Low frequency asymptotics for the spin-weighted spheroidal equation in the case of $s = 1/2$*

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The spin-weighted spheroidal equation in the case of $s = 1/2$ is thoroughly studied by using the perturbation method from the supersymmetric quantum mechanics. The first-five terms of the superpotential in the series of parameter β are given. The general form for the n-th term of the superpotential is also obtained, which could also be derived from the previous terms W_k, $k < n$. From these results, it is easy to obtain the ground eigenfunction of the equation. Furthermore, the shape-invariance property in the series of parameter β is investigated and is proven to be kept. This nice property guarantees that the excited eigenfunctions in the series form can be obtained from the ground eigenfunction by using the method from the supersymmetric quantum mechanics. We show the perturbation method in supersymmetric quantum mechanics could completely solve the spin-weight spheroidal wave equations in the series form of the small parameter β.

Keywords: spheroidal wave equation, supersymmetric quantum mechanics, superpotential, shape invariance

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1. Introduction

The spin-weighted spheroidal functions first appeared in the study of the stable problem of the Kerr black hole. For the perturbation ψ of the Kerr black hole, Teukolsky wrote the wave equation in the Kinnersley tetrad and Boyer–Lindquist coordinates[1−2]

$$\begin{aligned}&\left[\frac{\left(r^2+a^2\right)}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2}+\frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi}\\&+\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2}-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)\\&-2s\left[\frac{a\left(r-M\right)}{\Delta}+\frac{\mathrm{i}\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi}\\&-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right)\\&-2s\left[\frac{M\left(r^2-a^2\right)}{\Delta}-r-\mathrm{i}\,a\cos\theta\right]\frac{\partial\psi}{\partial t}\\&+\left(s^2\cot^2\theta-s\right)\psi=0,\end{aligned}\tag{1}$$

where a and M represent the angular momentum per unit mass and the mass of the rotating black hole, respectively, $\Delta = r^2 - 2Mr + a^2$, s is the helicity or the spin weight of the perturbation field. Teukolsky made a nice separation of Eq. (1) and obtained the radial equation $R(r)$ as

$$\begin{aligned}&\Delta^{-s}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{s+1}\frac{\mathrm{d}R}{\mathrm{d}r}\right)\\&\quad+\left[\frac{K^2-2\,\mathrm{i}\,s\left(r-M\right)K}{\Delta}+4\,\mathrm{i}\,s\omega r-\bar{\lambda}\right]R=0,\end{aligned}\tag{2}$$

and the angular equations $\Theta(\theta)$ as

$$\begin{aligned}&\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)+s+a^2\omega^2\cos^2\theta\right.\\&\qquad\left.-2as\omega\cos\theta-\frac{\left(m+s\cos\theta\right)^2}{\sin^2\theta}+E\right]\Theta=0,\end{aligned}\tag{3}$$

where $K=(r^2+a^2)\omega-am$ and $\bar{\lambda}=E+a^2\omega^2-2am\omega$. The spin weight s of the perturbation field could be $0,\pm\frac{1}{2},\pm1,\pm2$, which correspond to scalar, neutrino, electromagnetic and gravitational perturbations, respectively. The angular equation (Eq. (3)) is the so-called spin-weighted spheroidal wave equation and its regular solutions are the spin-weighted harmonics (SWSH). The spin-weighted spheroidal functions have been widely used in many fields, such as gravitational wave detection, quantum filed theory in curved space–time, black hole stable problem.[1−4] Further detailed studies on them are very important.[1−4] The Eq. (3), together with the boundary condition that requires Θ

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finite at $\theta = 0,\ \pi$, constitutes a kind of Sturm–liuvelle problem.

We will give a brief discussion on the results of SWSH that have been obtained in the literature. We will also show their advantages and compare them with our results. The SWSH has been studied in two limits: the high frequency and the low frequency limits. According to the standard conventions, the high frequency and the low frequency correspond to the large and the small parameter β, where $\beta = a\omega$, and ω is the frequency of the mode. Our work only focuses on the low frequency side, one could refer to Refs. [3]–[6] for the research works on the high frequency side.

Actually, the small β regime has been extensively investigated (Press and Teukolsky,[7] Leaver,[4] Fackerell and Grossman,[8] Ryuichi Fujita and Hideyuki Tagoshi,[9] etc.). In those works, the spin-weighted spheroidal functions are expanded in the series of parameter β. The differences between them are just the forms of the expansions (Fackerell and Grossman expanded them as series involving Jacobi polynomials, while Ryuichi Fujita and Hideyuki Tagoshi used the hypergeometric functions as the basis functions for the expansion, etc). The previous works all concentrated on the numerical calculation of the eigenvalues. Hence, the numerical calculations can be relatively easily implemented. Nevertheless, the detailed analytic properties of the eigenfunctions and the relations between them also need some more attention. While, the efforts on this aspect are comparatively less. Our investigation then is based on this background.

When $\beta = 0$, the spin-weighted spheroidal functions become the spherical wave equations, whose solutions are the famous associated Legendre's functions P_l^m. The associated Legendre's functions have many nice properties, including the recurrence relations between them. These properties enable us to easily obtain the P_l^m of higher value l from the lower ones. Though the spheroidal wave equations are the extension of the spherical wave equations, their corresponding recurrence relations are not found with the previous methods. Until recent years, with the introduction of supersymmetric quantum mechanics to the study of the spheroidal equations (that is, $s = 0$ case), it becomes possible to solve the problem.[10−16] In Refs. [10]–[17], it is the superpotential that is expanded in the series of the parameter β and new interesting results are obtained. The most valuable result is the recurrence relations, which reduce to that for the associated Legendre's functions when $\beta = 0$.[10−17] This article would apply the method to study the case of $s = 1/2$.

This paper is divided into two parts. In the first part, we would use the approximation methods from the supersymmetric quantum mechanics to calculate the ground eigenvalue and the ground eigenfunction for the spheroidal functions. In the second part, we investigate the shape-invariance property and obtain the excited eigenfunctions in the series form from the ground eigenfunctions. Of course, the relations between the ground and the excited eigenfunctions are the most interesting result in this work.

