Generalized synchronization of two unidirectionally coupled discrete stochastic dynamical systems*

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The existence of two kinds of generalized synchronization manifold in two unidirectionally coupled discrete stochastic dynamical systems is studied in this paper. When the drive system is chaotic and the modified response system collapses to an asymptotically stable equilibrium or asymptotically stable periodic orbit, under certain conditions, the existence of the generalized synchronization can be converted to the problem of a Lipschitz contractive fixed point or Schauder fixed point. Moreover, the exponential attractive property of generalized synchronization manifold is strictly proved. In addition, numerical simulations demonstrate the correctness of the present theory. The physical background and meaning of the results obtained in this paper are also discussed.

Keywords: generalized synchronization manifold, discrete stochastic dynamical system, Lipschitz smoothness, Hölder continuity

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1. Introduction

In 1995, Rulkov *et al.*[1] first described the generalized synchronization (GS), which arises when a functional relation different from the identity is established between the drive or forcing subsystem and the driven or response subsystems in coupled systems. The GS in chaotic systems proposed by Rulkov *et al.* has profound physical meanings, which is connected with some fundamental problems in system science, such as the formation of patterns in complex systems, self-organization phenomena, emergence and so on.[2] In the 1970s and the 1980s, various self-organization theories thrived in Europe, of which Prigoging's dissipative structure theory and Haken's synergetics are especially prominent. The American SFI (Santa Fe Institute) is established to discuss emergence phenomena, whose first two seminar themes are both "scientific synthesis in emergence". In the 1990s, they emphasized the emergence as Santafe theme or Santafe viewpoint, contacting emergence with self-organizational complexity explicitly and published many monographs. Haken's slaving theory of synergetics in all the discussions about the concept of the emergence of complex systems is the most fundamental and simple one. To explain the slaving theory, the centre manifold theory and the inertial manifold theory are established in mathematics. Haken's thought that slow variables slave fast variables is at the heart of these theories. Haken repeatedly stated that order parameters as variables describing the global macro were formed when microeconomic subsystems interacted with each other in critical conditions, which could not be investigated at all on the micro level (reduction failure theory). This discussion is the typical application of emergence. The self-organization theory gives the scientific method for describing some emergence phenomena in fact. The property that the whole has while the part dose not have is called emergent property. From the hierarchical structure point of view, the emergent property is the attributive, characteristic or behaviour function existing on the high level but not on the low level.

The GS theory is a relatively suitable mathematical theory for studying the emergence phenomenon. The GS is an interesting and important topic which has been the subject of many theoretical and numerical studies,[3−16] and has been observed in laboratory experiments.[17−21] Meanwhile, some methods have been proposed to study the GS, such as the auxiliary system method,[22−23] the modified system method,[24] and the Lyapunov exponent method.[25] Recently, Alvarez-Llamoza and Cosenza[26] extended

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the concept of GS from determined systems to stochastic systems and present the minimal conditions for the occurrence of GS of stochastic systems by the Lyapunov exponent method.

In this paper, we study the existence and the property of the GS of the stochastic system by the auxiliary system method and the modified system method. We theoretically derive the sufficient conditions for the existence of the Lipschitz smooth GS and the Hölder continuous GS in stochastic dynamical systems by using the Lipschitz contraction mapping principle and Schauder fixed point theory. The obtained theorems and numerical simulations in this paper generalize the results of Refs. [3] and [27] to discrete stochastic dynamical systems. Based on the modified system approach, the GS can be classified into three types: when the drive system is chaotic, the modified response system collapses to an asymptotically stable equilibrium, asymptotically stable periodic orbit and chaos, which are respectively called the first kind of GS, the second GS and the third GS. In this paper, we will interpret them from the physical point of view and prove the existence and property of the first two kinds of GS therein.

It is known that the GS theory is a relatively suitable mathematical theory for studying the emergence phenomenon. However, the big defect of the GS discussed at present in drive-response systems, the adaptive strategy, that is to say, the response system makes different responses to the drive system in different conditions, has not been introduced. The complete synchronization of the prisoner's dilemma problem was studied using the game theory in Ref. [28], but the authors only gave numerical results and did not establish the relatively complete theory. We attempt to give preliminary explanations of this emergence phenomenon of how the GS in intercoupling chaotic systems with different responses is generated in this paper. We find that it can be strictly interpreted in theory by applying Haken's thought of slaving theory that slow variables (order parameters) slave fast variables. In the following section, we explain it from the physical point of view and give strict mathematic proof.

