Generation-channel interference of high-order harmonics of an anharmonic oscillator driven by bi-chrime laser fields and its mechanism*

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A method to describe the generation channels of high-order harmonics is proposed. According to this method, the mechanism of generation-channel interference of high-order harmonics is revealed clearly. We take the anharmonic oscillator driven by bi-chrome fields as an example to illustrate that this method can be used to understand the effect of generation-channel interference.

Keywords: generation-channel interference, anharmonic oscillator, high-order harmonics

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1. Introduction

In recent decades, the high-order harmonic generation (HHG) has become an important field of research for studying the interaction between atom and intense laser field (laser intensity $I > 10^{13}$ W/cm^2). The HHG is the phenomenon that[1,2] when the atoms are exposed to an intense laser field, the secondary radiation, whose frequency is integral multiple of that of the original field, takes place. In this process, typically, only the odd harmonics are observed and the harmonic spectrum is characterized by an initial decline in the intensity with the increase of the harmonic order and finally a sharp cut-off, beyond which there is no further harmonic emission. These characteristics can be interpreted by the two-step (Simpleman) model.[3,4]

In 2007, Cai and Qiao[5] found an interesting phenomenon when they studied the HHG of hydrogen atom, i.e., the generation-channel interference of high-order harmonics. This phenomenon was further studied in their subsequent work.[6] In Ref. [5], the authors pointed out that in the harmonic spectrum of the hydrogen atom exposed to two laser fields of different frequencies, the amplitudes of some harmonics would vary observably when the ratio between the intensities of the two laser fields changed and would even completely vanish at some ratios. They referred to this phenomenon as the pathway interference of high-order harmonics.

One-dimensional (1D) harmonic and anharmonic oscillators are both very simple physical systems and each takes a very important part in the description of the interaction between light and material in both classical and quantum theories. In quantum mechanics, the harmonic and the anharmonic oscillators have been the touchstone of new theories.[7−9] The study of the HHG of the anharmonic oscillator is an interesting work. Balcou *et al.*[10] studied the HHG by using a slowly driven Duffing anharmonic oscillator and pointed out that the power spectrum displays a general character of the HHG. In Ref. [10], the Duffing anharmonic oscillator was treated by classical and quantum mechanics respectively and the characteristics of the HHG could be interpreted simply based on the classical theory. Furthermore, the authors studied the similarity between the classical damping and the quantum mechanical ionization and interpreted the existence of intrinsic phases between the high harmonics and the driving field. Averbukh and Moiseyev[11] studied the HHG in classical and quantum anharmonic oscillators driven by a monochromatic laser field. They used the results of classical theory to interpret the emission probability of the k-order harmonic and applied their method to the calculation of the HHG in various kinds of model potentials.

Though in Refs. [5] and [6] the concept of generation channel of high-order harmonics was given and the effect of generation-channel interference was il-

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lustrated, how this phenomenon happened was not explained. The mechanism of generation-channel interference was still inexplicable and mysterious in Refs. [5] and [6]. The main purpose of our paper is to put forward a method to describe the generation channels of high-order harmonics, based on which the mechanism of generation-channel interference of high-order harmonics becomes understandable. In this paper, we use the pseudo-spectral operator-split method to numerically solve the time-dependent Schrödinger equation of the anharmonic oscillator driven by bi-chrome laser fields and finally obtain the power spectrum of the high-order harmonics. In the spectrum, we also find obvious generation-channel interferences of the high-order harmonics. According to the analogy with the Fourier series expansion method used in classical theory, we put forward a mathematical description of the generation channels of the harmonics and thus reveal the mechanism of the effect of generation-channel interference. Because of the characteristic of our numerical method, the terms that produce the effect of interference cannot be separated from each other. This limitation reduces the persuasion of our argument to a certain extent. For the sake of confirming the conviction of our idea, we use a semi-analytical method to solve the classical dynamical equation of the anharmonics oscillator driven by bi-chrome laser fields approximately. From the solutions, we find that the generation channels of the harmonics are separated from each other and the mechanism of generation-channel interference of high-order harmonics is revealed naturally. Moreover, the results obtained from this semi-analytical method accord with that obtained from the quantum method approximately. From the point of view of the correspondence principle, the phenomenon of the generation-channel interference of high-order harmonics can be interpreted in quantum theory.

In Section 2, we review the concept of harmonics in classical theory, the generation channels of high-order harmonics of a hydrogen atom and the effect of generation-channel interference. In Section 3, we put forward a method to describe the generation channels and to interpret the generation-channel interference. We calculate the high-order harmonics of the anharmonic oscillator driven by bi-chrome fields by using quantum and classical theories respectively. In Section 5, we give a short summary of our work.

