Rapid Communication

Universal critical properties of the Eulerian bond-cubic model*

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We investigate the Eulerian bond-cubic model on the square lattice by means of Monte Carlo simulations, using an efficient cluster algorithm and a finite-size scaling analysis. The critical points and four critical exponents of the model are determined for several values of n. Two of the exponents are fractal dimensions, which are obtained numerically for the first time. Our results are consistent with the Coulomb gas predictions for the critical O(n) branch for $n < 2$ and the results obtained by previous transfer matrix calculations. For $n = 2$, we find that the thermal exponent, the magnetic exponent and the fractal dimension of the largest critical Eulerian bond component are different from those of the critical O(2) loop model. These results confirm that the cubic anisotropy is marginal at $n = 2$ but irrelevant for $n < 2$.

Keywords: phase transition and critical phenomena, Eulerian-bond cubic model, Monte Carlo simulation, fractal dimension

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1. Introduction

Generally, the Hamiltonian of the n-component spin cubic model[1,2] can be written as

$$\mathscr{H}/k_{\rm B}T = -\sum_{<i,j>}[K\boldsymbol{s}_i\cdot\boldsymbol{s}_j + M(\boldsymbol{s}_i\cdot\boldsymbol{s}_j)^2], \qquad (1)$$

where T is the temperature, $k_{\rm B}$ is the Boltzmann constant and the sum on $< i, j >$ includes all pairs of nearest-neighbor (NN) sites. The spin $\boldsymbol{s}_i$ is an n-component vector located on the i-th site, namely $\boldsymbol{s}_i = (s_{i1}, s_{i2}, \cdots, s_{in})$, such that one and only one of the n components has a nonzero value ±1. This model is also called face-cubic model because the spin can be regarded as a vector located at the centre of an n-dimensional hypercube, and pointing to the centre of one of the faces of the hypercube.

This model combines the Potts degrees of freedom[3,4] with Ising degrees of freedom. Therefore, the Hamiltonian can also be written as

$$\mathscr{H}/k_{\rm B}T = -\sum_{<i,j>}[Ks_is_j + M]\delta_{\sigma_i,\sigma_j}, \qquad (2)$$

where $s_i = \pm1$ is an Ising spin and $\sigma_i = 1, 2, \cdots, n$ is a Potts spin. The partition sum of the cubic model can be written as

$$Z = \sum_{\{s\},\{\sigma\}} {\rm e}^{-\mathscr{H}/k_{\rm B}T} = \sum_{\{s\},\{\sigma\}}\prod_{<i,j>} {\rm e}^{(Ks_is_j+M)\delta_{\sigma_i,\sigma_j}}. \qquad (3)$$

For the special case, $\cosh K = {\rm e}^{-M}$, this partition sum can be mapped to (see Refs. [1, 2] for details)

$$Z = \sum_{\{b\}}(nx)^{N_{\rm b}}n^{N_{\rm c}}, \qquad (4)$$

where $x = {\rm e}^M\sinh K/n$. This model is defined in terms of bond variables that can take the values absent and present. The bond configuration $\{b\}$ is restricted to be Eulerian, which means that each site is connected

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to an even number of bonds. $N_{\rm b}$ is the number of bonds and $N_{\rm c}$ is the number of components. Typically, a component is a group of sites connected by bonds, but it can also be an isolated site. The model may thus be called Eulerian bond-cubic model and one of its configurations is shown in Fig. 1. Furthermore, n is no longer restricted to be an integer number in Eq. (4), it can be any real number.

The Eulerian bond-cubic model has been studied by means of transfer matrix (TM) calculations and a finite-size scaling analysis in Ref. [2]. In the region $1 \leq n \leq 2$, critical points and three critical exponents were determined. To further investigate the nature of the phase transition and the critical behaviour, especially the geometric properties of critical configurations of the model, one may also employ Monte Carlo simulations. However, the problem arises to design an efficient Monte Carlo algorithm for this model, in view of the nonlocal weight $n^{N_{\rm c}}$. A local update of the Metropolis type algorithm requires an nonlocal search to determine the change of the number of components. This, together with the critical slowing-down, would make the simulation very time-consuming in the critical region.