2. Existence and property of the first Lipschitz smooth GS

We consider the following two unidirectionally coupled discrete stochastic dynamical system:

$$x_{n+1} = g(x_n), \tag{1}$$

$$y_{n+1} = \begin{cases} \omega(x_n, y_n), & \text{(with probability } p\text{)}, \\ f(y_n), & \text{(with probability } 1-p\text{)}, \end{cases} \tag{2}$$

where $F_K : (x_n, y_n) \longrightarrow (x_{n+1}, y_{n+1})$ is a map, $x_n \in R^m$, $y_n \in R^n$; and $g(x_n)$ and $f(y_n)$ describe the dynamics of the drive and the driven variables, respectively; the coupling relation between which is chosen to be $\omega(x_n, y_n) = (1-\epsilon)f(y_n) + \epsilon g(x_n)$, with ϵ being the coupling strength. Assume that $g(x)$ is the continuous function and $f(y)$ is the continuous smooth function. Suppose that there is a dissipative ball $B \subset R^{n+m}$, which satisfies $F_K(B) \subset \mathrm{Int}(B)$, $B = B_x \times B_y$ with $B_x \subset R^m$, $B_y \subset R^n$, and A_K is a maximal attractor in B, where $A_K = \bigcap_{n=0}^{\infty} F_K^n(B)$.

There exists a certain relationship, which can also be regarded as some self-organization, some pattern, namely some invariant manifold in mathematics, between two chaotic systems when they are being coupled with each other in some condition. The relationship is generated as the changing of control parameters, being just a relatively simple emergence, which is called GS, having many characteristics of emergence. We can give the definition of chaotic GS as follows.

Definition 1 Given two dynamical systems X and Y, if there exists a manifold $S = \{(X, Y), Y = \Phi(X)\}$, which includes at least one attractor, then X and Y carry out GS, where Φ and S are respectively called the GS map and the GS manifold.

Suppose that the auxiliary system is

$$\tilde{y}_{n+1} = \begin{cases} \omega(x_n, \tilde{y}_n), & \text{(with probability } p\text{)}, \\ f(\tilde{y}_n), & \text{(with probability } 1-p\text{)}, \end{cases} \tag{3}$$

where $\omega(x_n, \tilde{y}_n) = (1-\epsilon)f(\tilde{y}_n) + \epsilon g(x_n)$. Let the response system and the auxiliary system be completely synchronized, that is, $\lim_{n\to\infty} |y_n - \tilde{y}_n| = 0, (x_n, y_n) = F_K^n(x_0, y_0)$ and $(x_n, \tilde{y}_n) = F_K^n(x_0, \tilde{y}_0)$.

The thought of our defined modified system is the specification of the method of identifying fast variable and slow variable. Define the modified equation of response system as

$$y_{n+1} = (1-\epsilon)f(y_n). \tag{4}$$

According to the dynamical behaviour of the modified response system (4), the GS between systems (1) and (2) can be classified into three types: when the drive system (1) is chaotic and the modified response system (4) collapses to an asymptotically stable equilibrium, asymptotically stable periodic orbit and chaos, they are called the first kind of GS, the second GS and the third GS respectively.

The fact that the X system is chaotic shows that the X variable is a slow variable (order parameter) and the Y system collapses to an asymptotically stable equilibrium (the first kind of fast variable) or asymptotically stable periodic orbit (the second kind of fast variable). From Haken's synergetics, the result of X variable slaving Y variable in some conditions can be derived. We will mathematically analyse the cause and the property of the slaving process.

We first study the existence and the property of the first kind of GS in this section, in other words, we first mathematically analyse the cause and the property of the slaving process that X variable (order parameter) slaves Y variable (the first kind of fast variable).

Suppose that $\bar{y}$ is the asymptotically stable equilibrium of modify equation (4), i.e., $\bar{y} = (1-\epsilon)f(\bar{y})$, then all eigenvalues of the Jacobi matrix $D(1-\epsilon)f(\bar{y})$ will satisfy $|\lambda| < 1$.

First of all, we should give the characterization of convergence of the fast variable. The following two lemmas show the metric of this convergence in detail.

Lemma 1[29] Let X and Y be two Banach spaces, $U \subset X$ is an open set, the map $f : U \longrightarrow Y$ is continuously differentiable, and $x + th \in U$, $(\forall t \in [0,1])$, then $f(x+h) - f(x) = \int_0^1 Df(x+th)h\mathrm{d}t$.

Lemma 2 Suppose that $y_n, \tilde{y}_n, n > N$ is in δ-neighbourhood of $\bar{y}$, then when δ is sufficiently small,

$$|y_{n+1} - \tilde{y}_{n+1}| \leq k|y_n - \tilde{y}_n|, \quad 0 < k < 1. \tag{5}$$

Proof When δ is sufficiently small, $|y - \bar{y}| < \delta$, the norms of eigenvalues of $D(1-\epsilon)f(\bar{y})$ and $D(1-\epsilon)f(y)$ are all less than 1. By Lemma 1, we obtain

$$\begin{aligned}
&|y_{n+1} - \tilde{y}_{n+1}| \\
&= |\omega(x_n, y_n) - \omega(x_n, \tilde{y}_n)| \\
&= |(1-\epsilon)f(y_n) - (1-\epsilon)f(\tilde{y}_n)| \\
&= \int_0^1 D(1-\epsilon)f(y_n + t(\tilde{y}_n - y_n))|y_n - \tilde{y}_n|\mathrm{d}t \\
&\leq k|y_n - \tilde{y}_n|, \quad 0 < k < 1.
\end{aligned}$$

This completes the proof.

