl of Eqs. (12) and (13), the corresponding magnetic fields of the TE and TM terms turn out to be

$$\begin{aligned}&\boldsymbol{H}_{\rm TE}(\boldsymbol{r})\\ &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{R}{\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(q\cos\alpha - p\sin\alpha)J_1(kRb)}{b^3}\\ &\times (p\gamma\boldsymbol{i} + q\gamma\boldsymbol{j} - b^2\boldsymbol{k})\\ &\times \exp[{\rm i}k(px+qy+\gamma z)]\,{\rm d}p\,{\rm d}q, \qquad (14)\\ &\boldsymbol{H}_{\rm TM}(\boldsymbol{r})\\ &= -\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{R}{\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(p\cos\alpha + q\sin\alpha)J_1(kRb)}{\gamma b^3}\\ &\times (q\boldsymbol{i} - p\boldsymbol{j})\exp[{\rm i}k(px+qy+\gamma z)]\,{\rm d}p\,{\rm d}q, \qquad (15)\end{aligned}$$

where ε_0 and μ_0 are the electric permittivity and the magnetic permeability in vacuum, respectively. As the TE and TM terms are not circularly symmetrical, the electromagnetic fields of the TE and TM terms are represented in the normal three orthogonal coordinates. In fact, the plane wave diffracted by a circular aperture of the wavelength scale is no longer circularly symmetrical. To prove the above statements, the electric fields of the TE and TM terms are rewritten and expressed in variables ρ, φ and z of the cylindrical coordinates. The electric field of the TE term can be rewritten as follows:

$$\begin{aligned}&\boldsymbol{E}_{\rm TE}(\boldsymbol{r})\\ &= \frac{R}{\lambda}\int_0^{\infty}\int_0^{2\pi} J_1(kRb)\sin(\theta-\alpha)(\sin\theta\boldsymbol{i} - \cos\theta\boldsymbol{j})\\ &\times \exp[{\rm i}k\rho b\cos(\theta-\varphi)]\exp({\rm i}k\gamma z)\,{\rm d}b\,{\rm d}\theta, \qquad (16)\end{aligned}$$

where $\varphi = \tan^{-1}(y/x)$, $\theta = \tan^{-1}(p/q)$ and $\rho = (x^2+y^2)^{1/2}$. Performing the integral over the variable θ, we have

$$\boldsymbol{E}_{\rm TE}(\boldsymbol{r}) = E_{{\rm TE}x}(\boldsymbol{r})\boldsymbol{i} + E_{{\rm TE}y}(\boldsymbol{r})\boldsymbol{j}, \qquad (17)$$

where

$$\begin{aligned}E_{{\rm TE}x}(\boldsymbol{r}) &= \frac{kR}{2}\int_0^{\infty} J_1(kRb)\exp({\rm i}k\gamma z)[\cos\alpha J_0(k\rho b)\\ &+ \cos(2\varphi-\alpha)J_2(k\rho b)]\,{\rm d}b, \qquad (18)\\ E_{{\rm TE}y}(\boldsymbol{r}) &= \frac{kR}{2}\int_0^{\infty} J_1(kRb)\exp({\rm i}k\gamma z)[\sin\alpha J_0(k\rho b)\\ &+ \sin(2\varphi-\alpha)J_2(k\rho b)]\,{\rm d}b. \qquad (19)\end{aligned}$$

Under the integration process, the following integral formula is used:

$$\begin{aligned}J_n(k\rho b) &= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[{\rm i}k\rho b\cos(\theta-\varphi)\right.\\ &\left.+ {\rm i}n\left(\theta-\varphi-\frac{\pi}{2}\right)\right]{\rm d}\theta. \qquad (20)\end{aligned}$$

Similarly, the electric field of the TM term is

$$\boldsymbol{E}_{\rm TM}(\boldsymbol{r}) = E_{{\rm TM}x}(\boldsymbol{r})\boldsymbol{i} + E_{{\rm TM}y}(\boldsymbol{r})\boldsymbol{j} + E_{{\rm TM}z}(\boldsymbol{r})\boldsymbol{k}, \qquad (21)$$