For this reason, we make use of the colouring algorithm, which is proven to be useful for some models with nonlocal weights in their partition sums, such as the random-cluster[5,6] model and the O(n) loop model.[7] It was first proposed by Chayes and Machta,[8,9] and was originally combined with the Swendsen–Wang algorithm[10] to simulate the Potts model or the random-cluster model. Ding *et al.*[11] extended the application of the "coloring algorithm" to the simulations of the O(n) loop model on the honeycomb lattice using the Metropolis algorithm and greatly improved the efficiency of the algorithm. Deng *et al.*[12] further proposed two efficient cluster algorithms by combining the colouring trick with the Swendsen–Wang algorithm for loop models: the algorithm 1 and the algorithm 2. More applications of this colouring trick can be found in Refs. [13] and [14].

By using the algorithm 2,[12] the Eulerian bond-cubic model (4) has been preliminarily simulated. However, only critical points were reported. In this work, we develop a variant of the algorithm 1 to simulate the Eulerian bond-cubic model on the square lattice. We pay attention not only to the thermodynamic properties of the model, e.g. the critical points, the thermal exponent and the magnetic exponent, but also to the geometric properties such as the fractal dimensions of the critical configurations.

The paper is arranged as follows: In Section 2, we describe the cluster algorithm in details for the Eulerian bond-cubic model. In Section 3, we describe the variables to be sampled and their finite-size scaling behaviours. Our numerical results are given in Section 4 and a summary is given in Section 5.

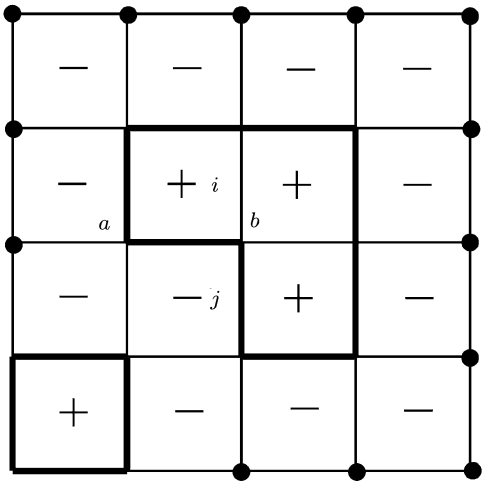


Fig. 1. A typical configuration of the Eulerian bond-cubic model and the definition of dual sites. The sites i and j are the dual sites of a and b; reversely, a and b are also the dual sites of i and j.

2. Algorithm

The partition sum of the Eulerian bond-cubic model (4) can be written as

$$Z = \sum_{\{b\}} \sum_{c_{\rm r}=0}^{N_{\rm c}} \binom{N_{\rm c}}{c_{\rm r}} (nx)^{N_{\rm b}} (n-1)^{c_{\rm r}} 1^{c_{\rm g}}, \qquad (5)$$

where $c_{\rm r}$ is the number of red components and $c_{\rm g}$ is the number of green components, with $c_{\rm r} + c_{\rm g} = n_{\rm c}$. Here, each Eulerian bond configuration is decomposed into a number of coloured-Eulerian-bond configurations. After this decomposition, the colouring algorithm can be applied in the procedure of the Monte Carlo simulation of the model. We make use of the Ising spins sitting on the dual lattice to represent configurations: if two NN Ising spins S_i and S_j on the dual lattice are different, the corresponding edge on the original lattice between S_i and S_j is occupied by a bond. The Eulerian bonds are precisely the domain wall of the Ising spins. There is a two-to-one correspondence between the Ising-spin configurations $\{S\}$ and the Eulerian bond configurations $\{b\}$, see Fig. 1 for example. In order to describe the algorithm more clearly, we define the dual sites: for an edge on the original lattice,

the dual sites of the two sites connected by the edge are the two sites on the dual lattice that sit at the two sides of the edge, reversely, the two sites connected by an edge on the original lattice are also the dual sites of the two sites that sit at the two sides of the edge. Then the Swendsen–Wang type algorithm with colouring trick for the Eulerian bond-cubic model can be described as:

1) Start from an arbitrary Ising-spin configuration, which corresponds to an Eulerian bond configuration on the original lattice.

2) Set colour to the components of the Eulerian bond configuration. For each component on the original lattice, set it as green (active) with probability $p = 1/n$, or red (inactive) with probability $1 - p$.

3) Construct the Swendsen–Wang clusters by placing percolation bonds on the dual lattice.

a) For each pair of NN sites i and j on the dual lattice, a percolation bond is placed between them with probability $p = 1$ if not all of the colours of their dual sites are green.

b) If all of the colours of their dual sites are green and $S_i = S_j$, connect them by a percolation bond with probability $p = 1 - nx$; otherwise, let the edge be vacant.

c) Each pair of NN sites on the dual lattice is considered to be in the same cluster if there is a percolation bond between them. These percolation clusters are called Swendsen–Wang clusters.