2. The first several terms of the superpotential

In order to employ the technique from the supersymmetric quantum mechanics (SUSYQM), we first transform Eq. (3) into the Schrödinger form[10−17]

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2} + \left[\frac{1}{4} + s + \beta^2\cos^2\theta - 2s\beta\cos\theta - \frac{(m + s\cos\theta)^2 - 1/4}{\sin^2\theta} + E\right]\Psi = 0 \quad (4)$$

by using the transformation

$$\Theta(\theta) = \frac{\Psi(\theta)}{\sqrt{\sin\theta}}, \quad (5)$$

with $\beta = a\omega$. The corresponding boundary conditions now become $\Psi|_{\theta=0} = \Psi|_{\theta=\pi} = 0$. For convenience, we apply the terminology of quantum mechanics, such as potential energy, ground state, excited state, in the following. Please note that it does not really refer to a real quantum system. The potential in Eq. (4) is

$$V(\theta, \beta, s) = -\left[\frac{1}{4} + s + \beta^2\cos^2\theta - 2s\beta\cos\theta - \frac{(m + s\cos\theta)^2 - 1/4}{\sin^2\theta}\right]. \quad (6)$$

The superpotential W is a core concept in the SUSYQM and is connected with the potential by[18]

$$W^2 - W' = V(\theta, \beta, s) - E_0. \quad (7)$$

Solving the above Reccita's equation, we could obtain the superpotential W, which will give the ground eigenfunction Ψ_0 as

$$\Psi_0 = N\exp\left(-\int W\,\mathrm{d}\theta\right), \quad (8)$$

according to the theory of the SUSYQM.[18] Hence, the key work is to solve Eq. (7). Practically, equation (7) is as difficult to deal with as Eq. (4). Therefore, we count on the perturbation methods to tackle

it. Expanding the superpotential W and the ground eigenvalue $E_{0;m}$ into series of parameter β[10−17]

$$W = \sum_{n=0}^{\infty} \beta^n W_n, \tag{9}$$

$$E_{0;m} = \sum_{n=0}^{\infty} E_{0,n;m}\beta^n, \tag{10}$$

and substituting them into Eq. (7), we obtain the following equations[17]

$$\begin{aligned} W_0' - W_0^2 &= E_{0,0;m} + s + \frac{1}{4} \\ &\quad - \frac{(m + s\cos\theta)^2 - 1/4}{\sin^2\theta} \\ &\equiv f_0(\theta), \end{aligned} \tag{11}$$

$$W_1' - 2W_0W_1 = -2s\cos\theta + E_{0,1;m} \equiv f_1(\theta), \tag{12}$$

$$W_2' - 2W_0W_2 = \cos^2\theta + W_1^2 + E_{0,2;m} \equiv f_2(\theta),$$

$$\begin{aligned} W_n' - 2W_0W_n &= \sum_{k=1}^{n-1} W_kW_{n-k} + E_{0,n;m} \\ &\equiv f_n(\theta), \quad (n \geq 3), \end{aligned} \tag{13}$$

by equating the coefficients of the same powers of β. Our main work is to find the solutions of the above equations in the case $s = 1/2$.

The solution of Eq. (11) is easily found[16]

$$\begin{aligned} E_{0,0;m} &= m^2 + m - 3/4, \\ W_0 &= -\frac{1/2 + (m + 1/2)\cos\theta}{sin\theta}. \end{aligned} \tag{14}$$

From the knowledge of differential equations, it is easy to obtain W_n from W_0 by

$$W_n(\theta) = \left(\tan\frac{\theta}{2}\sin^{2m+1}\theta\right)^{-1} A_n(\theta), \tag{15}$$

where

$$A_n(\theta) = \int f_n(\theta)\tan\frac{\theta}{2}\sin^{2m+1}\theta\,\mathrm{d}\theta. \tag{16}$$

To calculate Eqs. (15) and (16), we need the following integral formula:[19]

$$\begin{aligned} P(2m,\theta) &= \int \sin^{2m}\theta\,\mathrm{d}\theta \\ &= -\frac{\cot\theta}{2m+1}\left[\sum_{k=0}^{m-1} \bar{I}(2m,k)\sin^{2m-2k}\theta\right] \\ &\quad + \frac{(2m-1)!!}{(2m)!!}\theta, \end{aligned} \tag{17}$$

where

$$\bar{I}(2m,k) = \frac{(2m+1)(2m-1)\cdots(2m-2k+1)}{2m(2m-2)\cdots(2m-2k)}, \quad (k \geq 0). \tag{18}$$

After a simple derivation, we obtain the general formula

$$\begin{aligned} &P(2m+2n) \\ &= \frac{2m+1}{2m+2n+1}\bar{I}(2m+2n,n-1)P(2m,\theta) \\ &\quad - \cos\theta\sum_{l=1}^{n}\frac{\bar{I}(2m+2n,n-l)}{2m+2n+1}\sin^{2m+2l-1}\theta. \end{aligned} \tag{19}$$

The above formula can also be proved by mathematical induction. With the help of Eqs. (14) and (19), $A_1(\theta)$ is now simplified to

$$\begin{aligned} A_1(\theta) &= \int (E_{0,1;m} - \cos\theta)(1-\cos\theta)\sin^{2m}\theta\,\mathrm{d}\theta \\ &= (E_{0,1;m}+1)P(2m,\theta) - P(2m+2,\theta) \\ &\quad - \frac{E_{0,1;m}+1}{2m+1}\sin^{2m+1}\theta \\ &= \left(E_{0,1;m} + 1 - \frac{2m+1}{2m+2}\right)P(2m,\theta) \\ &\quad + \left(\frac{\cos\theta}{2m+2} - \frac{E_{0,1;m}+1}{2m+1}\right)\sin^{2m+1}\theta. \end{aligned} \tag{20}$$

Now we discuss the term $P(2m,\theta)$. According to Eqs. (8) and (17), we can see that $\Psi_0(\theta)$ is divergent at the boundaries $\theta = 0,\ \pi$, which does not meet the requirement that $\Psi(\theta)$ should be finite at $\theta = 0,\ \pi$. So the coefficient of the term $P(2m,\theta)$ in the above equation of A_1 must be zero. That is,

$$E_{0,1} = -\frac{1}{2m+2}. \tag{21}$$

Consequently, A_1 becomes

$$A_1(\theta) = -\frac{\sin^{2m+1}\theta}{2m+2} + \frac{\cos\theta\sin^{2m+1}\theta}{2m+2}. \tag{22}$$