2. Review of generation-channel interference

2.1. high-order harmonics in classical theory

Considering a classical anharmonic oscillator driven by a periodic field, the motion equation is

$$\ddot{x} + \omega_0^2 x + \lambda x^3 = p \sin \omega t, \tag{1}$$

where p is the amplitude of the driven force, ω is the corresponding frequency, ω_0 and λ are the parameters characterizing the system. For simplicity, the mass of the oscillator is assumed to be unity. This equation is the famous Duffing equation,[12] a typical non-linear differential equation. The equation has no full analytical solution but can be solved by using the method of progressive approximation. It has the following approximate periodic solution:[13]

$$\begin{aligned} x(t) = {} & a_1 \sin \omega t + a_2 \sin 3\omega t + a_3 \sin 5\omega t \\ & + a_4 \sin 7\omega t + a_5 \sin 9\omega t, \end{aligned} \tag{2}$$

where $a_1 \sin \omega t$ is the main harmonic term, $a_2 \sin 3\omega t$, $a_3 \sin 5\omega t$, $a_4 \sin 7\omega t, \ldots$ are the secondary harmonic terms. The non-linear term λx^3 in Eq. (1) causes the secondary harmonics. If the oscillator is charged and the periodic external field is light, then according to classical electrodynamics,[14] the anharmonic oscillator will emit harmonic radiation, whose frequency is odd multiple of that of the incident light.

If the external fields are two periodic fields with different frequencies, then Eq. (1) will become

$$\ddot{x} + \omega_0^2 x + \lambda x^3 = p_1 \sin \Omega_1 t + p_2 \sin \Omega_2 t, \tag{3}$$

where p_1 and p_2 are amplitudes of the two external fields, Ω_1 and Ω_2 are the corresponding frequencies. Equation (3) has no full analytical solution either. Its approximate solution has the following form:

$$\begin{aligned} x(t) = {} & a_1 \sin \Omega_1 t + a_2 \sin 3\Omega_1 t \\ & + a_3 \sin 5\Omega_1 t + a_4 \sin 7\Omega_1 t + \cdots \\ & + b_1 \sin \Omega_2 t + b_2 \sin 3\Omega_2 t \\ & + b_3 \sin 5\Omega_2 t + b_4 \sin 7\Omega_2 t + \cdots \\ & + c_1 \sin(2\Omega_1 - \Omega_2)t + c_2 \sin(2\Omega_1 + \Omega_2)t \\ & + c_3 \sin(2\Omega_2 - \Omega_1)t \\ & + c_4 \sin(2\Omega_2 + \Omega_1)t + \cdots \end{aligned} \tag{4}$$

As can be seen, it contains not only the harmonic terms, such as $\sin 3\Omega_1 t$, $\sin 5\Omega_1 t$, $\sin 3\Omega_2 t$, $\sin 5\Omega_2 t, \ldots$ but also the combination harmonic terms $\sin(2\Omega_1 - \Omega_2)t$, $\sin(2\Omega_1 + \Omega_2)t$, $\sin(2\Omega_2 - \Omega_1)t$, $\sin(2\Omega_2 + \Omega_1)t, \ldots$

2.2. Generation channel of high-order harmonics and effect of generation-channel interference

In Ref. [5], the high-order harmonics of a hydrogen atom driven by two linear-polarized laser fields were studied. The bi-chrome laser fields used in the reference are given by

$$E(t) = E_0 \sin^2(\pi t/\tau_{\rm p})[\sin(\omega_1 t) + \beta \sin(\omega_2 t)], \quad (5)$$

where E_0 is the amplitude of the first laser field, $\tau_{\rm p}$ is the duration of the laser pulse, β is the ratio between the amplitudes of the two laser fields and ω_1 and ω_2 are the frequencies of the first and the second lasers, respectively. For convenience, the authors of Ref. [5] defined a reference frequency ω_0 and assumed $\omega_1 = 3\omega_0$ and $\omega_2 = 5\omega_0$. By direct numerically integrating the time-dependent Schrödinger equation, they obtained the wave function of the hydrogen atom driven by bi-chrome laser fields and thus obtained the dipole moment in the length form given by

$$d_{\rm a}(t) = \langle \Psi(\boldsymbol{r}, t)| - \frac{z}{r^3} + E(t)\,|\Psi(\boldsymbol{r}, t)\rangle\,. \quad (6)$$

The power spectrum of the high-order harmonics can be acquired from the Fourier transformation of the dipole moment as

$$P(\omega) = \left| \frac{1}{t_{\rm f} - t_{\rm i}} \frac{1}{\omega^2} \int_{t_{\rm i}}^{t_{\rm f}} d_{\rm a}(t)\, {\rm e}^{-{\rm i}\,\omega t}\, {\rm d}t \right|^2 . \quad (7)$$

They also pointed out that only the harmonics whose frequencies are the odd multiple of the reference frequency could be found in the power spectrum, namely, the frequencies of the harmonics must satisfy

$$\omega = (2n+1)\omega_0, \quad n = 0, 1, 2, \ldots$$

For the harmonic $\omega = 15\omega_0$, there are two main generation channels, i.e.,

$$\mathrm{H} + 5\gamma(3\hbar\omega_0) \rightarrow \mathrm{H} + \gamma(15\hbar\omega_0), \quad (8a)$$
$$\mathrm{H} + 3\gamma(5\hbar\omega_0) \rightarrow \mathrm{H} + \gamma(15\hbar\omega_0). \quad (8b)$$