where

$$\begin{aligned}E_{{\rm TM}x}(\boldsymbol{r}) &= \frac{kR}{2}\int_0^{\infty} J_1(kRb)\exp({\rm i}k\gamma z)[\cos\alpha J_0(k\rho b)\\ &- \cos(2\varphi-\alpha)J_2(k\rho b)]\,{\rm d}b, \qquad (22)\\ E_{{\rm TM}y}(\boldsymbol{r}) &= \frac{kR}{2}\int_0^{\infty} J_1(kRb)\exp({\rm i}k\gamma z)[\sin\alpha J_0(k\rho b)\\ &- \sin(2\varphi-\alpha)J_2(k\rho b)]\,{\rm d}b, \qquad (23)\\ E_{{\rm TM}z}(\boldsymbol{r}) &= -{\rm i}kR\cos(\varphi-\alpha)\int_0^{\infty}\frac{b}{\gamma}J_1(kRb)\\ &\times \exp({\rm i}k\gamma z)J_1(k\rho b)\,{\rm d}b. \qquad (24)\end{aligned}$$

The electric field of the diffracted plane wave yields

$$\boldsymbol{E}(\boldsymbol{r}) = E_x(\boldsymbol{r})\boldsymbol{i} + E_y(\boldsymbol{r})\boldsymbol{j} + E_z(\boldsymbol{r})\boldsymbol{k}, \qquad (25)$$

where

$$\begin{aligned}E_x(\boldsymbol{r}) &= kR\cos\alpha\int_0^{\infty} J_1(kRb)\\ &\times \exp({\rm i}k\gamma z)J_0(k\rho b)\,{\rm d}b, \qquad (26)\\ E_y(\boldsymbol{r}) &= kR\sin\alpha\int_0^{\infty} J_1(kRb)\\ &\times \exp({\rm i}k\gamma z)J_0(k\rho b)\,{\rm d}b, \qquad (27)\\ E_z(\boldsymbol{r}) &= -{\rm i}kR\cos(\varphi-\alpha)\int_0^{\infty}\frac{b}{\gamma}J_1(kRb)\\ &\times \exp({\rm i}k\gamma z)J_1(k\rho b)\,{\rm d}b. \qquad (28)\end{aligned}$$

Because the expressions of the electric fields of the TE term, the TM term and the diffracted plane wave include the variable φ, they are not circularly symmetrical. Therefore, the cylindrical coordinates have no advantage over the Cartesian coordinates. Of course, by using the following transformation formulas:

$$\begin{aligned}&\boldsymbol{e}_\rho = \cos\varphi\boldsymbol{i} + \sin\varphi\boldsymbol{j},\\ &\boldsymbol{e}_\varphi = -\sin\varphi\boldsymbol{i} + \cos\varphi\boldsymbol{j},\\ &\boldsymbol{e}_z = \boldsymbol{k}, \qquad (29)\end{aligned}$$

we can represent the electromagnetic fields of the TE term, the TM term and the diffracted plane wave in the cylindrical coordinates, where $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_z$ are three unit vectors in the cylindrical coordinates.

In the far-field regime, the condition $kr = k(x^2+y^2+z^2)^{1/2} \to \infty$ is satisfied. By means of the method of stationary phase,[18–20] the analytical expressions of the TE term are found to be

$$\begin{aligned}\boldsymbol{E}_{\rm TE}(\boldsymbol{r}) &= -\frac{{\rm i}Rz\sin(\varphi-\alpha)J_1(kR\rho/r)}{\rho^2 r}\\ &\times (y\boldsymbol{i} - x\boldsymbol{j})\exp({\rm i}kr), \qquad (30)\end{aligned}$$

$$\boldsymbol{H}_{\rm TE}(\boldsymbol{r}) = -{\rm i}\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{Rz\sin(\varphi-\alpha)J_1(kR\rho/r)}{\rho^2 r^2} \times (xz\boldsymbol{i}+yz\boldsymbol{j}-\rho^2\boldsymbol{k})\exp({\rm i}kr). \quad (31)$$

The analytical expressions of the TM term turn out to be

$$\boldsymbol{E}_{\rm TM}(\boldsymbol{r}) = -{\rm i}\frac{R\cos(\varphi-\alpha)J_1(kR\rho/r)}{\rho^2 r} \times (xz\boldsymbol{i}+yz\boldsymbol{j}-\rho^2\boldsymbol{k})\exp({\rm i}kr), \quad (32)$$

$$\boldsymbol{H}_{\rm TM}(\boldsymbol{r}) = {\rm i}\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{R\cos(\varphi-\alpha)J_1(kR\rho/r)}{\rho^2} \times (y\boldsymbol{i}-x\boldsymbol{j})\exp({\rm i}kr). \quad (33)$$