Therefore, it is easy to obtain the first order $W_1(\theta)$ from Eq. (15),

$$W_1(\theta) = -\frac{1}{2m+2}\sin\theta. \tag{23}$$

By tedious calculation the same as that for $E_{0,1;m}$ and $W_1(\theta)$ and through choosing the quantities $E_{0,2;m} - E_{0,4;m}$ to make the Ψ_0 satisfy the boundary conditions, one could obtain $W_2(\theta), W_3(\theta), W_4(\theta)$. The results are

$$E_{0,2;m} = -\frac{4m^2+10m+5}{(2m+2)^3}, \tag{24}$$

$$E_{0,3;m} = -\frac{4(2m+1)^2(2m+3)}{(2m+2)^5(2m+4)}, \tag{25}$$

$$E_{0,4;m} = -\frac{2(2m+1)^2(2m+3)(2m^2+9m+2)}{(2m+2)^7(2m+4)}, \tag{26}$$

$$W_2(\theta) = b_{2,1}\sin\theta + a_{2,1}\sin\theta\cos\theta, \tag{27}$$

$$W_3(\theta) = b_{3,1}\sin\theta + b_{3,2}\sin^3\theta + a_{3,1}\sin\theta\cos\theta, \tag{28}$$

$$W_4(\theta) = b_{4,1}\sin\theta + b_{4,1}\sin^3\theta + a_{4,1}\sin\theta\cos\theta + a_{4,2}\sin^3\theta\cos\theta, \quad (28)$$

where

$$b_{2,1} = -\frac{2m+1}{(2m+2)^3}, \quad a_{2,1} = \frac{2m+1}{(2m+2)^2},$$
$$b_{3,1} = \frac{4(2m+1)}{(2m+2)^5(2m+4)}, \quad (29)$$
$$b_{3,2} = -\frac{2(2m+1)}{(2m+2)^3(2m+4)},$$
$$a_{3,1} = -\frac{4(2m+1)}{(2m+2)^4(2m+4)}, \quad (30)$$
$$b_{4,1} = \frac{2(2m+1)(2m^2+9m+2)}{(2m+2)^7(2m+4)},$$
$$a_{4,1} = -\frac{2(2m+1)(2m^2+9m+2)}{(2m+2)^6(2m+4)}, \quad (31)$$
$$b_{4,2} = -\frac{6m(2m+1)}{(2m+2)^5(2m+4)},$$
$$a_{4,2} = \frac{2m(2m+1)}{(2m+2)^4(2m+4)}. \quad (32)$$

Now, we compare above results with previous ones. In Ref. [3], Berti got the following results for the energy:

$$f_0 = l(l+1) - s(s+1), \quad f_1 = -\frac{2ms^2}{l(l+1)}, \quad (33)$$
$$f_2 = h(l+1) - h(l) - 1,$$
$$f_3 = \frac{2h(l)ms^2}{(l-1)l^2(l+1)} - \frac{2h(l+1)ms^2}{(l)(l+1)^2(l+2)}, \quad (34)$$
$$f_4 = ms^2\left[\frac{4h(l+1)}{l^2(l+1)^4(l+2)^2} - \frac{4h(l)}{(l-1)^2(l)^4(l+1)^2}\right] - \frac{(l+2)h(l+1)h(l+2)}{2(l+1)(2l+3)}, + \frac{h^2(l+1)}{2(l+1)} + \frac{h(l)h(l+1)}{2l^2+2l} - \frac{h^2(l)}{2l} + \frac{(l-1)h(l-1)h(l)}{4l^2-2l}, \quad (35)$$

where

$$h(l) = \frac{[l^2 - \frac{1}{4}(\alpha+\beta)^2][l^2 - \frac{1}{4}(\alpha-\beta)^2](l^2-s^2)}{2(l-\frac{1}{2})l^3(l+\frac{1}{2})}, \quad (36)$$
$$\frac{1}{2}(\alpha+\beta) = \max(|m|,|s|),$$
$$\frac{1}{2}(\alpha+\beta) = \frac{ms}{\max(|m|,|s|)}. \quad (37)$$

In our case ($s = 1/2$, $l = m$, $|m| > |s|$), we apply these parameters in Eqs. (33)–(37), the results are in agreement with Eqs. (24) and (25). Therefore, we conclude that the method of the SUSYQM is correct and effective.

3. The general n-th term of the superpotential

From W_1, W_2, W_3, and W_4, we can calculate W_5, W_6, ... in the same way. Instead, we hypothetically summarize a general formula for W_n as

$$W_n(\theta) = \sum_{k=1}^{[n/2]} a_{n,k}\sin^{2k-1}\theta\cos\theta + \sum_{k=1}^{[(n+1)/2]} b_{n,k}\sin^{2k-1}\theta, \quad (38)$$

and use mathematical induction to prove the result. The [ ] denotes the floor function. We will also provide the formula for obtaining W_n from those W_i with $i < n$.

First it is easy to see that W_1, W_2, W_3, and W_4 are consistent with the assumption for W_n, which means the assumption is true when $n = 1$. Assuming that all W_N with $N \le n-1$ are in the form of Eq. (38), we will try to solve the differential equation for W_n and verify that it also can be written in the form of Eq. (38). Back to Eqs. (13), (15), and (16), we need to simplify the term $\sum_{k=1}^{n-1} W_k W_{n-k}$ to calculate W_n. When $1 \le k \le n-1$, then $1 \le n-k \le n-1$, $W_k(\theta)$ and $W_{n-k}(\theta)$ could be written in the form of Eq. (38), which are

$$W_k(\theta) = \sum_{i=1}^{[\frac{k}{2}]} a_{k,i}\sin^{2\,\mathrm{i}-1}\theta\cos\theta + \sum_{i=1}^{[\frac{k+1}{2}]} b_{k,i}\sin^{2\,\mathrm{i}-1}\theta,$$
$$W_{n-k}(\theta) = \sum_{j=1}^{[\frac{n-k}{2}]} a_{n-k,j}\sin^{2j-1}\theta\cos\theta + \sum_{j=1}^{[\frac{n-k+1}{2}]} b_{n-k,j}\sin^{2j-1}\theta. \quad (39)$$

We notice the following facts are true for $i < n$,

$$a_{i,j} = 0, \quad j < 1 \ \text{ or } \ j > \left[\frac{\mathrm{i}}{2}\right];$$
$$b_{i,j} = 0, \quad j < 1 \ \text{ or } \ j > \left[\frac{\mathrm{i}+1}{2}\right]. \quad (40)$$

Thus

$$\sum_{k=1}^{n-1} W_k W_{n-k} = \bar{A}_1 + \bar{A}_2 + \bar{A}_3 + \bar{A}_4, \quad (41)$$

where

$$\bar{A}_1 = \sum_{k=1}^{n-1}\sum_{i=1}^{[\frac{k+1}{2}]}\sum_{j=1}^{[\frac{n-k+1}{2}]} b_{k,i}b_{n-k,j}\sin^{2\,\mathrm{i}+2\mathrm{j}-2}\theta,$$
$$\bar{A}_2 = \sum_{k=1}^{n-1}\sum_{i=1}^{[\frac{k}{2}]}\sum_{j=1}^{[\frac{n-k}{2}]} a_{k,i}b_{n-k,j}\sin^{2\,\mathrm{i}+2\mathrm{j}-2}\theta\cos^2\theta,$$

$$\bar{A}_3 = \cos\theta \sum_{k=1}^{n-1} \sum_{i=1}^{[\frac{k+1}{2}]} \sum_{j=1}^{[\frac{n-k}{2}]} b_{k,i} a_{n-k,j} \sin^{2\,\mathrm{i}+2\mathrm{j}-2}\theta,$$

$$\bar{A}_4 = \cos\theta \sum_{k=1}^{n-1} \sum_{i=1}^{[\frac{k}{2}]} \sum_{j=1}^{[\frac{n-k+1}{2}]} a_{k,i} b_{n-k,j} \sin^{2\,\mathrm{i}+2\mathrm{j}-2}\theta. \quad (42)$$