Equation (8a) denotes that the hydrogen atom emits one photon with frequency $15\omega_0$ after it has absorbed 5 photons with frequency ω_1 ($\omega_1 = 3\omega_0$); and equation (8b) indicates that the hydrogen atom emits one photon with frequency $15\omega_0$ after it has absorbed 3 photons with frequency ω_2 ($\omega_2 = 5\omega_0$). Of course, there are other generation channels for the harmonic $\omega = 15\omega_0$, however their contributions are very small and can be neglected in practice. For the harmonic $\omega = 13\omega_0$, it also has two main generation channels as follows:

$$\mathrm{H} + \gamma(3\hbar\omega_0) + 2\gamma(5\hbar\omega_0) \rightarrow \mathrm{H} + \gamma(13\hbar\omega_0), \quad (9a)$$
$$\mathrm{H} + 6\gamma(3\hbar\omega_0) - \gamma(5\hbar\omega_0) \rightarrow H + \gamma(13\hbar\omega_0). \quad (9b)$$

Equation (9a) denotes that the hydrogen atom emits one photon with frequency $13\omega_0$ after it has absorbed 1 photon with frequency ω_1 and 2 photons with frequency ω_2; and Eq. (9b) shows that the hydrogen atom emits 1 photon with frequency $13\omega_0$ after it has absorbed 6 photons with frequency ω_1 and sent 1 photon with frequency ω_2. Different from Eqs. (8a) and (8b), equations (9a) and (9b) show a kind of mixed process. For the harmonic $\omega = 13\omega_0$, there are still many other similar generation channels, the whole process of which satisfies the various principles of conservation, such as parity conservation, energy conservation, angular moment conservation. They pointed out that by making a slight variation of the frequency of one of the two laser fields, for example $\omega_2 = 5\omega_0 \rightarrow \omega_2 = 5.1\omega_0$, the channel or channels taking chief part in a certain order harmonic generation progress can be identified.

By tuning the amplitude (denoted by parameter β) of the second laser field, they found that the amplitudes of some harmonics would change significantly; and certain harmonics will completely disappear at some values of β. They called the phenomenon, in which the amplitudes of the harmonics change with β, the pathway interference of the high-order harmonics and called the extreme case, in which the harmonics completely disappear, the harmonic collapse. Take the harmonic $\omega = 15\omega_0$ for example, when $\beta = 0.35$, it will completely disappear . It is similar to the interference of light. For the harmonics $\omega = 29\omega_0$, $\omega = 33\omega_0, \ldots$, there also exist such kind of interference phenomena.

3. Spectra of driven anharmonic oscillator and effect of generation-channel interference

3.1. Method for describing the generation channels and the effect of generation-channel interference of high-order harmonics

The generation channels and the effect of generation-channel interference of high-order harmonics have been put forward in Ref. [5], while the mechanism of generation-channel interference has not been

explored. It can be seen without any difficulty that these concepts mainly stand on the viewpoint of photon. The main purpose of our paper is to consider the concept of the generation channel and give a mathematical description of the generation channels. Our mathematical description of the generation channels is quantitative and the mechanism of the generation-channel interference will be revealed naturally according to the description. The starting point is the Fourier series expansion of the orbit of the periodic motion.

In Ref. [5], when the dipole moment in the length form (Eq. (6)) was obtained, the Fourier transformation (Eq. (7)) was made as usual and thus the power spectrum of high-order harmonics was obtained. If no Fourier transformation but the Fourier series expansion of the dipole moment is made, then we will have the following equation:

$$\begin{aligned} d_{\rm a}(t) = {} & c_1 \sin\omega_0 t + c_2 \sin 3\omega_0 t + c_3 \sin 5\omega_0 t \\ & + c_4 \sin 7\omega_0 t + \cdots . \end{aligned} \tag{10}$$

We can also obtain the power spectrum of the harmonics if we make the Fourier transformation of each harmonic term separately. There are two bases on which we can make this expansion. One is similar to that of classical theory and the other one is the discrete line form, which is also the form obtained from the Fourier transformation of $d_{\rm a}(t)$. We can make the Fourier series expansion to the discrete line form, because according to the Fourier transformation we know that the Fourier transformation of sine function or cosine function is in the line form. The value of the dipole moment $d_{\rm a}(t)$ in Eq. (6) is obtained by numerically integrating the time-dependent Schrödinger equation on the conditions $\omega_1 = 3\omega_0$, $\omega_2 = 5\omega_0$ and $\omega_0 = 0.0143$ arb. units. Now, if we do not make the special presupposition $\omega_1 = 3\omega_0$, $\omega_2 = 5\omega_0$, then similar to Eq. (4), equation (10) will have the following form:

$$\begin{aligned} d_{\rm a}(t) = {} & a_1 \sin\omega_1 t + a_2 \sin 3\omega_1 t \\ & + a_3 \sin 5\omega_1 t + a_4 \sin 7\omega_1 t + \cdots \\ & + b_1 \sin\omega_2 t + b_2 \sin 3\omega_2 t \\ & + b_3 \sin 5\omega_2 t + b_4 \sin 7\omega_2 t + \cdots \\ & + d_1 \sin(2\omega_1 - \omega_2)t + d_2 \sin(2\omega_1 + \omega_2)t \\ & + d_3 \sin(2\omega_2 - \omega_1)t \\ & + d_4 \sin(2\omega_2 + \omega_1)t + \cdots \end{aligned} \tag{11}$$

Assume $\omega_1 = 3\omega_0$ and $\omega_2 = 5\omega_0$, then equation (11) will accord with Eq. (10). It illustrates that there is a corresponding relation between the classical (Eq. (4)) and the quantum (Eq. (6)) results.