We suppose that the drive system (1) and response system (2) satisfy the following two conditions:

Supposition 1 In systems (1) and (2), assume that f, g satisfy

$$\begin{aligned}
&|f(y_n) - f(\tilde{y}_n)| \leq r_1|y_n - \tilde{y}_n|, \\
&|g^{-1}(x_n) - g^{-1}(\tilde{x}_n)| \leq r_2|x_n - \tilde{x}_n|, \\
&r_1 \leq k, \quad r_2 > 1.
\end{aligned} \tag{6}$$

Supposition 2 In systems (2), assume that ω satisfies

$$|\omega(x_n, y_n) - \omega(\tilde{x}_n, y_n)| \leq \eta|x_n - \tilde{x}_n|, \quad \eta > 0. \tag{7}$$

The above two suppositions are very natural, which show that the two response strategies of the response system satisfy the usual Lipschitz property and also depict the extension property of the chaotic X system as a slow variable.

Based on the above, we present the theoretical result about the existence of the first Lipschitz smooth GS manifold as Theorem 1.

Theorem 1 Under conditions (5), (6) and (7), for $x_n \in B_x, y \in B_y$, if

$$[p(kL+\eta) + (1-p)r_1L]r_2 \leq L,$$

$[p(1-\epsilon)r_1 + (1-p)r_1]M + \mathrm{Sup}_{x_n \in B_x}|[p(1-\epsilon)r_1 + (1-p)r_1]\bar{y} + (1-p\epsilon)f(\bar{y}) + p\epsilon g(x_n)| \leq M$, where M, L are positive constants, then there exists Lipschitz smooth GS manifold $S = \{(x,y) \mid y = h(x)\}$ between systems (1) and (2).

The proof of Theorem 1 can be found in appendix A.

The thought of the proof of Theorem 1 is very intuitive. Nearly asymptotically stable equilibrium of the modified response system, whether the chaotic orbit of drive system (X system) possibly establishes invariant manifold, depends on the balance between the attractive force towards the equilibrium and chaotic repulsive force of the drive system, especially as a characterization of fast variable (for Supposition 1). We estimate the balance between the attractive force and the repulsive force by using contractive fixed point theory. Because the effective and universal method of calculating GS manifold has not been found now, there is no simple method with evident physical concept of constructively proving GS.

Theorem 2 Under the conditions of Theorem 1, the GS manifold is exponentially attractive.

Proof Let

$$x_{n+1} = g(x_n),$$

$$h(x_{n+1}) = \begin{cases} \omega(x_n, h(x_n)), & \text{(with probability } p\text{)}, \\ f(h(x_n)), & \text{(with probability } 1-p\text{)}. \end{cases}$$

From Lemma 1, we obtain

$$\begin{aligned} & |y_{n+1} - h(x_{n+1})| \\ \leq\ & k|y_n - h(x_n)| \\ \leq\ & \cdots \\ \leq\ & k^n|y_0 - h(x_0)| \to 0, \quad (n \to \infty). \end{aligned}$$

This proves the GS manifold is attractive.

Haken's slaving principle as a physical theory is very profound, but from a mathematical point of view, it is relatively rough. The proof of Theorem 2 shows that if the convergent strength is very big, the absorbing set of GS manifold is a great region and this manifold is smooth, but if the convergent strength is not very big, the absorbing set of GS manifold is very complex, probably it is a fractal set and this manifold is only Hölder continuous.

3. Existence of the first Hölder continuous GS

From the proof of Theorem 1, we note that the condition $|y_{n+1} - \tilde{y}_{n+1}| \leq k|y_n - \tilde{y}_n|, 0 < k < 1$ of Lemma 2 is too strict. Commonly, for not all $k_n < 1$, the condition should be

$$|y_{n+1} - \tilde{y}_{n+1}| \leq k_n|y_n - \tilde{y}_n|,\ 0 < k_n < \tilde{k},\ \tilde{k} > 1. \quad (8)$$

Certainly, under such conditions, Theorem 1 does not hold as the Lipschitz contraction mapping principle cannot be applied any more. However, we can turn the problem of the existence of GS manifold into the problem of Schauder fixed point in Hölder continuous function space. Here, Supposition 1 becomes the following Supposition 3.

Supposition 3 In systems (1) and (2), assume that f and g satisfy

$$\begin{aligned} & |f(y_n) - f(\tilde{y}_n)| \leq r_1|y_n - \tilde{y}_n|, \\ & |g^{-1}(x_n) - g^{-1}(\tilde{x}_n)| \leq r_2|x_n - \tilde{x}_n|, \\ & r_1 \leq \tilde{k}, \quad r_2 > 1. \end{aligned} \quad (9)$$

Definition 2 The function $f(x)$ has α order Hölder exponential property, if $|f(x_1) - f(x_2)| \leq L_\alpha|x_1 - x_2|^\alpha, 0 < \alpha < 1$, where $|\cdot|$ denotes the norm in Eculidean space. In other words, $f(x)$ has α order Hölder continuity.