4) Flip every Swendsen–Wang cluster with probability $p = 1/2$.

5) Sample the variables of interest, erase the colours and restart at step 2).

The algorithm can be modified to be Wolff-type[15] by constructing only one cluster which is then flipped with probability $p = 1$ in steps 3) and 4).

3. Sampled variables and their finite-size scaling behaviour

A typical high-temperature (small x) configuration of the Eulerian bond-cubic model with a given n has only a few small components, as shown in Fig. 2(a). When x becomes larger, the typical configuration has more bonds and bigger components. A component that spans the whole lattice will emerge when x reaches or exceeds the critical point x_c (see Figs. 2(b) and 2(c)).

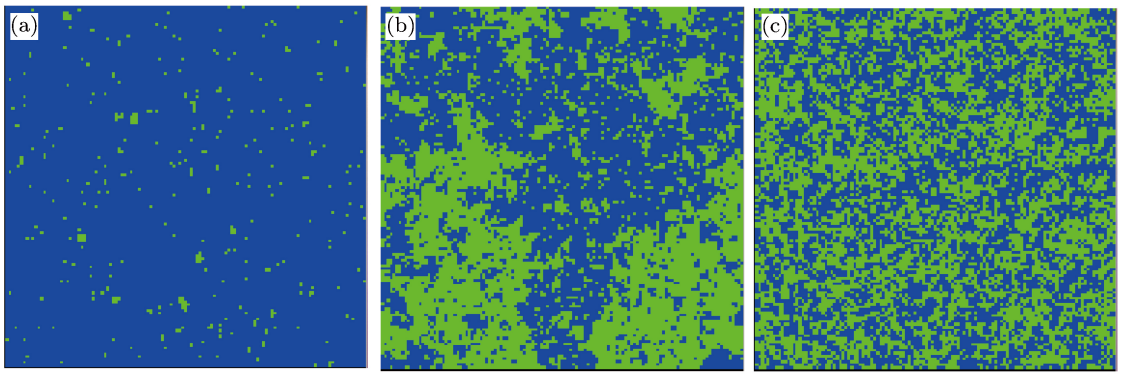


Fig. 2. (colour online) Three typical configurations for $n = 1.5$ Eulerian bond-cubic model with system size $L = 128$: (a) at high temperature $x = 0.3$, (b) at critical point $x = 0.444245$, (c) at low temperature $x = 0.6$. Colour Green means the spin on the dual lattice is '+', blue means '−'; the domain wall of the Ising clusters are exactly the Eulerian bond configurations.

The behaviour of the components in these configurations is very similar to that of clusters in percolation phenomena,[16] so we call the spanning component a percolating component. The percolation probability P_s is defined as

$$P_s = \langle P \rangle \ , \tag{6}$$

where P is 1 if there exists a percolating component on the configuration, 0 otherwise. The $\langle \cdots \rangle$ means the average over the canonical ensemble. For an infinite system, P_s is 1 for $x > x_c$ and 0 for $x < x_c$, which is a Θ function. However, for a finite system, the value of P_s changes continuously when x passes x_c, as shown in Fig. 3.

On the other hand, the phase transition of this model can be described by the Ising spins on the dual lattice. When x is small, most Ising spins have the same sign, thus the system is in a long-range ordered

state (ferromagnetic) and has a nonzero spontaneous magnetization, as shown in Fig. 2(a). When x is large, the Ising spins will be in a disordered state (paramagnetic), and the magnetization will be zero, as shown in Fig. 2(c). The phase transition of the system is a ferromagnetic one. Concretely, the magnetization m is defined as

$$m = \langle \mathscr{M} \rangle, \tag{7}$$

with

$$\mathscr{M} = \left| \frac{\sum\limits_i S_i}{V} \right|, \tag{8}$$

where $V = L^d$ is the system volume and d is the dimension of the lattice. In the current paper, $d = 2$. Figure 4 shows the magnetization m versus x for $n = 1.5$ Eulerian bond-cubic model with system size $L = 256$. Figures 2, 3 and 4 give a general description of the phase transition of the model.