The terms in Eq. (11) corresponding to $c_7 \sin 15\omega_0 t$ in Eq. (10) are $a_3 \sin 5\omega_1 t$, $b_2 \sin 3\omega_2 t$, $d_{\rm i} \sin(-5\omega_1 + 6\omega_2)t$, $d_j \sin(10\omega_1 - 3\omega_2)t, \ldots$. Obviously, when $\omega_1 = 3\omega_0$ and $\omega_2 = 5\omega_0$, the above sine terms will all become $\sin 15\omega_0 t$. So we call these terms the generation channels of harmonic $15\omega_0$. The coefficients a_3, b_2, $d_{\rm i}$, $d_j, \ldots$ denote the amplitudes of the corresponding channels and the expression $|a_3 + b_2 + d_{\rm i} + d_j + \cdots|^2$ denotes the amplitude of the harmonic $15\omega_0$. This case is very similar to the interference of two or more waves. Thus the mechanism of generation-channel interference of high-order harmonics is understandable according to the above arguments.

In the following, we will take the anharmonic oscillator driven by bi-chrome fields as an example to show that our ideas are effective for describing the generation channels and for understanding the effect of generation-channel interference. Our calculations will be carried out in quantum and classical theories respectively.

3.2. Treatment of the anharmonic oscillator driven by bi-chrome fields in quantum theory

The Schrödinger equation corresponding to Eq. (3) is (in atomic units)

$$\mathrm{i}\,\frac{\partial}{\partial t}\Psi(x,t) = H\Psi(x,t),$$

with

$$\begin{aligned} H &= -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}\omega_0^2 x^2 + \frac{1}{4}\lambda x^4 - xE(t), \\ E(t) &= E_0(\sin\Omega_1 t + \beta \sin\Omega_2 t), \end{aligned} \tag{12}$$

where $E(t)$ denotes the resultant field of the two laser fields that have the same polarization orientation, Ω_1 and Ω_2 are the corresponding frequencies. In the electric dipole approximation, we only consider the electrical vector component. The symbol β is an adjustable parameter, which denotes the ratio between the amplitudes of the two lasers.

We use the pseudo-spectral operator-split method[15,16] to numerically integrate Eq. (12). Choosing the ground state wave function of the system without the external laser fields as the initial wave function, we obtain the time-dependent wave function $\Psi(x,t)$ by using the time evolution. Then the dipole moment in length form is given by

$$d_{\rm l}(t) = \langle \Psi(x,t)|\, x\, |\Psi(x,t)\rangle\,. \tag{13}$$

The power spectrum of the high-order harmonics can be acquired from the Fourier transformation of Eq. (13) as

$$P(\omega) = \left| \frac{1}{t_{\rm f} - t_{\rm i}} \int_{t_{\rm i}}^{t_{\rm f}} d_{\rm l}(t)\, {\rm e}^{-{\rm i}\,\omega t}\, {\rm d}t \right|^2 . \qquad (14)$$

In the calculation, we choose $\omega_0 = 0.2$ arb. units, $\lambda = 0.016$ arb. units, $E_0 = 1.0$ arb. units, $\Omega_1 = 3\Omega_0$, $\Omega_2 = 5\Omega_0$ and $\Omega_0 = 0.01$ arb. units, where Ω_0 is the reference frequency.

Figure 1(a) is the power spectrum of the high-order harmonics, which is obtained on the condition that only the first laser field is present. The abscissa denotes the frequency in units of $\Omega_0 = 0.01$ arb. units; the ordinate denotes the intensity of the harmonic. Because the frequency of the first laser is $\Omega_1 = 3\Omega_0$, the harmonics appearing in the graph are the first, the third, the fifth, the seventh and the ninth orders. It is worth emphasizing that the corresponding classical Eq. (1) in this condition also gives a similar result (Eq. (2)).

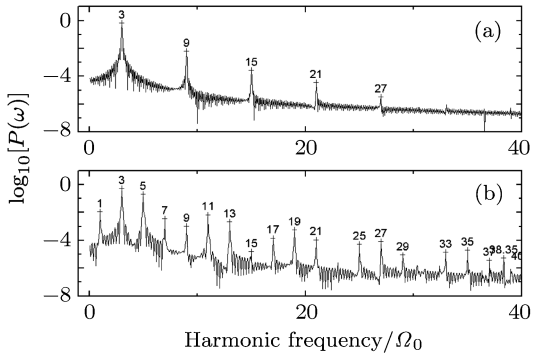


Fig. 1. Spectra of high-order harmonics of the anharmonic oscillator at (a) $\beta = 0$ and (b) $\beta = 0.7$.

Figure 1(b) shows the power spectrum of the high-order harmonics at $\beta = 0.7$. From this figure, we can find that there are not only the harmonics whose frequencies are odd multiple of that of the first or the second laser such as $3\Omega_0$, $9\Omega_0$, $15\Omega_0$, $5\Omega_0$, $25\Omega_0$, and $35\Omega_0$, but also the harmonics whose frequencies are the combinations of Ω_1 and Ω_2. It accords with the result (Eq. (4)) of the classical theory (Eq. (3)) qualitatively. So we can conclude that there is a one–one correspondence relation between the spectrum of the classical anharmonic oscillator and that of the quantum one. It should be emphasized that this one–one correspondence relation is only qualitative. The quantitative relation should be obtained by practical calculations.