We introduce another two Lemmas in Ref. [30].

Lemma 3 (Schauder fixed point Theorem) Let $\hat{M}$ be a bounded closed convex set of Banach space B and $F : \hat{M} \to \hat{M}$ is a compact map, in other words, (a) F is continuous; (b) F maps a bounded set onto a relative compact set in $\hat{M}$. Then there is $z_0 \in \hat{M}$ such that $F(z_0) = z_0$.

Lemma 4 (Arzela Theorem) Let Ω be a bounded region in R^n and $\tilde{M}$ be a subset of $C_0(\bar{\Omega})$. If (a) there exists a constant K such that $\mathrm{Sup}_{x\in\Omega}|f(x)| \leq K, \forall f \in \tilde{M}$; (b) for arbitrary given $\varepsilon > 0$, there exists $\delta > 0$, $|x - y| < \delta, x, y \in \Omega$, such that $|f(x) - f(y)| < \varepsilon, \forall f \in \tilde{M}$, then there is a function sequence $\{f_n\}$ in $\tilde{M}$, which converges to f in $C_0(\bar{\Omega})$.

Based on the above mentioned, we obtain the following Theorem.

Theorem 3 Under conditions (7)–(9), for $x_n \in B_x, y \in B_y$, we assume that

$$[p(\tilde{k}L_\alpha + \eta(2\rho)^{1-\alpha}) + (1-p)r_1L_\alpha](r_2)^\alpha \leq \tilde{L}_\alpha, \quad (10)$$

$$\begin{aligned} & [p(1-\epsilon)r_1 + (1-p)r_1]M \\ & + \mathrm{Sup}_{x_n\in B_x}|[p(1-\epsilon)r_1 + (1-p)r_1]\bar{y} \\ & + (1-p\epsilon)f(\bar{y}) + p\epsilon g(x_n)| \leq \tilde{k}M + m, \end{aligned} \quad (11)$$

$$M + \tilde{L}_\alpha \leq R, \quad \tilde{k}M + m + \tilde{L}_\alpha \leq R, \quad (12)$$

where the $M, m, \eta, \rho, \tilde{L}_\alpha$ and R are positive constants, then there will be a Hölder continuous GS manifold $S = \{(x, y) \mid y = h(x)\}$ between systems (1) and (2).

The proof of Theorem 3 may be found in appendix B.

The thought of the proof of Theorem 3 is also very intuitive. Nearly asymptotically stable equilibrium of the modified response system, whether the chaotic orbit of drive system (X system) possibly establishes invariant manifold depends on the balance between the attractive force towards the equilibrium and the chaotic repulsive force of the drive system. Now we need only to use a relatively weak Schauder fixed theorem instead of contractive fixed point theorem and no longer obtain the uniqueness of the GS manifold.

We do not discuss the attractive property of the GS. In fact, the attractive set should be fractal, as we proved in Ref. [31].

4. Existence of the second kind of GS

In this section, consider that system (1) is chaotic and system (4) has asymptotically stable periodic solutions. Let $\bar{y}^{(2)}$ be an asymptotic stable 2 periodic points of the modified equation (4), i.e.,

$$\bar{y}^{(2)} = (1-\epsilon)f((1-\epsilon)f(\bar{y}^{(2)})), \tag{13}$$

then each eigenvalue λ of Jacobi matrix $D(1-\epsilon)f((1-\epsilon)f(\bar{y}^{(2)})$ satisfies $|\lambda| < 1$. Systems (1) and (2) can be converted into the following systems:

$$x_{n+2} = g(g(x_n)) = g^{(2)}(x_n), \tag{14}$$

$$\begin{aligned} & y_{n+2} \\ &= \begin{cases} \omega(x_{n+1}, y_{n+1}), & \text{(with probability } p\text{)}, \\ f(y_{n+1}), & \text{(with probability } 1-p\text{)}, \end{cases} \\ &= \begin{cases} \omega(g(x_n), \omega(x_n, y_n)), & \text{(with probability } p\text{)}, \\ f(f(y_n)), & \text{(with probability } 1-p\text{)}, \end{cases} \\ &= \begin{cases} \omega^{(2)}(x_n, y_n), & \text{(with probability } p\text{)}, \\ f^{(2)}(y_n), & \text{(with probability } 1-p\text{)}. \end{cases} \end{aligned} \tag{15}$$

Thus the second kind of GS of systems (1) and (2) can be turned into the first GS of systems (14) and (15). Accordingly the conditions in Section 2 become the following inequalities:

$$|y_{n+2} - \tilde{y}_{n+2}| \le k|y_n - \tilde{y}_n|, \quad 0 < k < 1. \tag{16}$$

Assumption 4 In systems (14) and (15), assume that $f^{(2)}$ and $(g^{(2)})^{-1}$ satisfy

$$\begin{aligned} & |f^{(2)}(y_n) - f^{(2)}(\tilde{y}_n)| \le r_1|y_n - \tilde{y}_n|, \\ & |(g^{(2)})^{-1}(x_n) - (g^{(2)})^{-1}(\tilde{x}_n)| \le r_2|x_n - \tilde{x}_n|, \\ & r_1 \le k, \quad r_2 > 1. \end{aligned} \tag{17}$$