The energy flux distributions of the TE and TM terms in the far-field plane are given by

$$\begin{aligned}&\langle S_z\rangle_{\rm TE}\\ &= \frac{1}{2}{\rm Re}\left[\boldsymbol{E}_{\rm TE}(\boldsymbol{r})\times\boldsymbol{H}^*_{\rm TE}(\boldsymbol{r})\right]_z\\ &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{R^2z^3\sin^2(\varphi-\alpha)J_1^2(kR\rho/r)}{2\rho^2r^3}, \quad (34)\\ &\langle S_z\rangle_{\rm TM}\\ &= \frac{1}{2}{\rm Re}\left[\boldsymbol{E}_{\rm TM}(\boldsymbol{r})\times\boldsymbol{H}^*_{\rm TM}(\boldsymbol{r})\right]_z\\ &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{R^2z\cos^2(\varphi-\alpha)J_1^2(kR\rho/r)}{2\rho^2r}, \quad (35)\end{aligned}$$

where Re means taking the real part and the asterisk denotes the complex conjugation. Because the electromagnetic fields of the TE and TM terms are orthogonal to each other in the far-field, the energy flux distribution of the diffracted plane wave in the far-field plane is found to be

$$\begin{aligned}&\langle S_z\rangle\\ &= \langle S_z\rangle_{\rm TE}+\langle S_z\rangle_{\rm TM}\\ &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{R^2zJ_1^2(kR\rho/r)}{2\rho^2r^3}[z^2+\rho^2\cos^2(\varphi-\alpha)]. \quad (36)\end{aligned}$$

The powers of the TE and TM terms in the far-field plane are given by

$$\begin{aligned}P_{\rm TE} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle S_z\rangle_{\rm TE}\,{\rm d}x\,{\rm d}y,\\ P_{\rm TM} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle S_z\rangle_{\rm TM}\,{\rm d}x\,{\rm d}y. \quad (37)\end{aligned}$$

Inserting Eq. (34) into Eq. (37), we can obtain

$$\begin{aligned}&P_{\rm TE}\\ &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{R^2z^3}{2}\int_0^{\infty}\int_0^{2\pi}\frac{\sin^2(\varphi-\alpha)J_1^2(kR\rho/r)}{\rho r^3}\,{\rm d}\rho{\rm d}\varphi\\ &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\pi R^2z^3}{2}\int_0^{\infty}\frac{J_1^2(kR\rho/r)}{\rho r^3}\,{\rm d}\rho\\ &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\pi R^2z^3}{2}\int_0^{\infty}\frac{J_1^2[kR\rho/(z^2+\rho^2)^{1/2}]}{\rho(z^2+\rho^2)^{3/2}}\,{\rm d}\rho\\ &= \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\pi R^2}{2}\int_0^{\infty}\frac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{3/2}}\,{\rm d}\tau. \quad (38)\end{aligned}$$

In the above equation, ρ/z has been replaced by the variable τ. Because the far-field reference plane is fixed, the limits of integration of variable τ are the same as those of the variable ρ. The $P_{\rm TE}$ is independent of the linearly polarized angle. The power of the TE term in the far-field plane is independent of the axial propagation distance, which denotes that the power of the TE term keeps conservative upon propagation. Similarly, the power of the TM term in the far-field plane turns out to be

$$P_{\rm TM} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\pi R^2}{2}\int_0^{\infty}\frac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{1/2}}\,{\rm d}\tau. \quad (39)$$

The ratio of the power of the TE term to that of the diffracted plane wave is described by

$$\begin{aligned}C_{\rm TE} &= \frac{P_{\rm TE}}{P_{\rm TE}+P_{\rm TM}}\\ &= \frac{\displaystyle\int_0^{\infty}\frac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{3/2}}\,{\rm d}\tau}{\displaystyle\int_0^{\infty}\frac{(2+\tau^2)J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{3/2}}\,{\rm d}\tau}. \quad (40)\end{aligned}$$

In the far-field, therefore, $C_{\rm TE}$ is only determined by the ratio R/λ. The ratio of the power of the TM term to that of the diffracted plane wave yields

$$C_{\rm TM} = \frac{P_{\rm TM}}{P_{\rm TE}+P_{\rm TM}} = 1-C_{\rm TE}. \quad (41)$$

We now investigate the far-field divergence angles of the TE term, the TM term and the diffracted plane wave, where the far-field divergence angle is defined by

$$\begin{aligned}\tan\theta_x &= \lim_{z\to\infty}\frac{W_x(z)}{z},\\ \tan\theta_y &= \lim_{z\to\infty}\frac{W_y(z)}{z}. \quad (42)\end{aligned}$$