Figures 2(a)–2(c) show the power spectra of the high-order harmonics of the anharmonic oscillator driven by bi-chrome laser fields on the conditions $\beta = 0.5$, 1.0, and 1.5, respectively. They illustrate the effect of generation-channel interference of the high-order harmonics.

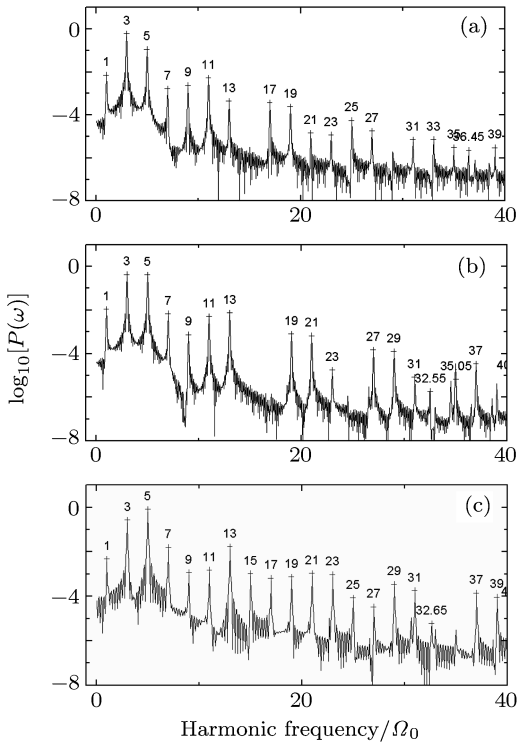


Fig. 2. Spectra of high-order harmonics of the anharmonic oscillator at (a) $\beta = 0.5$, (b) $\beta = 1.0$, and (c) $\beta = 1.5$.

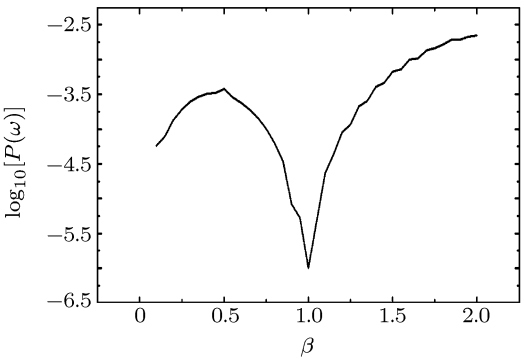


Fig. 3. Dependence of the intensity of the 17-th harmonic on β.

We focus on the 17-th harmonic. In Fig. 2(a) ($\beta = 0.5$), the 17-th harmonic is obviously visible and has a considerable amplitude. While in Fig. 2(b) ($\beta = 1.0$), the 17-th harmonic is submerged in the background of noise. The 17-th harmonic re-emerges in Fig. 2(c) ($\beta = 1.5$) with a considerable amplitude. Figure 3 illustrates the dependence of the 17-th harmonic on β. It can be seen that the amplitude of the 17-th harmonic increases with the increase of β and

reaches a maximum at about $\beta = 0.5$, then begins to decrease with the increase of β and reaches a minimum at about $\beta = 1.0$ and then begins to rise again. This is the effect of generation-channel interference found in Ref. [5].

We have shown how to understand this effect in the previous sections. According to Eq. (11) and supposing that the dipole moment $d_{\rm a}(t)$ obtained by numerically integrating the Schrödinger equation can be expanded in Fourier series, then the series will contain the various harmonics in the form $A_i \sin(\pm m\Omega_1 \pm n\Omega_2)$, there m and n should satisfy the following conditions according to the principle of parity conservation:

$$m + n = \text{odd}, \quad m, n \in \mathbf{N}. \tag{15}$$

When $\Omega_1 = 3\Omega_0$ and $\Omega_2 = 5\Omega_0$, various (m, n) values that satisfy $\pm m\Omega_1 \pm n\Omega_2 = 17\Omega_0$ will give the generation channel $A_i \sin(\pm m\Omega_1 \pm n\Omega_2)$ corresponding to the 17-th harmonic, e.g., $A_1 \sin(-\Omega_1 + 4\Omega_2)t$, $A_2 \sin(4\Omega_1 + \Omega_2)t$, $A_3 \sin(9\Omega_1 - 2\Omega_2)t$, ... So, the intensity of the 17-th harmonic is proportional to $|A_1 + A_2 + A_3 + \cdots|^2$. The A_i depends on the parameters ω_0 and λ of the system and E_0, Ω_1 and Ω_2 of the external field, namely,

$$A_i = A_i(\omega_0, \lambda, E_0, \Omega_1, \Omega_2). \tag{16}$$

As we can solve Eq. (14) numerically but not analytically, the coefficient of $\sin(17\Omega_0 t)$ contained in $d_{\rm a}(t)$ is only a number equal to $|A_1 + A_2 + A_3 + \cdots|$, which denotes the amplitude of the 17-th harmonic directly, namely, we have obtained the final result after the interference.