Assumption 5 In system (15), assume that

$$|\omega^{(2)}(x_n, y_n) - \omega^{(2)}(\tilde{x}_n, y_n)| \le \eta|x_n - \tilde{x}_n|, \eta > 0. \tag{18}$$

Theorem 4 Under the conditions of inequalities (16)–(18), for $x_n \in B_x$ and $y \in B_y$, if $[p(kL+\eta)+(1-p)r_1L]r_2 \le L$, $[p(1-\epsilon)r_1+(1-p)r_1]M + \mathrm{Sup}_{x_n\in B_x}|[p(1-\epsilon)r_1+(1-p)r_1]\bar{y} + (1-p\epsilon)f(\bar{y}) + p\epsilon g(x_n)| \le M$, where M and L are positive constants, then systems (1) and (2) have the second Lipschitz smooth GS manifold $S = \{(x,y) \mid y = h(x)\}$.

When for not all $k_n < 1$, condition (8) in Section 3 becomes

$$|y_{n+2} - \tilde{y}_{n+2}| \le k_n|y_n - \tilde{y}_n|, \quad 0 < k_n < \tilde{k}, \tag{19}$$

and Supposition 4 becomes

$$\begin{aligned} & |f^{(2)}(y_n) - f^{(2)}(\tilde{y}_n)| \le r_1|y_n - \tilde{y}_n|, \\ & |(g^{(2)})^{-1}(x_n) - (g^{(2)})^{-1}(\tilde{x}_n)| \le r_2|x_n - \tilde{x}_n|, \\ & r_1 \le \tilde{k}, \quad r_2 > 1. \end{aligned} \tag{20}$$

We obtain the following theorem.

Theorem 5 Under conditions of inequalities (18)–(20), for $x_n \in B_x, y \in B_y$, assume that

$$\begin{aligned} & [p(\tilde{k}L_\alpha + \eta(2\rho)^{1-\alpha}) + (1-p)r_1L_\alpha](r_2)^\alpha \le \tilde{L}_\alpha, \\ & [p(1-\epsilon)r_1 + (1-p)r_1]M \\ & + \mathrm{Sup}_{x_n\in B_x}|[p(1-\epsilon)r_1 + (1-p)r_1]\bar{y}^2 \\ & + (1-p\epsilon)f(\bar{y}^2) + p\epsilon g(x_n)| \le \tilde{k}M + m, \\ & M + \tilde{L}_\alpha \le R, \quad \tilde{k}M + m + \tilde{L}_\alpha \le R, \end{aligned}$$

where $M, m, \eta, \rho, \tilde{L}_\alpha$ and R are positive constants, then systems (1) and (2) have the second Hölder continuous GS manifold $S = \{(x,y) \mid y = h(x)\}$.

5. Numerical simulations

In order to further illustrate the theoretical results established in the previous section, we take the Logistic system as the fundamental model. The drive system and the response system are respectively described as

$$f(n+1) = af(n)(1-f(n)), \tag{21}$$

$$g_{n+1} = \begin{cases} (1-k)bg(n)(1-g(n)) + kaf(n)(1-f(n)), & \text{(with probability } p\text{)}, \\ bg(n)(1-g(n)), & \text{(with probability } 1-p\text{)}. \end{cases} \tag{22}$$

Accordingly, the modified equation and auxiliary system of response system can be respectively described as

$$w(n+1) = (1-k)bw(n)(1-w(n)), \tag{23}$$

and

$$h_{n+1} = \begin{cases} (1-k)bh(n)(1-h(n)) + kaf(n)(1-f(n)), & \text{(with probability } p\text{)}, \\ bh(n)(1-h(n)), & \text{(with probability } 1-p\text{)}. \end{cases} \tag{24}$$

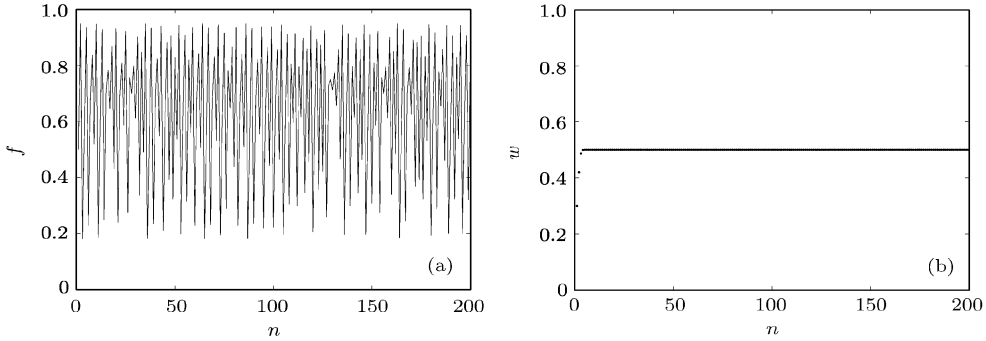


Fig. 1. (a) Chaotic diagram of drive system (21) and (b) graphical representation of modified response system (23).