The $W_x(z)$ and $W_y(z)$ are the beam widths in the z-plane and are given by

$$W_j^2(z) = 4\frac{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}j^2\langle S_z\rangle\,{\rm d}x\,{\rm d}y}{\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle S_z\rangle\,{\rm d}x\,{\rm d}y}, \quad (43)$$

where $j = x$ or y (hereafter). We first consider the far-field divergence angle of the TE term. The second-order moments of the TE term in the far-field are

$$W_{x\mathrm{TE}}^2(z) = 4\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\langle S_z\rangle_{\mathrm{TE}}\,\mathrm{d}x\,\mathrm{d}y}{P_{\mathrm{TE}}} = z^2(1+2\sin^2\alpha)\frac{\int_0^{\infty}\dfrac{\tau J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{(1+\tau^2)^{3/2}}\,\mathrm{d}\tau}{\int_0^{\infty}\dfrac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{3/2}}\,\mathrm{d}\tau}, \tag{44}$$

$$W_{y\mathrm{TE}}^2(z) = 4\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2\langle S_z\rangle_{\mathrm{TE}}\,\mathrm{d}x\,\mathrm{d}y}{P_{\mathrm{TE}}} = z^2(1+2\cos^2\alpha)\frac{\int_0^{\infty}\dfrac{\tau J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{(1+\tau^2)^{3/2}}\,\mathrm{d}\tau}{\int_0^{\infty}\dfrac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{3/2}}\,\mathrm{d}\tau}. \tag{45}$$

The far-field divergence angles of the TE term are found to be

$$\theta_{x\mathrm{TE}} = \tan^{-1}\left\{\left[(1+2\sin^2\alpha)\frac{\int_0^{\infty}\dfrac{\tau J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{(1+\tau^2)^{3/2}}\,\mathrm{d}\tau}{\int_0^{\infty}\dfrac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{3/2}}\,\mathrm{d}\tau}\right]^{1/2}\right\}, \tag{46}$$

$$\theta_{y\mathrm{TE}} = \tan^{-1}\left\{\left[(1+2\cos^2\alpha)\frac{\int_0^{\infty}\dfrac{\tau J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{(1+\tau^2)^{3/2}}\,\mathrm{d}\tau}{\int_0^{\infty}\dfrac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{3/2}}\,\mathrm{d}\tau}\right]^{1/2}\right\}. \tag{47}$$

The far-field divergence angles of the TE term are determined by the linearly polarized angle and the ratio R/λ. Similarly, the far-field divergence angles of the TM term are

$$\theta_{x\mathrm{TM}} = \tan^{-1}\left\{\left[(1+2\cos^2\alpha)\frac{\int_0^{\infty}\dfrac{\tau J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{(1+\tau^2)^{1/2}}\,\mathrm{d}\tau}{\int_0^{\infty}\dfrac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{1/2}}\,\mathrm{d}\tau}\right]^{1/2}\right\}, \tag{48}$$

$$\theta_{y\mathrm{TM}} = \tan^{-1}\left\{\left[(1+2\sin^2\alpha)\frac{\int_0^{\infty}\dfrac{\tau J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{(1+\tau^2)^{1/2}}\,\mathrm{d}\tau}{\int_0^{\infty}\dfrac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{1/2}}\,\mathrm{d}\tau}\right]^{1/2}\right\}. \tag{49}$$

There is a relation among the beam widths of the TE term, the TM term and the diffracted plane wave as

$$\begin{aligned} W_j^2(z) &= 4\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j^2\langle S_z\rangle\,\mathrm{d}x\,\mathrm{d}y}{P_{\mathrm{TE}}+P_{\mathrm{TM}}} \\ &= 4\left(\frac{P_{\mathrm{TE}}}{P_{\mathrm{TE}}+P_{\mathrm{TM}}}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j^2\langle S_z\rangle_{\mathrm{TE}}\,\mathrm{d}x\,\mathrm{d}y}{P_{\mathrm{TE}}} + \frac{P_{\mathrm{TM}}}{P_{\mathrm{TE}}+P_{\mathrm{TM}}}\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j^2\langle S_z\rangle_{\mathrm{TM}}\,\mathrm{d}x\,\mathrm{d}y}{P_{\mathrm{TM}}}\right) \\ &= C_{\mathrm{TE}}W_{j\mathrm{TE}}^2(z) + C_{\mathrm{TM}}W_{j\mathrm{TM}}^2(z). \end{aligned} \tag{50}$$