Though we cannot solve Eq. (12) analytically, we can analytically solve its classical counterpart (Eq. (3)) by an approximate method. From the approximately analytical result, we will find that there is also generation-channel interference of high-order harmonics in classical theory and that the generation-channel interfering with each other is visible.

3.3. Treatment of the anharmonic oscillator driven by bi-chrome fields in classical theory

We consider Eq. (3) again and rewrite it in the following form:

$$\ddot{x} = -\omega_0^2 x - \lambda x^3 + p_1 \sin \Omega_1 t + p_2 \sin \Omega_2 t. \tag{17}$$

Equation (17) can be solved by using the method of progressive approximations[17] as

$$\ddot{x}_n = -\omega_0^2 x_{n-1} - \lambda x_{n-1}^3 + p_1 \sin \Omega_1 t + p_2 \sin \Omega_2 t. \tag{18}$$

We choose

$$x_0 = A_1 \sin \Omega_1 t + A_2 \sin \Omega_2 t \tag{19}$$

as the zero-order approximation.

Substituting Eq. (19) into Eq. (18) and reordering the result, we have

$$\begin{aligned}
\ddot{x}_1 = & -\left[\omega_0^2 A_2 + \frac{3}{4}\lambda(A_2^3 + 2A_2A_1^2) - p_2\right] \sin \Omega_2 t \\
& + \frac{1}{4}\lambda A_1^3 \sin 3\Omega_1 t + \frac{1}{4}\lambda A_2^3 \sin 3\Omega_2 t \\
& - \frac{3}{4}\lambda A_1^2 A_2 [\sin(2\Omega_1 - \Omega_2)t - \sin(2\Omega_1 + \Omega_2)t] \\
& - \frac{3}{4}\lambda A_2^2 A_1 [\sin(2\Omega_2 - \Omega_1)t \\
& - \sin(2\Omega_2 + \Omega_1)t].
\end{aligned} \tag{20}$$

Integrating Eq. (20) twice and omitting the integral constants, we obtain the periodic solution

$$\begin{aligned}
x_1 = & \frac{1}{\Omega_1^2}\left[\omega_0^2 A_1 + \frac{3}{4}\lambda(A_1^3 + 2A_1A_2^2) - p_1\right] \sin \Omega_1 t \\
& + \frac{1}{\Omega_2^2}\left[\omega_0^2 A_2 + \frac{3}{4}\lambda(A_2^3 + 2A_2A_1^2) - p_2\right] \sin \Omega_2 t \\
& - \frac{1}{36\Omega_1^2}\lambda A_1^3 \sin 3\Omega_1 t - \frac{1}{36\Omega_2^2}\lambda A_2^3 \sin 3\Omega_2 t \\
& + \frac{3}{4}\lambda A_1^2 A_2 \left[\frac{1}{(2\Omega_1 - \Omega_2)^2} \sin(2\Omega_1 - \Omega_2)t \right. \\
& \left. - \frac{1}{(2\Omega_1 + \Omega_2)^2} \sin(2\Omega_1 + \Omega_2)t\right] \\
& + \frac{3}{4}\lambda A_2^2 A_1 \left[\frac{1}{(2\Omega_2 - \Omega_1)^2} \sin(2\Omega_2 - \Omega_1)t \right. \\
& \left. - \frac{1}{(2\Omega_2 + \Omega_1)^2} \sin(2\Omega_2 + \Omega_1)t\right].
\end{aligned} \tag{21}$$

Comparing Eq. (21) with Eq. (19) yields

$$\begin{cases}
(\Omega_1^2 - \omega_0^2)A_1 - \dfrac{3}{4}\lambda A_1(A_1^2 + 2A_2^2) = p_1, \\
(\Omega_2^2 - \omega_0^2)A_2 - \dfrac{3}{4}\lambda A_2(A_2^2 + 2A_1^2) = p_2.
\end{cases} \tag{22}$$

Let

$$\begin{aligned}
b_1 &= \frac{1}{\Omega_1^2}\left[\omega_0^2 A_1 + \frac{3}{4}\lambda(A_1^3 + 2A_1A_2^2) - p_1\right], \\
b_2 &= \frac{1}{\Omega_2^2}\left[\omega_0^2 A_2 + \frac{3}{4}\lambda(A_2^3 + 2A_2A_1^2) - p_2\right], \\
b_3 &= -\frac{1}{36\Omega_1^2}\lambda A_1^3, \quad b_4 = -\frac{1}{36\Omega_2^2}\lambda A_2^3, \\
b_5 &= \frac{3}{4}\lambda A_1^2 A_2 \frac{1}{(2\Omega_1 - \Omega_2)^2}, \\
b_6 &= -\frac{3}{4}\lambda A_1^2 A_2 \frac{1}{(2\Omega_1 + \Omega_2)^2}, \\
b_7 &= \frac{3}{4}\lambda A_2^2 A_1 \frac{1}{(2\Omega_2 - \Omega_1)^2},
\end{aligned}$$

$$b_8 = -\frac{3}{4}\lambda A_2^2 A_1 \frac{1}{(2\Omega_2+\Omega_1)^2}, \tag{23}$$

then

$$\begin{aligned} x_1 = {} & b_1\sin\Omega_1 t + b_2\sin\Omega_2 t + b_3\sin 3\Omega_1 t + b_4\sin 3\Omega_2 t \\ & + b_5\sin(2\Omega_1-\Omega_2)t + b_6\sin(2\Omega_1+\Omega_2)t \\ & + b_7\sin(2\Omega_2-\Omega_1)t + b_8\sin(2\Omega_2+\Omega_1)t. \end{aligned} \tag{24}$$