Exchanging the order of the sums in all above equations and with $p = i + j$, they are simplified to

$$\begin{aligned}
\bar{A}_1 &= \sum_{p=2}^{\bar{c}_1} \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} b_{k,p-j} b_{n-k,j} \sin^{2p-2}\theta, \\
\bar{A}_2 &= \sum_{p=2}^{\bar{c}_2-1} \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} a_{k,p-j} a_{n-k,j} \sin^{2p-2}\theta, \\
&\quad - \sum_{p=3}^{\bar{c}_2} \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} a_{k,p-1-j} a_{n-k,j} \sin^{2p-2}\theta, \\
\bar{A}_3 &= \cos\theta \sum_{p=2}^{\bar{c}_3} \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} b_{k,p-j} a_{n-k,j} \sin^{2p+2}\theta, \\
\bar{A}_4 &= \cos\theta \sum_{p=2}^{\bar{c}_4} \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} a_{k,p-j} b_{n-k,j} \sin^{2p+2}\theta, \quad (43)
\end{aligned}$$

where

$$\begin{aligned}
\bar{c}_1 &= \left[\frac{k+1}{2}\right] + \left[\frac{n-k+1}{2}\right], \\
\bar{c}_2 &= \left[\frac{k}{2}\right] + \left[\frac{n-k}{2}\right] + 1, \\
\bar{c}_3 &= \left[\frac{k+1}{2}\right] + \left[\frac{n-k}{2}\right], \\
\bar{c}_4 &= \left[\frac{k}{2}\right] + \left[\frac{n-k+1}{2}\right]. \quad (44)
\end{aligned}$$

With the help of Eq. (40), it is easy to see

$$\bar{c}_1 = \bar{c}_2 = \frac{n}{2} + 1,$$

$$\bar{c}_3 = \bar{c}_4 = \frac{n}{2}, \quad \text{when } n \text{ is even}, \quad (45)$$

$$\bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \frac{n+1}{2}, \quad \text{when } n \text{ is odd}. \quad (46)$$

Hence

$$\sum_{k=1}^{n-1} W_k W_{n-k} = \sum_{p=2}^{[\frac{n}{2}]+1} \left[(h_{n,p} + g_{n,p}\cos\theta\right] sin^{2p-2}\theta, \quad (47)$$

where $h_{n,p}$ and $g_{n,p}$ are constant coefficients:

$$\begin{aligned}
h_{n,p} &= \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} (b_{k,p-j} b_{n-k,j} + a_{k,p-j} a_{n-k,j} \\
&\quad - a_{k,p-1-j} a_{n-k,j}), \quad (48) \\
g_{n,p} &= \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} (b_{k,p-j} a_{n-k,j} + a_{k,p-j} b_{n-k,j}). \quad (49)
\end{aligned}$$

We notice the facts that when n is even

$$\begin{aligned}
g_{n,p} &= 0, \quad p < 2 \text{ or } p > \frac{n}{2}; \\
h_{n,p} &= 0, \quad p < 2 \text{ or } p > \frac{n}{2} + 1, \quad (50)
\end{aligned}$$

and when n is odd,

$$g_{n,p} = h_{n,p} = 0, \quad p < 2 \text{ or } p > \frac{n+1}{2}. \quad (51)$$

We have used the facts given in Eq. (40) and substituted the quantities $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$, $\bar{c}_4$ by their maximum $[\frac{n}{2}]+1$ in the above equation. To obtain the final conclusion, we will take Eqs. (50) and (51) into consideration. Substituting $f_n(z) = \sum_{k=1}^{n-1} W_k W_{n-k} + E_{0,n;m}$ into Eq. (16) and using Eq. (17), we obtain

$$\begin{aligned}
A_n(\theta) &= \int \sum_{p=2}^{[\frac{n}{2}]+1} (h_{n,p} + g_{n,p}\cos\theta) \sin^{2p-2}\theta (1-\cos\theta) \sin^{2m}\theta \,\mathrm{d}\theta + \int E_{0,n;m} \times (1-\cos\theta)\sin^{2m}\theta \,\mathrm{d}\theta \\
&= \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p} - h_{n,p}}{2m+2p-1} \sin^{2m+2p-1}\theta - \frac{E_{0,n;m}}{2m+1} \sin^{2m+1}\theta + \sum_{p=2}^{[\frac{n}{2}]+1} (h_{n,p} - g_{n,p}) P(2m+2p-2,\theta) \\
&\quad + \sum_{p=2}^{[\frac{n}{2}]+1} g_{n,p} P(2m+2p,\theta) + E_{0,n;m} P(2m,\theta). \quad (52)
\end{aligned}$$

With Eq. (19), we have

$$\begin{aligned}
A_n(\theta) &= \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p} - h_{n,p}}{2m+2p-1} \sin^{2m+2p-1}\theta - \frac{E_{0,n;m}}{2m+1} \sin^{2m+1}\theta - \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} \frac{g_{n,p} - h_{n,p}}{2m+2p-1} \sin^{2m+2p-1}\theta \\
&\quad - \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p} \sum_{l=1}^{p} \frac{\bar{I}(2m+2p, p-l)}{2m+2p+1} \sin^{2m+2l-1}\theta + bP(2m,\theta), \quad (53)
\end{aligned}$$

where the quantities $e_{n,p}$ and the coefficient b of $P(2m,\theta)$ are

$$e_{n,p} = \frac{2m+2p}{2m+2p-1}\left(h_{n,p} - \frac{g_{n,p}}{2m+2p}\right), \tag{54}$$

$$b = E_{0,n;m} + \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p}\frac{(2m+1)\bar{I}(2m+2p,p-1)}{2m+2p+1}, \tag{55}$$

respectively. As stated before, the coefficient b must be zero to make the eigenfunction finite at the boundary, hence

$$E_{0,n;m} = -\sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p}\frac{(2m+1)\bar{I}(2m+2p,p-1)}{2m+2p+1}, \tag{56}$$

which can be used to further simplify terms in Eq. (53). A term in Eq. (53) becomes

$$\begin{aligned}
&\cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p}\sum_{l=1}^{p}\frac{\bar{I}(2m+2p,p-l)}{2m+2p+1}\sin^{2m+2l-1}\theta \\
&= \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p}\sum_{l=2}^{p}\frac{\bar{I}(2m+2p,p-l)}{2m+2p+1}\sin^{2m+2l-1}\theta \\
&\quad + \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-1)}{2m+2p+1}\sin^{2m+1}\theta \\
&= \cos\theta \sum_{l=2}^{[\frac{n}{2}]+1}\sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-l)}{2m+2p+1}\sin^{2m+2l-1}\theta \\
&\quad + \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-1)}{2m+2p+1}\sin^{2m+1}\theta.
\end{aligned} \tag{57}$$