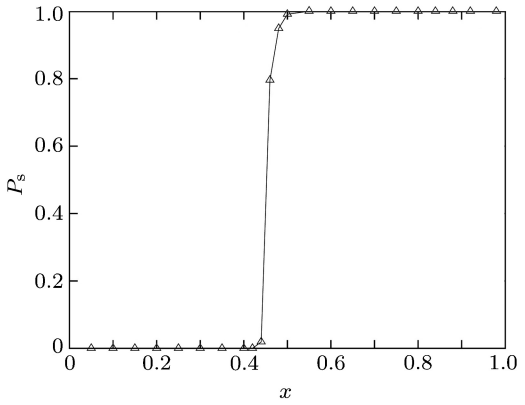


Fig. 3. The percolation probability P_s versus x for $n = 1.5$ Eulerian bond-cubic model with system size $L = 256$.

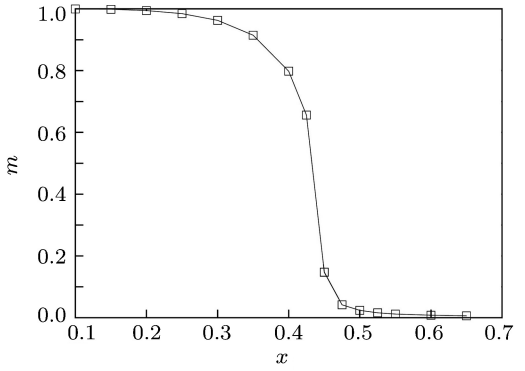


Fig. 4. The magnetization m versus x for $n = 1.5$ Eulerian bond-cubic model with system size $L = 256$.

The critical point can be determined by the percolation probability P_s and the Binder ratio of m

$$Q = \frac{\langle \mathscr{M}^2 \rangle^2}{\langle \mathscr{M}^4 \rangle}. \tag{9}$$

According to the renormalization theory, the percolation probability P_s, the magnetization m and the Binder ratio Q display the following finite-size scaling behaviour:[17,18]

$$P_\mathrm{s} = P_\mathrm{s}^{(0)} + a_1(x - x_\mathrm{c})L^{y_\mathrm{t}} + a_2(x - x_\mathrm{c})^2 L^{2y_\mathrm{t}} + \cdots + b_1 L^{y_1} + b_2 L^{y_2} + \cdots, \tag{10}$$

$$m = L^{y_\mathrm{m}-d}[m_0 + a_1(x - x_\mathrm{c})L^{y_\mathrm{t}} + a_2(x - x_\mathrm{c})^2 L^{2y_\mathrm{t}} + \cdots + b_1 L^{y_1} + b_2 L^{y_2} + \cdots, \tag{11}$$

$$Q = Q_0 + a_1(x - x_\mathrm{c})L^{y_\mathrm{t}} + a_2(x - x_\mathrm{c})^2 L^{2y_\mathrm{t}} + \cdots + b_1 L^{y_1} + b_2 L^{y_2} + \cdots, \tag{12}$$

where $y_\mathrm{t} > 0$ and $y_\mathrm{m} > 0$ are the thermal exponent and magnetic exponent respectively and y_1, $y_2, \cdots, < 0$ are the correction-to-scaling exponents. x_c is the critical point and m_0, a_i, b_i are unknown constants. We can see that P_s and Q have a similar finite-size scaling behaviour. Both Eqs. (11) and (12) can be used to determine the critical point and the thermal exponent, as will be shown in more detail in Section 4.

At the critical point x_c, Eq. (11) reduces to

$$m = L^{y_\mathrm{m}-d}(m_0 + b_1 L^{y_1} + b_2 L^{y_2} + \cdots). \tag{13}$$

Both Eq. (11) and Eq. (13) can be used to determine the magnetic exponent y_m. However, Eq. (13) is used more in practice because it has fewer parameters thus is more convenient in a data analysis.

Besides the critical exponents y_t and y_m, we are also interested in the fractal structure of the critical configurations. There are two fractals on the critical configuration: the largest Eulerian bond components and the largest Ising cluster. The Ising cluster is defined as a group of NN Ising spins in the same sign. We define the percolation strength P_∞^α and the average size χ^α based on the components and Ising clusters,

$$P_\infty^\alpha = \left\langle \frac{c_\infty}{\sum\limits_i c_i} \right\rangle, \tag{14}$$

$$\chi^\alpha = \left\langle \frac{\sum\limits_i c_i^2}{\sum\limits_i c_i} \right\rangle, \tag{15}$$

where the superscript $\alpha = \mathrm{s}$, b or c. For $\alpha = \mathrm{s}$, c_i is the number of sites in the i-th component; for $\alpha = \mathrm{b}$, c_i is the number of bonds in the i-th component; and for $\alpha = \mathrm{c}$, c_i is the number of Ising spins in the i-th Ising cluster. c_∞ is the size of the largest component or Ising cluster on the configuration. The subscript ∞ is used because only the largest component or the largest

Ising cluster may have an nonzero fraction comparing with the system volume in the thermodynamic limit. The P_∞^α can be considered to be the order parameter of the phase transition, playing a role of magnetization in the Ising model. The Greek letter χ is used to denote the average size, because it has the property that is very similar to the magnetic susceptibility of the Ising model.