From Eq. (18), we can obtain the secondary approximate solution

$$\ddot{x}_2 = -\omega_0^2 x_1 - \lambda x_1^3 + p_1\sin\Omega_1 t + p_2\sin\Omega_2 t. \tag{25}$$

Substituting Eq. (24) into Eq. (25) and re-ordering the result, we can find that it contains various harmonics whose frequencies are the combinations of Ω_1 and Ω_2 in $\ddot{x}_2$. This operation can be accomplished with the help of the software Mathematica. Because it contains too many terms in $\ddot{x}_2$, we do not list all of them explicitly but list only some order harmonics specifically when necessary. For example, when $\Omega_1 = 3\Omega_0$, $\Omega_2 = 5\Omega_0$, the $\sin 17\Omega_0 t$ terms contained in Eq. (25) are

$$\begin{aligned} & -\frac{3}{4}\lambda(2b_1b_2b_4 + 2b_1b_4b_5 + 2b_3b_4b_6 - 2b_2b_4b_7 \\ & + 2b_5b_6b_7 - b_1b_7^2 + b_4^2b_3 + 2b_2b_5b_3 + 2b_4b_6b_8 \\ & + 2b_1b_7b_8 + b_3b_8^2)\sin(-\Omega_1+4\Omega_2)t \end{aligned}$$

and

$$\begin{aligned} & \frac{3}{4}\lambda(2b_1b_2b_3 - b_3^2b_5 + b_1^2b_6 - 2b_1b_3b_6 + 2b_2b_5b_6 \\ & - b_2b_6^2 - 2b_3b_4b_7 + 2b_3b_5b_7 - 2b_2b_3b_8 \\ & + 2b_1b_5b_8 - 2b_6b_7b_8)\sin(4\Omega_1+\Omega_2)t. \end{aligned}$$

Let

$$\begin{aligned} C_1 = {} & -\frac{3}{4}\lambda(2b_1b_2b_4 + 2b_1b_4b_5 + 2b_3b_4b_6 \\ & - 2b_2b_4b_7 + 2b_5b_6b_7 - b_1b_7^2 + b_4^2b_3 + 2b_2b_5b_3 \\ & + 2b_4b_6b_8 + 2b_1b_7b_8 + b_3b_8^2), \\ C_2 = {} & \frac{3}{4}\lambda(2b_1b_2b_3 - b_3^2b_5 + b_1^2b_6 - 2b_1b_3b_6 \\ & + 2b_2b_5b_6 - b_2b_6^2 - 2b_3b_4b_7 + 2b_3b_5b_7 \\ & - 2b_2b_3b_8 + 2b_1b_5b_8 - 2b_6b_7b_8), \end{aligned}$$

then the $\sin 17\Omega_0 t$ term contained in (25) is $(C_1 + C_2)\sin 17\Omega_0 t$. According to the laws of classical electrodynamics, the relationship between the intensity of the 17-order harmonic and C_1, C_2 is

$$I \propto |C_1 + C_2|^2. \tag{26}$$

We can determine A_1, A_2, $b_{\rm i}$ $(i = 1, 2, \ldots, 8)$, C_1 and C_2 each as a function of ω_0, λ, p_1, p_2, Ω_1, Ω_2. Finally, we have

$$\begin{aligned} I_{17} \propto {} & |C_1(\omega_0, \lambda, p_1, p_2, \Omega_1, \Omega_2) \\ & + C_2(\omega_0, \lambda, p_1, p_2, \Omega_1, \Omega_2)|^2. \end{aligned} \tag{27}$$

Therefore the generation-channel interfering with each other is visible. Unfortunately, Eqs. (22) are coupled binary cubic equations, namely, nonlinear equations. They can only be solved numerically. In the following, we will give the values of C_1 and C_2 with p_2 in the interval of [0, 2].

Let $\omega_0 = 0.2$ arb. units, $\lambda = 0.016$ arb. units, $p_1 = 1.0$ arb. units, $\Omega_1 = 3\Omega_0$ and $\Omega_2 = 5\Omega_0$, where $\Omega_0 = 0.01$ arb. units is the reference frequency; these parameters are the same as those in Eq. (12). We fix p_1 and choose p_2 as the tunable parameter corresponding to β in Eq. (12). Our purpose is to determine the values of C_1 and C_2 at different p_2. We omit the calculation process and give only the results in Table 1.