Taking Eqs. (56) and (57) into Eq. (53), we obtain

$$A_n(\theta) = F_1 + \cos\theta F_2, \tag{58}$$

where F_1 and F_2 are

$$\begin{aligned}
F_1 &= \sum_{p=2}^{[\frac{n}{2}]+1}\frac{g_{n,p}-h_{n,p}}{2m+2p-1}\sin^{2m+2p-1}\theta \\
&\quad - \frac{E_{0,n;m}}{2m+1}\sin^{2m+1}\theta,
\end{aligned} \tag{59}$$

$$\begin{aligned}
F_2 &= -\sum_{p=2}^{[\frac{n}{2}]+1}\frac{g_{n,p}-h_{n,p}}{2m+2p-1}\sin^{2m+2p-1}\theta \\
&\quad - \sum_{p=2}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-1)}{2m+2p+1}\sin^{2m+1}\theta \\
&\quad - \sum_{l=2}^{[\frac{n}{2}]+1}\sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-l)}{2m+2p+1}\sin^{2m+2l-1}\theta.
\end{aligned} \tag{60}$$

With the help of Eq. (15), it is easy to obtain

$$\begin{aligned}
W_n &= \frac{\cos\theta(F_1+F_2) + (F_1+F_2-\sin^2\theta F_2)}{\sin^{2m+2}\theta} \\
&= \frac{\cos\theta(F_1+F_2)}{\sin^{2m+2}\theta} \\
&\quad + \frac{F_1+F_2-\sin^2\theta(F_1+F_2)+F_1\sin^2\theta}{\sin^{2m+2}\theta}.
\end{aligned} \tag{61}$$

With the help of Eq. (56), we obtain

$$\begin{aligned}
&F_1+F_2 \\
&= -\sum_{l=2}^{[\frac{n}{2}]+1}\sum_{p=l}^{[\frac{n}{2}]+1} e_{n,p}\frac{\bar{I}(2m+2p,p-l)}{2m+2p+1}\sin^{2m+2l-1}\theta.
\end{aligned} \tag{62}$$

Consequently, we have

$$\begin{aligned}
W_n &= \cos\theta\sum_{l=1}^{[\frac{n}{2}]} a_{n,l}\sin^{2l-1}\theta \\
&\quad + \sum_{l=1}^{[\frac{n}{2}]+1} b_{n,l}\sin^{2l-1}\theta,
\end{aligned} \tag{63}$$

where $a_{n,l}$ and $b_{n,l}$ are

$$a_{n,l} = -\sum_{p=l+1}^{[\frac{n}{2}]+1} e_{n,p}\frac{(2m+2l)\bar{I}(2m+2p,p-l)}{(2m+2l+1)(2m+2p+1)}, \tag{64}$$

$$b_{n,1} = a_{n,1} - \frac{E_{0,n;m}}{2m+1}, \tag{65}$$

$$b_{n,l} = a_{n,l} - a_{n,l-1} + \frac{g_{n,l}-h_{n,l}}{2m+2l-1}, \quad l \geq 2. \tag{66}$$

Comparing Eq. (63) with Eq. (38), the difference lies in the summation upper limit of the second term. When n is odd, $[\frac{n+1}{2}] = [\frac{n}{2}]+1$, which means Eq. (38) is the same as Eq. (63). When n is even, we could use Eqs. (40), (50) and (51) to obtain

$$a_{n,\frac{n}{2}+1} = 0, \tag{67}$$

$$b_{n,[\frac{n}{2}]+1} = b_{n,\frac{n}{2}+1} = -a_{n,\frac{n}{2}} - \frac{h_{n,\frac{n}{2}+1}}{2m+n+1} = 0. \tag{68}$$

Hence the upper limit for $b_{n,p}$ becomes $\frac{n}{2} = [\frac{n+1}{2}]$. Our induction about the general formula for W_n is completed. According to Eqs. (64), (66), and (56), we can get some interesting results

$$b_{n,l} = \frac{1}{2m+2l}(g_{n,l}-a_{n,l}), \quad l \geq 2, \tag{69}$$

$$a_{n,1} = \frac{2m+2}{(2m+1)(2m+3)}E_{0,n;m}. \tag{70}$$

We can compare the above results with $W_3(\theta)$ and $W_4(\theta)$ and verify the general formula for $W_n(\theta)$. It is found that the general formula for $W_n(\theta)$ is accurate. The details of the proof can be found in Ref. [20].

4. The ground state eigenfunction

Based on the above conclusions, the superpotential W could be written as

$$W = W_0 + \sum_{n=1}^{\infty} \beta^n W_n. \tag{71}$$

The ground eigenfunction becomes

$$\begin{aligned}\Psi_0 \;=\; & N(1-\cos\theta)^{\frac{1}{2}} \sin^m\theta \\ & \times \exp\Bigg[-\sum_{n=1}^{\infty} \beta^n \Bigg(\sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \frac{\sin^{2k}\theta}{2k} \\ & + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} P(2k-1,\theta) \Bigg) \Bigg], \end{aligned} \tag{72}$$

where N is an integral constant. Back to the Θ, the above ground eigenfunction becomes

$$\begin{aligned}\Theta_0 \;=\; & N(1-\cos\theta)^{\frac{1}{2}} \sin^{m-\frac{1}{2}}\theta \\ & \times \exp\Bigg[-\sum_{n=1}^{\infty} \beta^n \Bigg(\sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \frac{\sin^{2k}\theta}{2k} \\ & + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} P(2k-1,\theta) \Bigg) \Bigg], \end{aligned} \tag{73}$$

and the ground energy is

$$E_{0;m} = m(m+1) - \frac{3}{4} + \sum_{n=1}^{\infty} E_{0,n;m} \beta^n, \tag{74}$$

with $E_{0,n;m}$ determined by Eq. (56).

5. The excited eigenfunctions

Following the ideas in Ref. [21], we hope to extend the recurrence relations to Eq. (4) by using the methods from the supersymmetric quantum mechanics. The superpotential W connects the two partner potentials $V^{\mp}$ by

$$V^{\mp}(\theta) = W^2(\theta) \mp W'(\theta). \tag{75}$$

The shape-invariance property means that the partner potentials $V^{\pm}(x)$ are similar in shape and differ only in the parameters, that is

$$V^{+}(\theta; a_1) = V^{-}(\theta; a_2) + R(a_1), \tag{76}$$

where a_1 is a set of parameters, a_2 is a function of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of θ.