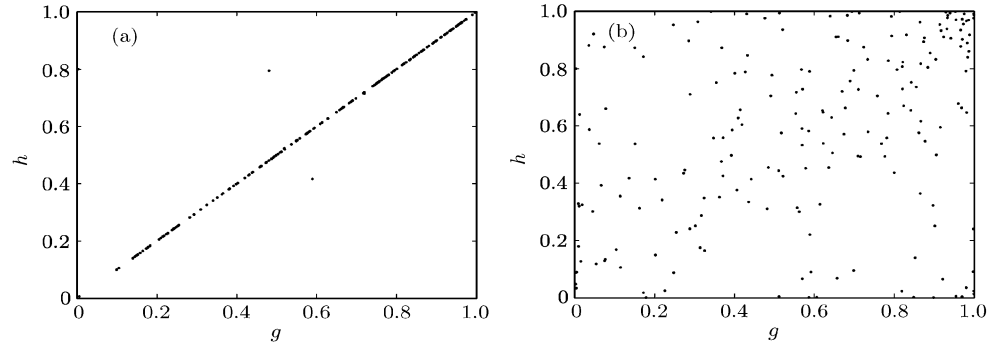


Fig. 2. Graphical representations of relationship between the response system (22) and auxiliary system (24) with coupling strength $k = 0.5$ for (a) $p = 0.8$ (complete synchronization) and (b) $p = 0.3$ (unsynchronized).

5.1. Existence of the first GS

We choose the parameter values and initial values respectively as $a = 3.8, b = 4,\ f(1) = 0.5, w(1) = 0.3, g(1) = 0.003$ and $h(1) = 0.8$ and the coupling strength as $k = 0.5$. Under the above conditions, drive system (21) is chaotic while modified response system (23) approaches an asymptotically stable equilibrium (see Fig. 1). In these cases, when probability $p = 0.8$, response system (22) and auxiliary system (24) carry out complete synchronization, i.e., drive system (21) and response system (22) carry out GS and when probability $p = 0.3$, response system (22) and auxiliary system (24) are unsynchronized, which is shown in Fig. 2.

5.2. Existence of the second GS

With the above initial values and the parameter values and coupling strength being respectively

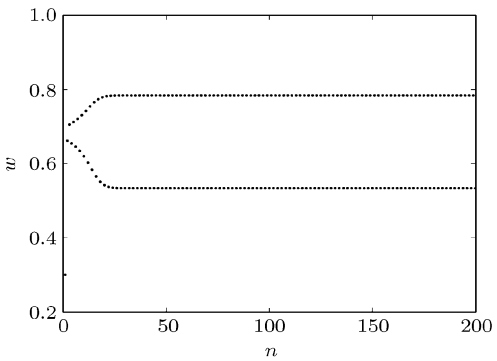


Fig. 3. Graphical representation of the modified response system (23) with $b = 3.5$ and $k = 0.1$.

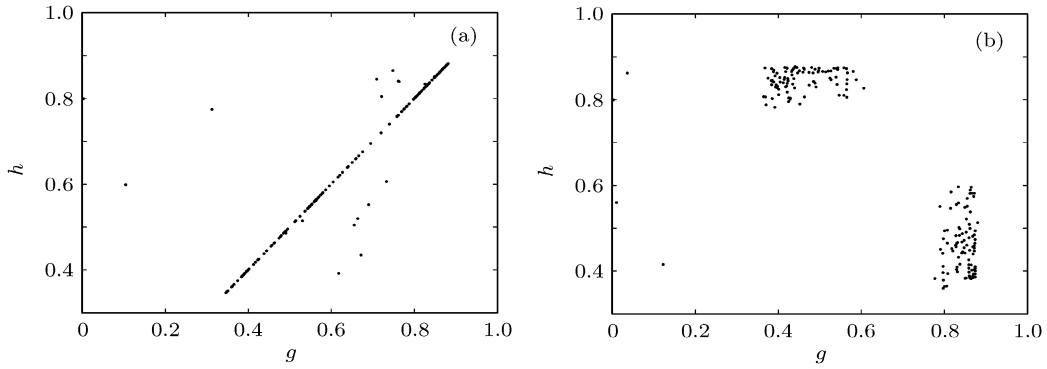


Fig. 4. Graphical representations of relationship between the response system (22) and auxiliary system (24) with coupling strength $k = 0.1$ for (a) $p = 0.9$ (complete synchronization) and (b) $p = 0.3$ (unsynchronized).

$a = 3.8, b = 3.5$, $k = 0.1$, drive system (21) is still chaotic (see Fig. 1(a)) while modified response system (23) collapses to asymptotically stable periodic orbit (see Fig. 3). In these cases, when probability $p = 0.9$, response system (22) and auxiliary system (24) realize complete synchronization and when probability $p = 0.3$, response system (22) and auxiliary system (24) are unsynchronized. The results are shown in Fig. 4.