Table 1. Corresponding values of C_1 and C_2 when p_2 is in the interval of [0, 2], all in the unit of 10^6 arb. units.

p_2	C_1	C_2	$\lvert C_1+C_2\rvert^2$
0.0	0.0	0.0	0.0
0.1	–0.0104	1.05	1.08
0.2	–0.0609	2.22	4.62
0.3	–0.189	3.59	11.56
0.4	–0.446	5.26	23.14
0.5	–0.892	7.28	40.83
0.6	–1.60	9.68	65.29
0.7	–2.66	12.4	94.67
0.8	–4.09	15.1	121.0
0.9	–5.80	17.0	125.44
1.0	–7.38	17.3	98.41
1.1	–8.55	16.4	61.47
1.2	–9.44	15.2	32.83
1.3	–10.2	14.0	14.44
1.4	–10.9	13.0	4.24
1.5	–11.6	12.1	0.02
1.6	–12.3	11.3	0.96
1.7	–12.9	10.6	5.52
1.8	–13.6	9.97	13.47
1.9	–14.4	9.44	24.3
2.0	–15.1	8.95	38.07

Table 1 lists the values of C_1, C_2 and $|C_1 + C_2|$ when $p_1 = 1.0$ arb. units (the intensity of the first laser field) and p_2 (the intensity of the second laser field) is in the interval of [0, 2]. Figure 4 illustrates the dependence of $|C_1 + C_2|^2$ on p_2. As can be seen, when $p_2 = 0$, $|C_1 + C_2|^2 = 0$, namely, the intensity of the 17-th harmonic is zero, which accords with the result of the quantum theory (Fig. 1(a)). The $|C_1 + C_2|^2$ increases with the increase of p_2 and reaches a maximum at a certain value of p_2, then begins to decrease until it reaches a minimum and then begins to increase again. We conclude that according to the approximately semi-analytical solution of Eq. (3), there

is also the effect of generation-channel interference in the classical theory and the generation-channel interfering with each other is visible. By comparing Fig. 4 with Fig. 3, we find that though the shapes of the two curves are not exactly the same, their variation trends are the same. They both illustrate the effect of generation-channel interference of high-order harmonics. The same conclusion holds for other order harmonics (see Fig. 5).

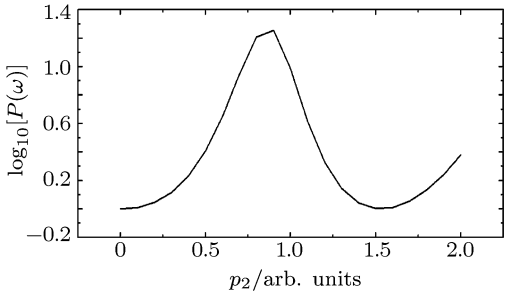


Fig. 4. Dependence of $|C_1 + C_2|^2$ of the 17-th harmonic on p_2.

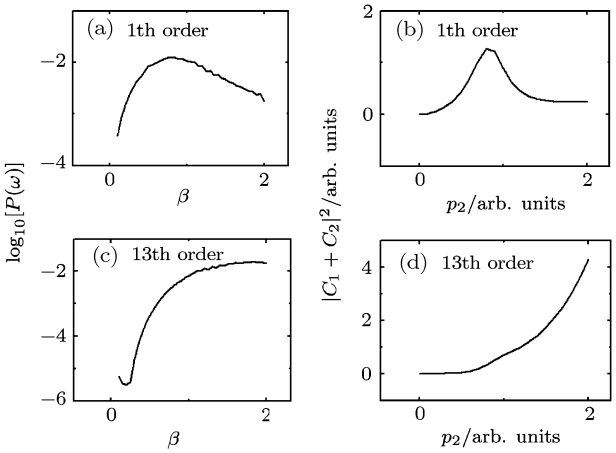


Fig. 5. The generation-channel interference of the 1-th and the 13-th order harmonics calculated by quantum and classical theories, respectively. The dependence of the intensity of the 1-th order harmonic on β calculated (a) by quantum theory and (b) by classical theory. Panels (c) and (d) exhibit the corresponding results of the 13-th order harmonic.

It is worthy to notice that in the quantum-mechanical calculations, the method we used is directly numerically integrating the Schrödinger equation. For a certain order harmonic, this calculation gives the contributions of all possible generation channels. So the result has a good reliability. In the classical calculations, the result has only reached the accuracy of second-order approximation. So it is only a crude estimate. However based on the final result of the classical calculations, the effect of generation-channel interference is still obvious and the variation trend of the interference is similar to that of the quantum-mechanical result.

4. Summary

In this paper, we review the concept of high-order harmonics in classical theory and emphasize that this concept is helpful for understanding the phenomenon of high-order harmonics in quantum theory. Then we review the concept of generation channel and the effect of generation-channel interference, which were first put forward in Ref. [5] but without a proper interpretation. This paper gives a method to describe the generation channels and to interpret the generation-channel interference. The method is established by analysing the high-order harmonics in classical theory. According to the classical theory, the generation channels can be described quantitatively and the mechanism of generation-channel interference is revealed naturally. Because the analytical solution of the time-dependent Schrödinger equation cannot be obtained and only the numerical solution can be given, for a certain order harmonic, we cannot give its generation channels separately but can only give the sum of all possible generation channels, namely, the interference result of all possible generation channels. In order to enhance the persuasion of our viewpoint, we calculate the high-order harmonics of the classical anharmonic oscillator driven by bi-chrome laser fields by using semi-analytical method. In the classical case, the main generation channels of a certain harmonic can be given separately and quantitatively. Therefore the generation channel interfering with each other becomes visible, just like the interference of wave. From the point of view of the correspondence principle, it is reasonable to understand the corresponding phenomenon in quantum theory by using this method.

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