In order to study the shape-invariance property of the spin-weighted spheroidal equation, we must introduce the parameters $A_{i,j}$ and $B_{i,j}$ into the superpotential W, which gives

$$\begin{aligned} & W(A_{n,j}, B_{n,j}, \theta) \\ = \; & -A_{0,0}(m+\frac{1}{2})\cot\theta - \frac{1}{2} B_{0,0} \csc\theta \\ & + \sum_{n=1}^{\infty} \beta^n W_n(A_{n,j}, B_{n,j}, \theta), \end{aligned} \tag{77}$$

where

$$\begin{aligned} & W_n(A_{n,j}, B_{n,j}, \theta) \\ = \; & \sum_{j=1}^{[\frac{n+1}{2}]} \bar{b}_{n,j} \sin^{2\mathrm{j}-1}\theta + \cos\theta \sum_{j=1}^{[\frac{n}{2}]} \bar{a}_{n,j} \sin^{2\mathrm{j}-1}\theta, \end{aligned} \tag{78}$$

with

$$\bar{a}_{n,j} = A_{n,j} a_{n,j}, \quad \bar{b}_{n,j} = B_{n,j} b_{n,j}. \tag{79}$$

Then, $V^{\pm}(A_{n,j}, B_{n,j}, \theta)$ and $V_n^{\pm}(A_{n,j}, B_{n,j}, \theta)$ are defined as

$$\begin{aligned} V^{\pm}(A_{n,j}, B_{n,j}, \theta) \;=\; & W^2(A_{n,j}, B_{n,j}, \theta) \pm W' \\ = \; & \sum_{n=0}^{\infty} \beta^n V_n^{\pm}(A_{i,j}, B_{n,j}, \theta). \end{aligned} \tag{80}$$

The key is to find some quantities $C_{i,j}$ and $D_{i,j}$ that make the relations

$$\begin{aligned} & V_n^{+}(A_{i,j}, B_{n,j}, \theta) \\ = \; & V_n^{-}(C_{i,j}, D_{n,j}, \theta) + R_{n;m}(A_{i,j}, B_{n,j}) \end{aligned} \tag{81}$$

hold with $R_{n;m}(A_{i,j}, B_{n,j}) = R_{n;m}$ as pure quantities. First, we show the above relations hold in the special cases $n = 0, 1, 2$. It is easy to obtain

$$V_0^{+}(A_{0,0}, B_{0,0}, \theta) = V_0^{-}(C_{0,0}, D_{0,0}, \theta) + R_{0;m}, \tag{82}$$

$$V_1^{+}(A_{1,1}, B_{1,1}, \theta) = V_1^{-}(C_{1,1}, D_{1,1}, \theta) + R_{1;m}, \tag{83}$$

$$V_2^{+}(B_{2,1}, A_{2,1}, \theta) = V_2^{-}(C_{2,1}, D_{2,1}, \theta) + R_{2;m}. \tag{84}$$

In order to retain the relations in Eq. (81), we just need to have

$$C_{0,0} = A_{0,0} + \frac{2}{2m+1}, \tag{85}$$

$$D_{0,0} = B_{0,0}, \tag{86}$$

$$D_{1,1} = \frac{(2m+1)A_{0,0} - 1}{(2m+1)A_{0,0} + 3} B_{1,1}, \tag{87}$$

and

$$\begin{aligned} D_{2,1} = \; & \frac{(2m+1)A_{0,0} - 1}{(2m+1)A_{0,0} + 3} B_{2,1} \\ & + \frac{6 B_{0,0} B_{2,1}}{[(2m+1)A_{0,0} + 3][(2m+1)A_{0,0} + 4]} \end{aligned}$$

$$-\frac{8[(2m+1)A_{0,0}+1]B_{0,0}B_{1,1}^2}{[(2m+1)A_{0,0}+3]^3[(2m+1)A_{0,0}+4]}, \quad (88)$$

$$C_{2,1} = \frac{8[(2m+1)A_{0,0}+1]B_{1,1}^2}{[(2m+1)A_{0,0}+3]^3[(2m+1)A_{0,0}+4]} + \frac{(2m+1)A_{0,0}-2}{(2m+1)A_{0,0}+4}A_{2,1}, \quad (89)$$

with

$$R_{0;m}(A_{0,0}) = (2m+1)A_{0,0}+1, \quad (90)$$

$$R_{1;m}(A_{0,0},B_{0,0},B_{1,1}) = -\frac{4B_{0,0}B_{1,1}}{(2m+1)A_{0,0}+3}, \quad (91)$$

$$R_{2;m}(A_{0,0},B_{0,0},B_{1,1},B_{2,1},A_{2,1}) = \left[-\frac{4B_{0,0}B_{2,1}}{(2m+1)A_{0,0}+3} + AB_{1,1}^2 + BA_{2,1}\right], \quad (92)$$

where

$$A = \frac{8B_{0,0}^2 - 8[(2m+1)A_{0,0}-1][(2m+1)A_{0,0}+3]}{[(2m+1)A_{0,0}+3]^3[(2m+1)A_{0,0}+4]},$$
$$B = \frac{6B_{0,0}^2 - 2[(2m+1)A_{0,0}-1][(2m+1)A_{0,0}+3]}{[(2m+1)A_{0,0}+3][(2m+1)A_{0,0}+4]}. \quad (93)$$

For the general case of n, we use the induction method to proceed. The above result shows that the formula (81) is true for $n = 0,\ 1,\ 2$. Suppose it is also true for $N \le n-1$, we need to prove that it is true for $N = n$. First, we simplify the expressions of $V_n^{\pm}(A_{i,j},B_{i,j},\theta)$. With the help of

$$W^2 = \sum_{n=0}^{\infty}\beta^n\sum_{k=0}^{n}W_k(A_{i,j},B_{n,j},\theta) \times W_{n-k}(A_{i,j},B_{n,j},\theta), \quad (94)$$

we have the formulas for $V_n^{\pm}$ in the case $n \ge 1$ as

$$V_n^-(A_{i,j},B_{n,j},\theta) = P_n^- + \bar{P}_n, \quad (95)$$
$$V_n^+(A_{i,j},B_{n,j},\theta) = P_n^+ + \bar{P}_n. \quad (96)$$

There are three parts

$$P_n^{\pm} = 2W_0W_n(A_{n,j},B_{n,j},\theta) \pm W_n'(A_{i,j},B_{n,j},\theta), \quad (97)$$
$$\bar{P}_n = \sum_{k=1}^{n-1}[W_k(A_{i,j},B_{n,j},\theta)W_{n-k}(A_{i,j},B_{n,j},\theta)], \quad (98)$$

in the above equations. The three parts can be simplified separately.