P_∞^α and χ^α have the following finite-size scaling behaviours similar to that of the magnetization:

$$P_\infty^\alpha = L^{y-d}(P_\infty^{(0)} + a_1(x-x_\mathrm{c})L^{y_\mathrm{t}} + a_2(x-x_\mathrm{c})^2L^{2y_\mathrm{t}} + \cdots + b_1L^{y_1} + b_2L^{y_2} + \cdots), \quad (16)$$

$$\chi^\alpha = L^{2y-d}(\chi_0 + a_1(x-x_\mathrm{c})L^{y_\mathrm{t}} + a_2(x-x_\mathrm{c})^2L^{2y_\mathrm{t}} + \cdots + b_1L^{y_1} + b_2L^{y_2} + \cdots). \quad (17)$$

When $\alpha =$ s or b, $y = y_\mathrm{H}$ is the fractal dimension of the largest component of the critical Eulerian bond configuration, which can be viewed as the hull of the largest Ising cluster; when $\alpha =$ c, then $y = y_\mathrm{c}$ is the fractal dimension of the largest critical Ising cluster. $P_\infty^{(0)}$, χ^0, a_i and b_i are unknown constants and the $y_i < 0$ are irrelevant exponents. At the critical point x_c, Eqs. (16) and (17) reduce to

$$P_\infty^\alpha = L^{y-d}(P_\infty^{(0)} + b_1L^{y_1} + b_2L^{y_2} + \cdots), \quad (18)$$

$$\chi^\alpha = L^{2y-d}(\chi_0 + b_1L^{y_1} + b_2L^{y_2} + \cdots). \quad (19)$$

We will use Eqs. (18) and (19) to determine y_H and y_c.

4. Results

Using the Swendsen–Wang type or Wolff type algorithm described in Section 2, we performed Monte Carlo simulations for the Eulerian bond-cubic model on the square lattice. We apply 10^4 Swendsen–Wang/Wolff cycles to equilibrate the system and average over 2×10^7 samples, where each sample is taken after every 3 cycles. The sizes of the simulated systems range from $L = 8$ to $L = 256$. Figure 5 shows part of the data of P_s versus x for $n = 1.5$ Eulerian bond-cubic model near the critical point x_c. We fit the data according to Eq. (10) using the nonlinear Levenberg–Marqurdt least-squares algorithm, which yields the thermal exponent $y_\mathrm{t} = 0.747(6)$ and the critical point $x_\mathrm{c} = 0.4443(2)$.

Figure 6 shows part of the data of Q versus x for $n = 1.5$ Eulerian bond-cubic model. Fitting the data according to Eq. (12), we obtain the thermal exponent $y_\mathrm{t} = 0.748(3)$ and the critical point $x_\mathrm{c} = 0.444245(8)$. We can see that results for y_t and x_c from the fit to the data for P_s agree well with the ones from the fit to the data for Q.

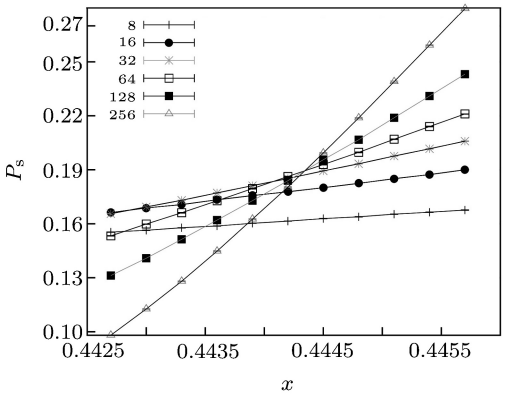


Fig. 5. The percolation probability P_s versus x for various system sizes of the $n = 1.5$ Eulerian bond-cubic model.