By using the parameters C_{TE} and C_{TM} as well as the far-field divergence angles of the TE and TM terms, we can calculate the far-field divergence angle of the diffracted plane wave by

$$\theta_j = \tan^{-1}\left[\left(C_{\mathrm{TE}}\tan^2\theta_{j\mathrm{TE}} + C_{\mathrm{TM}}\tan^2\theta_{j\mathrm{TM}}\right)^{1/2}\right]. \tag{51}$$

3. Numerical calculations and analyses

Figure 1 presents C_{TE} and C_{TM} as a function of the ratio R/λ. When the ratio R/λ tends to zero, C_{TE} reaches the minimum value of 1/4 and C_{TM} has the

maximum value of 3/4. By using the L'Hospital rule, the value of $C_{\rm TE}$ at the singularity $R/\lambda = 0$ can be derived as follows:

$$\begin{aligned} & C_{\rm TE} \\ = & \frac{\displaystyle\int_0^\infty \frac{J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{3/2}}\,{\rm d}\tau}{\displaystyle\int_0^\infty \frac{(2+\tau^2)J_1^2[kR\tau/(1+\tau^2)^{1/2}]}{\tau(1+\tau^2)^{3/2}}\,{\rm d}\tau} \\ = & \frac{\displaystyle\int_0^\infty \frac{\tau}{(1+\tau^2)^{5/2}}\,{\rm d}\tau}{\displaystyle\int_0^\infty \frac{\tau}{(1+\tau^2)^{5/2}}\,{\rm d}\tau + \int_0^\infty \frac{\tau}{(1+\tau^2)^{3/2}}\,{\rm d}\tau}. \end{aligned} \quad (52)$$

By using the mathematical integral formula

$$\int_0^\infty \frac{\tau^{2n-1}}{(1+\tau^2)^m}\,{\rm d}\tau = \frac{\Gamma(n)\Gamma(m-n)}{2\Gamma(m)}, \quad (53)$$

where $\Gamma(.)$ is a Gamma function, the value of $C_{\rm TE}$ at the singularity $R/\lambda = 0$ is found to be 1/4. As a result, the value of $C_{\rm TM}$ at the singularity $R/\lambda = 0$ is 3/4. When the ratio R/λ is large enough, $C_{\rm TE}$ reaches the maximum value of 1/2 and $C_{\rm TM}$ takes the minimum value of 1/2. Generally, $C_{\rm TE}$ is smaller than $C_{\rm TM}$, which means that the contribution of the power of the TM term to that of the diffracted plane wave is larger than the contribution of the power of the TE term to that of the diffracted plane wave. The fluctuation in the curves is caused by the diffraction.

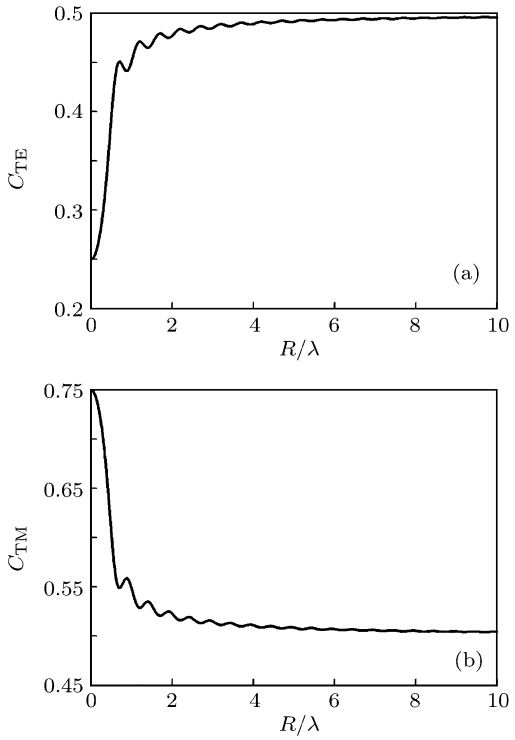


Fig. 1. $C_{\rm TE}$ and $C_{\rm TM}$ as a function of the ratio R/λ: (a) $C_{\rm TE}$, (b) $C_{\rm TM}$.