$$P_n^- = \cos\theta\sum_{p=1}^{[\frac{n+1}{2}]}P_{n,p}^-(A_{i,j},B_{n,j})\sin^{2p-2}\theta + \sum_{p=1}^{[\frac{n}{2}]+1}Q_{n,p}^-(A_{i,j},B_{n,j})\sin^{2p-2}\theta, \quad (99)$$

where

$$P_{n,p}^-(A_{n,j},B_{n,j}) = -B_{0,0}A_{n,p}a_{n,p} + [(1-2p)-(2m+1)A_{0,0}]B_{n,p}b_{n,p}, \quad (100)$$

$$Q_{n,p}^-(A_{n,j},B_{n,j}) = -B_{0,0}B_{n,p}b_{n,p} + [(1-2p)-(2m+1)A_{0,0}]A_{n,p}a_{n,p} + [(2p-2)+(2m+1)A_{0,0}]A_{n,p-1}a_{n,p-1}. \quad (101)$$

Similarly,

$$P_n^+ = \cos\theta\sum_{p=1}^{[\frac{n+1}{2}]}P_{n,p}^+(A_{i,j},B_{n,j})\sin^{2p-2}\theta + \sum_{p=1}^{[\frac{n}{2}]+1}Q_{n,p}^+(A_{i,j},B_{n,j})\sin^{2p-2}\theta, \quad (102)$$

where

$$P_{n,p}^+(A_{n,j},B_{n,j}) = -B_{0,0}A_{n,p}a_{n,p} + [(2p-1)-(2m+1)A_{0,0}]B_{n,p}b_{n,p}, \quad (103)$$

$$Q_{n,p}^+(A_{n,j},B_{n,j}) = -B_{0,0}B_{n,p}b_{n,p} + [-(2m+1)A_{0,0}+(2p-1)]A_{n,p}a_{n,p} + [(2-2p)+(2m+1)A_{0,0}]A_{n,p-1}a_{n,p-1}. \quad (104)$$

Notice that

$$P_{n,p}^-(A_{n,j},B_{n,j}) = P_{n,p}^+(A_{n,j},B_{n,j}) = Q_{n,p}^-(A_{n,j},B_{n,j}) = Q_{n,p}^+(A_{n,j},B_{n,j}) = 0, \quad p < 1 \text{ or } p > \left[\frac{n+1}{2}\right]. \quad (105)$$

From Eq. (17), it is easy to obtain

$$\bar{P}_n = \sum_{p=1}^{[\frac{n+1}{2}]}[G_{n,p}(A_{i,j},B_{i,j})\cos\theta + H_{n,p}(A_{i,j},B_{i,j})]\sin^{2p-2}\theta, \quad (106)$$

where

$$G_{n,p}(A_{i,j},B_{i,j}) = \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}(\bar{b}_{k,p-j}\bar{a}_{n-k,j} + \bar{a}_{k,p-j}\bar{b}_{n-k,j}), \quad (107)$$

$$H_{n,p}(A_{i,j},B_{i,j}) = \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}(\bar{b}_{k,p-j}\bar{b}_{n-k,j} + \bar{a}_{k,p-j}\bar{a}_{n-k,j} - \bar{a}_{k,p-1-j}\bar{a}_{n-k,j}), \quad (108)$$

with

$$\bar{a}_{i,j} = A_{i,j}a_{i,j}, \quad \bar{b}_{i,j} = B_{i,j}b_{i,j}. \tag{109}$$

When n is even,

$$\begin{aligned} &G_{n,p}(A_{i,j}, B_{i,j}) = 0, \quad p < 2 \text{ or } p > \frac{n}{2}; \\ &H_{n,p}(A_{i,j}, B_{i,j}) = 0, \quad p < 2 \text{ or } p > \frac{n}{2} + 1, \end{aligned} \tag{110}$$

and when n is odd,

$$\begin{aligned} &G_{n,p}(A_{i,j}, B_{i,j}) = H_{n,p}(A_{i,j}, B_{i,j}) = 0, \\ &\quad p < 2 \text{ or } p > \frac{n+1}{2}. \end{aligned} \tag{111}$$

With the help of Eqs. (110) and (111), we can obtain

$$\begin{aligned} V_n^-(A_{i,j}, B_{n,j}) = &\cos\theta \sum_{k=2}^{[\frac{n+1}{2}]} [G_{n,p}(A_{i,j}, B_{i,j}) + P_{n,p}^-(A_{n,j}, B_{n,j})] \sin^{2p-2}\theta \\ &+ \sum_{k=2}^{[\frac{n}{2}]+1} [H_{n,p}(A_{i,j}, B_{i,j}) + Q_{n,p}^-(A_{n,j}, B_{n,j})] \sin^{2p-2}\theta + P_{n,1}^-(A_{n,j}, B_{n,j}) + Q_{n,1}^-(A_{n,j}, B_{n,j}), \end{aligned} \tag{112}$$

$$\begin{aligned} V_n^+(A_{i,j}, B_{n,j}) = &\cos\theta \sum_{k=2}^{[\frac{n+1}{2}]} [G_{n,p}(A_{i,j}, B_{i,j}) + P_{n,p}^+(A_{n,j}, B_{n,j})] \sin^{2p-2}\theta \\ &+ \sum_{k=2}^{[\frac{n}{2}]+1} [H_{n,p}(A_{i,j}, B_{i,j}) + Q_{n,p}^+(A_{n,j}, B_{n,j})] \sin^{2p-2}\theta + P_{n,1}^+(A_{n,j}, B_{n,j}) + Q_{n,1}^+(A_{n,j}, B_{n,j}). \end{aligned} \tag{113}$$

By using

$$b_{n,0} = 0, \quad a_{n,0} = 0, \quad n = 1, 2, \ldots, \tag{114}$$

we could rewrite $V_n^+(A_{i,j})$ as

$$\begin{aligned} &V_n^+(A_{i,j}, B_{i,j}) = V_n^-(C_{i,j}, D_{i,j}) + R_{n;m}(A_{i,j}, B_{i,j}) \\ &= \cos\theta \sum_{k=2}^{[\frac{n+1}{2}]} [G_{n,p}(C_{i,j}, D_{i,j}) + P_{n,p}^-(C_{n,j}, D_{n,j})] \sin^{2p-2}\theta \\ &\quad + \sum_{k=2}^{[\frac{n}{2}]+1} [H_{n,p}(C_{i,j}, D_{i,j}) + Q_{n,p}^-(C_{n,j}, D_{n,j})] \sin^{2p-2}\theta \\ &\quad + P_{n,1}^-(C_{n,j}, D_{n,j}) + Q_{n,1}^-(C_{n,j}, D_{n,j}) + R_{n;m}(A_{i,j}, B_{i,j}), \end{aligned} \tag{115}$$

where

$$\begin{aligned} &R_{n;m}(A_{i,j}, B_{i,j}) \\ &= P_{n,1}^+(A_{n,j}, B_{n,j}) + Q_{n,1}^+(A_{n,j}, B_{n,j}) \\ &\quad - P_{n,1}^-(C_{n,j}, D_{n,j}) - Q_{n,1}^-(C_{n,j}, D_{n,j}), \end{aligned} \tag{116}$$