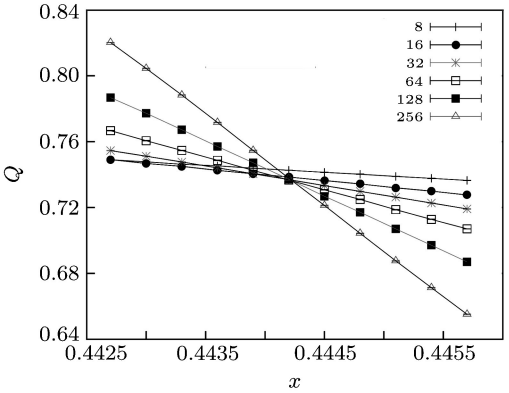


Fig. 6. The Binder ratio Q versus x for various system sizes of the $n = 1.5$ Eulerian bond-cubic model.

According to Coulomb gas theory, for $1 \le n < 2$, the Eulerian bond-cubic model belongs to the same universality class of the critical O(n) loop model, with critical exponents[19−22]

$$y_\mathrm{t} = 4 - \frac{4}{g}, \quad (20)$$

$$y_\mathrm{m} = 3 - \frac{3}{2g}, \quad (21)$$

$$y_\mathrm{H} = 1 + \frac{1}{2g}, \quad (22)$$

$$y_\mathrm{c} = 1 + \frac{g}{2} + \frac{3}{8g}, \quad (23)$$

with $n = -2\cos(\pi g)$, $1 \le g \le 2$, where g is the Coulomb gas coupling constant. According to Eq. (20), we have $y_\mathrm{t} = 0.748109\cdots$, for $n = 1.5$ Eulerian bond-cubic model, which is consistent with the

numerical result. For the critical point of the Eulerian bond-cubic model on the square lattice, there is no exact result except for the cases $n = 1$ and $n = 2$. $x_c = \sqrt{2} - 1$ for the case $n = 1$ and 0.5 for the case $n = 2$.[2] The critical points and some critical exponents of the Eulerian bond-cubic model are also determined by a finite-size scaling analysis based on numerical TM calculations in Ref. [2], where it is found $x_c = 0.44424(1)$ and $y_t = 0.749(2)$ for the $n = 1.5$ Eulerian bond-cubic model, which are in good agreement with our Monte Carlo results.

We have also simulated the cases $n = 1.0$, 1.1, 1.25, 1.7, 1.75, and 2.0, both the Coulomb gas predictions for the critical O(n) branch and the numerical results for y_t and x_c are summarized in Table 1.

Table 1. Thermal exponents and critical points of the Eulerian bond-cubic model on the square lattice.

n	y_t(T)	$y_t(P_s)$	$y_t(Q)$	y_t(TM)	$x_c(P_s)$	$x_c(Q)$	x_c(TM)	Q_0	$P_s^{(0)}$
1.0	1.000000	1.01(2)	0.998(2)	1.000000(1)	0.41422(1)	0.414214(2)	0.4142135(1)	0.8560(2)	0.0651(3)
1.1	0.957313	0.959(3)	0.957(3)	0.9572(5)	0.41916(1)	0.419155(4)	0.419154(2)	0.8359(2)	0.0812(4)
1.25	0.887399	0.890(4)	0.888(2)	–	0.42741(1)	0.427404(3)	–	0.8022(1)	0.110(1)
1.5	0.748109	0.747(6)	0.748(3)	0.749(2)	0.4443(2)	0.444245(8)	0.44424(1)	0.7454(3)	0.1904(5)
1.7	0.600379	0.598(4)	0.604(6)	0.595(5)	0.4624(3)	0.46213(2)	0.46214(1)	0.6686(3)	0.294(2)
1.75	0.552482	0.554(2)	0.57(3)	–	0.4679(4)	0.46754(2)	–	0.649(2)	0.329(1)
2.0	0.000000	0.52(3)	0.48(3)	0.50000(1)	0.4998(3)	0.50001(1)	0.5000000(1)	0.550(2)	0.554(5)

Note: in the parentheses, T represents theoretical predictions of the critical O(n) branch, P_s and Q denote Monte Carlo results from the fits to the data for P_s and Q, respectively; and TM means the results calculated by TM.[2]

One should pay more attention to the case of $n = 2$. The thermal exponent y_t is 0 for the critical branch of O(2) loop model, while it is 0.5 for the $n = 2$ Eulerian bond-cubic model. For the case $n = 2$, the Eulerian bond-cubic model reduces to a special case of the Ashkin-Teller model[20,23–25]. Our estimations of y_t agree with the exact results $y_t = 0.5$ for the Ashkin–Teller model.