Because the far-field divergence angles of the TE term, the TM term and the diffracted plane wave are determined by the ratio R/λ and the linearly polarized angle, we investigate the influences of the ratio R/λ and the linearly polarized angle on the far-field divergence angles. Figure 2 shows the far-field divergence angle of the TE term versus the ratio R/λ. With the increase of the ratio R/λ, the divergence angles of the TE term decrease. The fluctuation of the divergence angles of the TE term is relatively unconspicuous. The divergence angle of the TE term versus the linearly polarized angle is plotted in Fig. 3. The variation of the divergence angle of the TE term with α is periodic and the period is 180°. When $\alpha = 0$, $\theta_{x\rm TE}$ reaches the minimum value and $\theta_{y\rm TE}$ takes the maximum value. When $\alpha = 90°$, $\theta_{x\rm TE}$ has the maximum value and $\theta_{y\rm TE}$ reaches the minimum value. The divergence angles of the TE term with the smaller value of R/λ have the larger value.

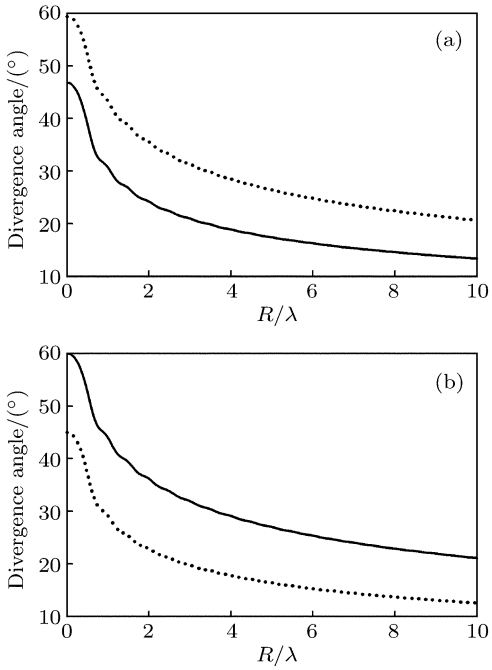


Fig. 2. The divergence angle of the TE term as a function of the ratio R/λ. The solid and the dotted curves correspond to the x- and y-directions, respectively. (a) $\alpha = 15°$, (b) $\alpha = 90°$.

The far-field divergence angle of the TM term versus the ratio R/λ is plotted in Fig. 4. As shown in Figs. 2 and 4, the divergence angle of the TE term versus the ratio R/λ is relatively smoothly changed, while the divergence angle of the TM term versus the ratio R/λ is reduced in an oscillatory way. This difference is stemmed from their different energy flux

distributions. When the variable τ tends to infinity, the expression of numerator in Eqs. (46) and (47) $\tau J_1^2[kR\tau/(1+\tau^2)^{1/2}]/(1+\tau^2)^{3/2} \approx J_1^2(kR)/\tau^2 \to 0$ and the expression of numerator in Eqs. (48) and (49) $\tau J_1^2[kR\tau/(1+\tau^2)^{1/2}]/(1+\tau^2)^{1/2} \approx J_1^2(kR)$. The variation of R will result in the oscillation of $J_1^2(kR)$, which leads to the oscillation of the divergence angle of the TM term.

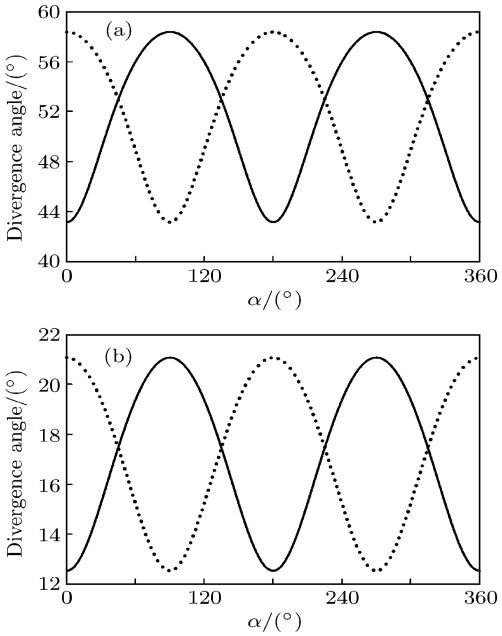


Fig. 3. The divergence angle of the TE term as a function of the linearly polarized angle. The solid and the dotted curves correspond to the x- and y-directions, respectively. (a) $R/\lambda = 0.25$, (b) $R/\lambda = 10$.