6. Conclusions and discussion

The study of the emergence phenomenon in system science is a difficult point of modern science. Now there has not been complete mathematical theory in the literature of system science, which gives only the descriptive theory in philosophy or system science. We attempt to establish emergence theory by applying the GS theory.

The following features are the main characteristics of emergence:

1) Emergence is a holistic phenomenon and characteristic property, but the holistic phenomenon and characteristic property is not necessarily an emergent property. The characteristic properties obtained by adding the part of characteristic properties are not emergent properties and only those that are dependent on a special relationship among the parts, the so-called constructive characteristics, are emergent property.

2) Emergence phenomenon has unexpected and surprising characteristics, namely the unpredictability and the counterintuitivity. Whether there is emergence phenomenon may be considered to be one of the remarkable characteristics in distinguishing a complex system from a simple system.

Several theorems established in this paper show that the GS has the above emergent characteristics, especially the complex properties, such as the stochastic property of our model, the non-smoothness and non uniqueness of the GS pointed out in Theorems 3–5, various periodic GSs, fractal characteristics of an attracting set, speak volumes for the unpredictability and the possible counterintuitivity of emergence phenomenon.

Appendix A

The proof of Theorem 1.

Proof Define a family of function H_{LM} which satisfies $\forall h \in H_{LM}, \forall x_1, x_2, x \in B_x$, $|h(x_1) - h(x_2)| \leq L|x_1 - x_2|$ and $|h(x)| \leq M$, and define the distance $\| h_1 - h_2 \| = \mathrm{Sup}_{x \in B_x} |h_1(x) - h_2(x)|$, then H_{LM} will be a complete metric space.

Define operator F as follows: $(\mathrm{F}h)(x_{n+1}) = p\omega(x_n, h(x_n)) + (1-p)f(h(x_n))$.

In order to apply Lipschitz contraction mapping principle, we need to prove the following two results. Firstly we prove that F is a map from H_{LM} onto H_{LM}.

$$
\begin{aligned}
&|(\mathrm{F}h)(x_{n+1}) - (\mathrm{F}h)(\tilde{x}_{n+1})| \\
&= |p\omega(x_n, h(x_n)) - p\omega(\tilde{x}_n, h(\tilde{x}_n)) \\
&\quad + (1-p)f(h(x_n)) - (1-p)f(h(\tilde{x}_n)| \\
&= |p\omega(x_n, h(x_n)) - p\omega(x_n, h(\tilde{x}_n)) + p\omega(x_n, h(\tilde{x}_n)) \\
&\quad - p\omega(\tilde{x}_n, h(\tilde{x}_n)) + (1-p)f(h(x_n)) \\
&\quad - (1-p)f(h(\tilde{x}_n)| \\
&\leq pk|h(x_n) - h(\tilde{x}_n)| + p\eta|x_n - \tilde{x}_n| \\
&\quad + (1-p)r_1|h(x_n) - h(\tilde{x}_n)| \\
&\leq [p(kL+\eta) + (1-p)r_1 L]|x_n - \tilde{x}_n| \\
&\leq [p(kL+\eta) + (1-p)r_1 L]r_2|x_{n+1} - \tilde{x}_{n+1}|
\end{aligned}
$$

$$
\begin{aligned}
&\le L|x_{n+1}-\tilde{x}_{n+1}|,\\
&|(\mathrm{F}h)(x_{n+1})|\\
&=|p\omega(x_n,h(x_n))+(1-p)f(h(x_n))|\\
&=|p[(1-\epsilon)f(h(x_n))+\epsilon g(x_n)]+(1-p)f(h(x_n))\\
&\quad -p(1-\epsilon)f(\bar{y})-(1-p)f(\bar{y})\\
&\quad +p(1-\epsilon)f(\bar{y})+(1-p)f(\bar{y})|\\
&\le |p(1-\epsilon)r_1(h(x_n)-\bar{y})+(1-p)r_1(h(x_n)-\bar{y})\\
&\quad +p\epsilon g(x_n)+(1-p\epsilon)f(\bar{y})|\\
&\le [p(1-\epsilon)r_1+(1-p)r_1]M\\
&\quad +\mathrm{Sup}_{x_n\in B_x}|[p(1-\epsilon)r_1+(1-p)r_1]\bar{y}+p\epsilon g(x_n)\\
&\quad +(1-p\epsilon)f(\bar{y})|\\
&\le M.
\end{aligned}
$$

Secondly, we show the contraction of F as follows:

$$
\begin{aligned}
&|(\mathrm{F}h)(x_{n+1})-(\mathrm{F}\tilde{h})(x_{n+1})|\\
&=|p\omega(x_n,h(x_n))-p\omega(x_n,\tilde{h}(x_n))\\
&\quad +(1-p)f(h(x_n))-(1-p)f(\tilde{h}(x_n)|\\
&\le pk|h(x_n)-\tilde{h}(x_n)|+(1-p)r_1|h(x_n)-\tilde{h}(x_n)|\\
&=[pk+(1-p)r_1]|h(x_n)-\tilde{h}(x_n)|\\
&\le k|h(x_n)-\tilde{h}(x_n)|\\
&\le k\parallel h-\tilde{h}\parallel .
\end{aligned}
$$

Hence, $\parallel \mathrm{F}h-\mathrm{F}\tilde{h}\parallel=\mathrm{Sup}|(\mathrm{F}h)(x_{n+1})-(\mathrm{F}\tilde{h})(x_{n+1})|\le k\parallel h-\tilde{h}\parallel$.