At the critical point, we sampled the magnetization m, percolation strength P_∞^c, P_∞^s, P_∞^b and the average cluster sizes χ^c, χ^s, χ^b.

The log–log plot of the data for magnetization m versus system size L for $n = 1.5$ Eulerian bond-cubic model at the critical point is shown in Fig. 7. Fitting the data according to Eq. (13), we obtain the magnetic exponent $y_m = 1.7803(3)$, which is consistent with the theoretical prediction $y_m = 1.78054, \cdots$, given by Eq. (21).

The log–log plot of the data for P_∞^c and χ^c versus L are shown in Fig. 8 and Fig. 9 respectively. Fitting the data, we obtain the fractal dimension of the largest critical Ising cluster $y_c = 1.9195(10)$ (fit from P_∞^c) and $y_c = 1.9195(11)$ (fit from χ^c), which are consistent with the theoretical prediction $y_c = 1.91989\cdots$, given by Eq. (23).

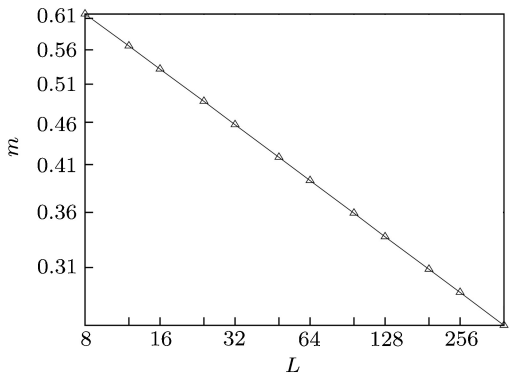


Fig. 7. The log–log plot of the magnetization m versus system size L for $n = 1.5$ Eulerian bond-cubic model. The error bars are much smaller than the size of the data points.

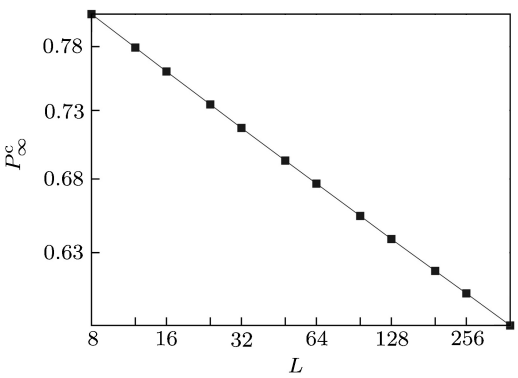


Fig. 8. The log–log plot of the percolation strength P_∞^c versus system size L for $n = 1.5$ Eulerian bond-cubic model. The error bars are much smaller than the size of the data points.

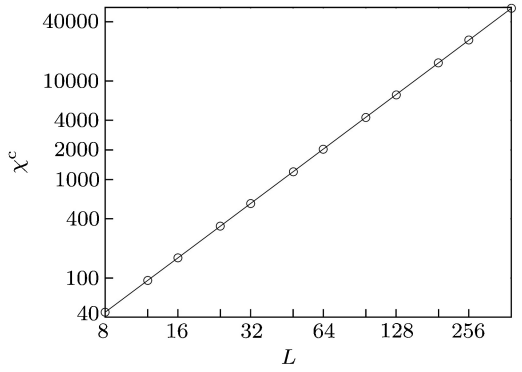


Fig. 9. The log–log plot of the average size $\chi^{\rm c}$ versus system size L for $n = 1.5$ Eulerian bond-cubic model. The error bars are much smaller than the size of the data points.

Fitting the data of $P_\infty^{\rm s}$, $\chi^{\rm s}$, $P_\infty^{\rm b}$ and $\chi^{\rm b}$, we obtain the fractal dimension of the largest critical component $y_{\rm H} = 1.410(5)$ (fit from $P_\infty^{\rm s}$), $y_{\rm H} = 1.405(4)$ (fit from $\chi^{\rm s}$), $y_{\rm H} = 1.410(4)$ (fit from $P_\infty^{\rm b}$) and $y_{\rm H} = 1.41(1)$ (fit from $\chi^{\rm b}$). All of them are consistent with the theoretical prediction $y_{\rm H} = 1.4064, \ldots$, given by Eq. (22).