with

$$P_{n,1}^-(C_{n,j}, D_{n,j}) = -[(2m+1)C_{0,0} + 1]C_{n,1}b_{n,1}, \tag{117}$$

$$Q_{n,1}^-(C_{n,j}, D_{n,j}) = -D_{0,0}C_{n,1}b_{n,1}, \tag{118}$$

and

$$P_{n,1}^+(A_{n,j}, B_{n,j}) = -[(2m+1)A_{0,0} - 1]B_{n,1}b_{n,1}, \tag{119}$$

$$Q_{n,1}^+(A_{n,j}, B_{n,j}) = -B_{0,0}B_{n,1}b_{n,1}. \tag{120}$$

To maintain the shape-invariance property for the n-th term, the following equations must be satisfied

$$\begin{aligned} &G_{n,p}(C_{i,j}, D_{i,j}) + P_{n,p}^-(C_{n,j}, D_{n,j}) \\ &= G_{n,p}(A_{i,j}, B_{i,j}) + P_{n,p}^+(A_{n,j}, B_{n,j}), \\ &H_{n,p}(C_{i,j}, D_{i,j}) + Q_{n,p}^-(C_{n,j}, D_{n,j}) \\ &= H_{n,p}(A_{i,j}, B_{i,j}) + Q_{n,p}^+(A_{n,j}, B_{n,j}), \\ &p = 2, 3, \ldots, \left[\frac{n+2}{2}\right]. \end{aligned}$$

Define

$$\begin{aligned} &G_{n,p}(A_{i,j}, B_{i,j}) - G_{n,p}(C_{i,j}, D_{i,j}) \\ &\quad + P_{n,p}^+(A_{n,j}, B_{n,j}) \equiv U_{n,p}, \end{aligned} \tag{121}$$

$$\begin{aligned} &H_{n,p}(A_{i,j}, B_{i,j}) - H_{n,p}(C_{i,j}, D_{i,j}) \\ &\quad + Q_{n,p}^+(A_{n,j}, B_{n,j}) \equiv \check{U}_{n,p}, \end{aligned} \tag{122}$$

then we have

$$\begin{aligned} P^{-}_{n,p}(C_{n,j},D_{n,j}) &= -\alpha_p D_{n,p}b_{n,p} - D_{0,0}C_{n,p}a_{n,p} \\ &= U_{n,p}, \\ Q^{-}_{n,p}(C_{n,j},D_{n,j}) &= -\alpha_p C_{n,p}a_{n,p} - D_{0,0}D_{n,p}b_{n,p} \\ &\quad + (\alpha_p - 1)C_{n,p-1}a_{n,p-1} \\ &= \check{U}_{n,p}, \end{aligned} \tag{123}$$

where $\alpha_p = [(2m+1)C_{0,0} + (2p-1)]$. In the above equations, the quantities $C_{i,j}, i < n,\ j < [i/2]$ and $D_{i,j}, i < n,\ j < [(i+1)/2]$ are known functions of the variables $A_{i,j}$, $B_{i,j}$, $i < n$, $j < n$. The only unknown quantities are the n quantities $C_{n,p}$, $p = 1, 2, \ldots, [n/2]$, $D_{n,p}$, $p = 1, 2, \ldots, [(n+1)/2]$. Therefore, we obtain

$$D_{n,p} = \frac{D_{0,0}a_{n,p}}{\alpha_p b_{n,p}}C_{n,p} - \frac{U_{n,p}}{\alpha_p b_{n,p}}, \tag{124}$$

$$\begin{aligned} C_{n,p-1} &= \frac{\left(\alpha_p + \dfrac{D^2_{0,0}}{\alpha_p}\right)a_{n,p}}{(\alpha_p - 1)a_{n,p-1}}C_{n,p} \\ &\quad + \frac{\check{U}_{n,p} - \dfrac{D_{0,0}}{\alpha_p}U_{n,p}}{(\alpha_p - 1)a_{n,p-1}}, \end{aligned} \tag{125}$$

$$p = 2, 3, \ldots, \left[\frac{n+2}{2}\right], \tag{126}$$

with $D_{n,[\frac{n+1}{2}]+1} = C_{n,[\frac{n}{2}]+1} = 0$. Here we give some notes: (1) when n is odd, one could first obtain $D_{n,[\frac{n+1}{2}]}$ from Eq. (124) and $C_{n,[\frac{n}{2}]}$ from Eq. (125) under the condition $C_{n,[\frac{n+1}{2}]} = 0$. Then, it is easy to calculate other quantities subsequently. (2) when n is even, one needs to calculate $C_{n,[\frac{n}{2}]}$ from Eq. (125) under the condition $C_{n,[\frac{n+1}{2}]} = 0$ first. Then, it is easy to calculate other quantities subsequently too.

The excited eigenvalues $E_{l;m}$ and the eigenfunctions Ψ_l are obtained by using the recurrence relation

$$E^{-}_{l;m} = E_{0;m} + \sum_{k=1}^{l} R(a_k, b_k), \tag{127}$$

$$E_{0;m} = m(m+1) - \frac{3}{4} + \sum_{n=1}^{\infty} E_{0,n;m}\beta^n, \tag{128}$$

$$R(a_k, b_k) = R_{0;m} + \sum_{n=1}^{\infty} \beta^n R_{n;m}(a_k, b_k), \tag{129}$$

$$a_1 = (A_{i,j}, B_{i,j}),\ a_2 = (C_{i,j}, D_{i,j}), \ldots; \tag{130}$$

$$\Psi_0 \propto \exp\left[-\int_{\theta_0}^{\theta} W(A_{n,j}, B_{n,j}, \theta)\,\mathrm{d}\theta\right], \tag{131}$$

$$\mathscr{A}^{\dagger} = -\frac{\mathrm{d}}{\mathrm{d}\theta} + W(A_{n,j}, B_{n,j}, \theta), \tag{132}$$

$$\Psi^{-}_{n} = \mathscr{A}^{\dagger}(A_{n,j}, B_{n,j}, \theta)\Psi^{-}_{n-1}(C_{n,j}, D_{n,j}, \theta), \tag{133}$$

$$n = 1, 2, 3, \ldots \tag{134}$$

6. Conclusion

In conclusion, we have first obtained the ground eigenvalue and the ground eigenfunction for the spin-weighted equation in the case of $s = 1/2$. Then, we have also proved the shape-invariance property for them and obtain their recurrence relation. From these results, we can obtain the excited eigenvalues and the excited eigenfunctions. These results will be helpful for their application. Besides, a similar process could be extended to the $s = 1,\ 2,\ 3/2$ spin-weighted functions, which are our further works.

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