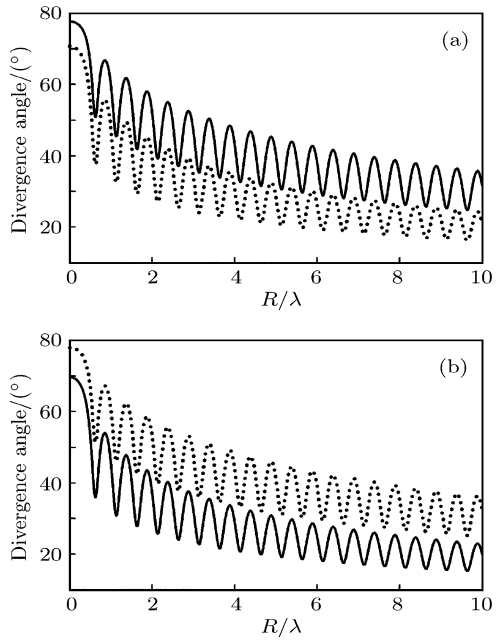


Fig. 4. The divergence angle of the TM term as a function of the ratio R/λ. The solid and the dotted curves correspond to the x- and y-directions, respectively. (a) $\alpha = 15°$, (b) $\alpha = 90°$.

Figure 5 represents the divergence angle of the TM term versus the linearly polarized angle. The variation of the divergence angle of the TM term with α is also periodic and the period is 180°. When $\alpha = 0$, $\theta_{x\mathrm{TM}}$ reaches the maximum value and $\theta_{y\mathrm{TM}}$ takes the minimum value. When $\alpha = 90°$, $\theta_{x\mathrm{TM}}$ has the minimum value and $\theta_{y\mathrm{TM}}$ reaches the maximum value. The divergent properties of the TE and TM terms can be interpreted as follows. The beam spot of the TE term is located at the orientation perpendicular to the direction of linearly polarized angle. Moreover, the energy flux distribution of the TE term is relatively centralized in the direction of linearly polarized angle. As a result, $\theta_{x\mathrm{TE}}$ takes the minimum value in the case of $\alpha = 0$, 180°, 360°, $\theta_{x\mathrm{TE}}$ reaches the maximum value in the case of $\alpha = 90°$, 270°, $\theta_{y\mathrm{TE}}$ reaches the maximum value in the case of $\alpha = 0$, 180°, 360° and $\theta_{y\mathrm{TE}}$ has the minimum value in the case of $\alpha = 90°$, 270°. The beam spot of the TM term is located at the direction of linearly polarized angle and the energy flux distribution of the TM term is relatively centralized in the orientation perpendicular to the direction of linearly polarized angle. Accordingly, $\theta_{x\mathrm{TM}}$ reaches the maximum value in the case of $\alpha = 0$, 180°, 360° and $\theta_{x\mathrm{TM}}$ takes the minimum value in the case of $\alpha = 90°$, 270°, $\theta_{y\mathrm{TM}}$ has the minimum value in the case of $\alpha = 0$, 180°, 360° and $\theta_{y\mathrm{TM}}$ reaches the maximum value in the case of $\alpha = 90°$, 270°.

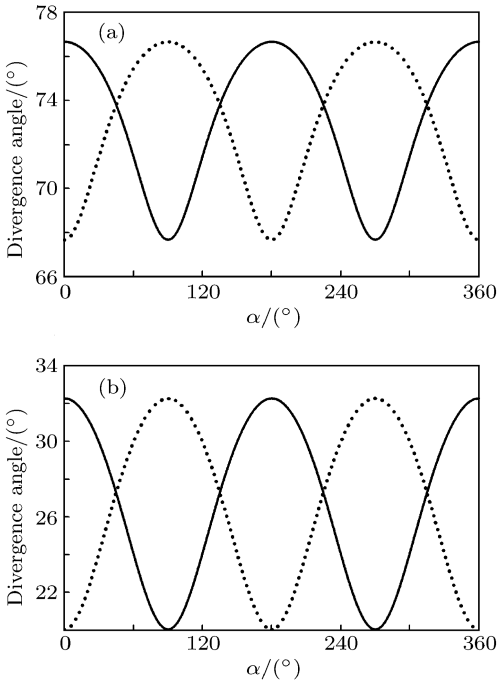


Fig. 5. The divergence angle of the TM term as a function of the linearly polarized angle. The solid and the dotted curves correspond to the x- and y-directions, respectively. (a) $R/\lambda = 0.25$, (b) $R/\lambda = 10$.