All the numerical results and the theoretical predictions of $y_{\rm m}$, $y_{\rm c}$ and $y_{\rm H}$ are listed in Tables 2 and 3. From these tables, we can see that the fractal dimension $y_{\rm c}$ of the largest critical Ising cluster of the $n = 2$ Eulerian bond-cubic model is the same as the critical O(2) value. However, the fractal dimension of the largest critical Eulerian bond component is $y_{\rm H} = 1.625$, which is obviously different from the critical O(2) value $y_{\rm H} = 1.5$, given by Eq. (22). In addition, the magnetic exponent $y_{\rm m}$ is different from the critical O(2) value 1.5. These results agree well with the exact results $y_{\rm m} = 1.625 = y_{\rm H}$ for the Ashkin–Teller model,[23−25] and again show the difference between the cubic symmetry and the O(n) symmetry in the case $n = 2$, when the cubic anisotropy becomes marginally relevant.

Table 2. The magnetic exponents $y_{\rm m}$ and fractal dimension $y_{\rm c}$ of the Eulerian bond-cubic model on the square lattice.

n	$y_{\rm m}$(T)	$y_{\rm m}$(MC)	$y_{\rm m}$(TM)	$y_{\rm c}$(T)	$y_{\rm c}(P_\infty^{\rm c})$	$y_{\rm c}(\chi^{\rm c})$
1.0	1.875	1.8751(2)	1.87501(1)	1.94792	1.9476(3)	1.9477(3)
1.1	1.85899	1.8590(1)	1.85895(5)	1.94257	1.9422(3)	1.9423(3)
1.25	1.83277	1.832(1)	–	1.93436	1.933(2)	1.933(2)
1.5	1.78054	1.7803(3)	1.7805(5)	1.91989	1.9195(10)	1.9195(11)
1.7	1.72514	1.724(1)	1.726(2)	1.90702	1.906(1)	1.909(3)
1.75	1.70786	1.709(2)	–	1.90347	1.902(2)	1.90(1)
2.0	1.5	1.6249(4)	1.62500(1)	1.875	1.875(1)	1.8750(1)

Note: in the parentheses, MC represents Monte Carlo results, $P_\infty^{\rm c}$ and $\chi^{\rm c}$ denote Monte Carlo results from the fits to the data for $P_\infty^{\rm c}$ and $\chi^{\rm c}$, respectively.

Table 3. The fractal dimension $y_{\rm H}$ of the Eulerian bond-cubic model on the square lattice.

n	$y_{\rm H}$(T)	$y_{\rm H}(P_\infty^{\rm s})$	$y_{\rm H}(\chi^{\rm s})$	$y_{\rm H}(P_\infty^{\rm b})$	$y_{\rm H}(\chi^{\rm b})$
1.0	1.375	1.376(1)	1.373(4)	1.378(4)	1.372(4)
1.1	1.3803	1.3806(9)	1.380(5)	1.382(4)	1.379(3)
1.25	1.3891	1.391(2)	1.390(2)	1.393(4)	1.389(1)
1.5	1.4064	1.410(5)	1.405(4)	1.410(4)	1.41(1)
1.7	1.4249	1.426(4)	1.43(2)	1.42(1)	1.430(6)
1.75	1.4307	1.43(1)	1.431(12)	1.427(11)	1.43(1)
2.0	1.5	1.625(1)	1.6253(5)	1.625(1)	1.624(1)

Note: in the parentheses, $P_\infty^{\rm s}$ represent Monte Carlo results from the fits to the data for $P_\infty^{\rm s}$ and $\chi^{\rm s}$ and $\chi^{\rm s}$ repectively.

5. Summary

We simulated the Eulerian bond-cubic model on the square lattice using an efficient cluster algorithm. Two fractal dimensions of the critical configurations as well as the critical points, the thermal and magnetic exponents of the model are determined by means of a finite-size scaling analysis. The two fractal dimensions are for the first time obtained numerically. The estimations of the critical points and the thermal and magnetic exponents are in good agreement with those obtained by means of TM calculations[2] for several values of n, including the case $n = 2$.

Our results for all critical exponents are consistent with the Coulomb gas predictions of the critical O(n) branch for $n < 2$. However, for $n = 2$, the thermal exponent, the magnetic exponent and the fractal dimensions $y_{\rm H}$ are different from the critical O(2) values and the model reduces to a special case of the Ashkin–Teller model. Our study confirms that the phase transition of the Eulerian bond-cubic model belongs to the critical O(n) universality class for $n < 2$. The cubic anisotropy is irrelevant for $n < 2$, but becomes marginal when $n = 2$.

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