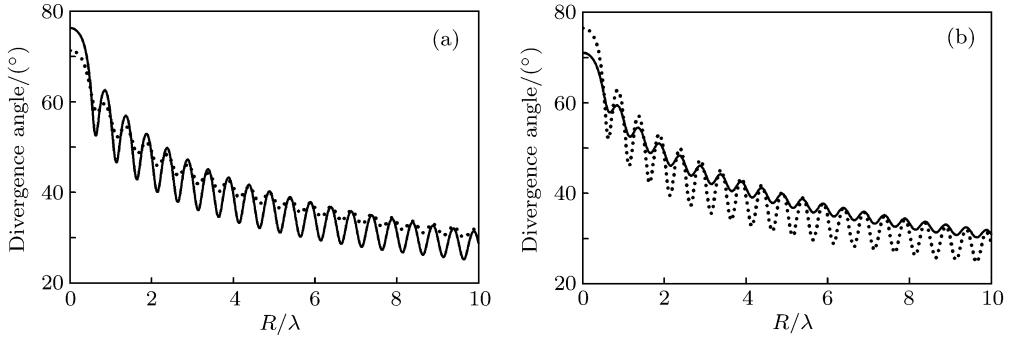


Fig. 6. The divergence angle of the diffracted planar wave as a function of the ratio R/λ. The solid and the dotted curves correspond to the x- and y-directions, respectively. (a) $\alpha = 15°$, (b) $\alpha = 90°$.

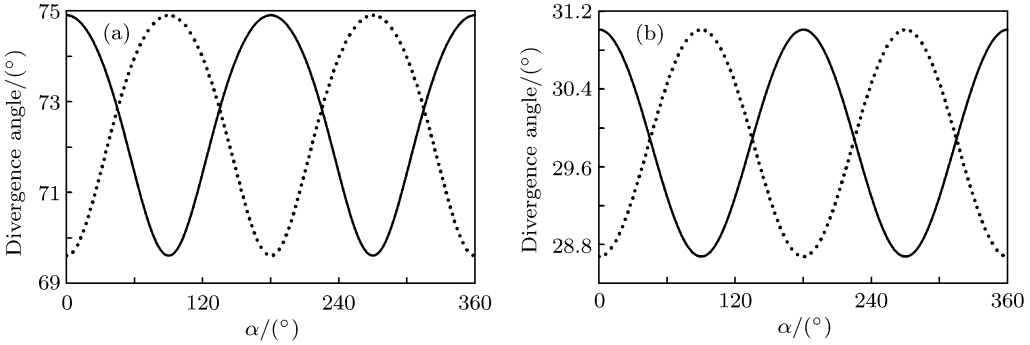


Fig. 7. The divergence angle of the diffracted planar wave as a function of the linearly polarized angle. The solid and the dotted curves correspond to the x- and y-directions, respectively. (a) $R/\lambda = 0.25$, (b) $R/\lambda = 10$.

Figure 6 represents the far-field divergence angle of the diffracted plane wave versus the ratio R/λ. Figure 6 can be regarded as the modulation of Figs. 2 and 4. With the increase of the ratio R/λ, the divergence angles of the diffracted plane wave decrease. The divergence angle of the diffracted plane wave versus the linearly polarized angle is depicted in Fig. 7. When R/λ is large enough such as $R/\lambda = 10$, the variational range of the divergence angle is small. Therefore, the influence of the linearly polarized state on the divergence angle of the diffracted plane wave is omitted in the framework of the scalar paraxial approximation.

4. Conclusions

By using the stationaryphase method combined with the angular-spectrum representation, the analytical expressions for the TE and TM terms of a vectorial plane wave diffracted by a circular aperture are derived in the far-field. The expressions of the energy flux distributions of the TE term, the TM term and the diffracted plane wave are also presented in the far-field. The contributions of the power of the TE and TM terms to that of the diffracted plane wave are examined in the far-field. The far-field divergence angles of the TE term, the TM term and the diffracted plane wave, which are related to the energy flux distributions and the powers of the TE term, the TM term and the diffracted plane wave, are derived and expressed in the integral form. The far-field divergence angles of the TE term, the TM term and the diffracted plane wave are determined by the ratio R/λ and the linearly polarized angle. The influences of the ratio R/λ and the linearly polarized angle on the far-field divergence angles of the TE term, the TM term and the diffracted plane wave are numerically investigated. The different energy flux distributions of the TE and TM terms result in the discrepancy of their divergence angles. This research may promote the recognition of the optical propagation through a circular aperture.[21,22]

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