Thus F is a contractive operator on H_{LM} and there is a fixed point $h=\mathrm{F}h$, i.e., $y=h(x)$ is a GS manifold between systems (1) and (2).

Now we prove that the GS manifold is invariant. Let $y_0=h(x_0), x_1=g(x_0)$, and

$$
y_1=\begin{cases}\omega(x_0,y_0) & \text{(with probability } p),\\ f(y_0) & \text{(with probability } 1-p),\end{cases}
$$

then we will have $y_1=h(x_1)$. In fact, $h(x_1)=(\mathrm{F}h)(x_1)=p\omega(x_0,h(x_0))+(1-p)f(h(x_0))$. Therefore, if $(x_0,y_0)\in S$, then $(x_1,y_1)\in S$ and $(x_n,y_n)\in S$.

Appendix B

The proof of Theorem 3.

Proof Define a family of function $H_{L\alpha}$ which satisfies $\forall h\in H_{L\alpha},\forall x_1,x_2,x\in B_x,\ |h(x_1)-h(x_2)|\le L_\alpha|x_1-x_2|^\alpha$, and $|h(x)|\le M$, and define the distance $\parallel h_1-h_2\parallel=\mathrm{Sup}_{x\in B_x}|h_1(x)-h_2(x)|$, then $H_{L\alpha}$ will be a complete metric space. Define operator F as $(\mathrm{F}h)(x_{n+1})=p\omega(x_n,h(x_n))+(1-p)f(h(x_n))$.

In order to apply the Schauder fixed point theorem, we need to prove the following three results. Firstly we prove several inequalities.

Let the diameter of B_x be less than ρ and $\parallel h-\tilde{h}\parallel\le 1$, then

$$
\begin{aligned}
&|(\mathrm{F}h)(x_{n+1})-(\mathrm{F}h)(\tilde{x}_{n+1})|\\
&=|p\omega(x_n,h(x_n))-p\omega(\tilde{x}_n,h(\tilde{x}_n))\\
&\quad +\ (1-p)f(h(x_n))-(1-p)f(h(\tilde{x}_n)|\\
&=|p\omega(x_n,h(x_n))-p\omega(x_n,h(\tilde{x}_n))+p\omega(x_n,h(\tilde{x}_n))\\
&\quad -\ p\omega(\tilde{x}_n,h(\tilde{x}_n))+(1-p)f(h(x_n))\\
&\quad -\ (1-p)f(h(\tilde{x}_n)|\\
&\le p\tilde{k}|h(x_n)-h(\tilde{x}_n)|+p\eta|x_n-\tilde{x}_n|\\
&\quad +\ (1-p)r_1|h(x_n)-h(\tilde{x}_n)|\\
&\le p\tilde{k}L_\alpha|x_n-\tilde{x}_n|^\alpha+p\eta(2\rho)^{1-\alpha}|x_n-\tilde{x}_n|^\alpha\\
&\quad +\ (1-p)r_1L_\alpha]|x_n-\tilde{x}_n|^\alpha\\
&=[p(\tilde{k}L_\alpha+\eta(2\rho)^{1-\alpha})+(1-p)r_1L_\alpha]|x_n-\tilde{x}_n|^\alpha\\
&\le [p(\tilde{k}L_\alpha+\eta(2\rho)^{1-\alpha})\\
&\quad +\ (1-p)r_1L_\alpha](r_2)^\alpha|x_{n+1}-\tilde{x}_{n+1}|^\alpha\\
&\le \tilde{L}_\alpha|x_{n+1}-\tilde{x}_{n+1}|^\alpha, \qquad \text{(B1)}\\
&|(\mathrm{F}h)(x_{n+1})|\\
&=|p\omega(x_n,h(x_n))+(1-p)f(h(x_n))|\\
&=|p[(1-\epsilon)f(h(x_n))+\epsilon g(x_n)]+(1-p)f(h(x_n))\\
&\quad -\ p(1-\epsilon)f(\bar{y})-(1-p)f(\bar{y})\\
&\quad +\ p(1-\epsilon)f(\bar{y})+(1-p)f(\bar{y})\\
&\le p(1-\epsilon)r_1(h(x_n)-\bar{y})+(1-p)r_1(h(x_n)-\bar{y})\\
&\quad +\ p\epsilon g(x_n)+(1-p\epsilon)f(\bar{y})|\\
&\le [p(1-\epsilon)r_1+(1-p)r_1]M\\
&\quad +\ \mathrm{Sup}_{x_n\in B_x}|[p(1-\epsilon)r_1+(1-p)r_1]\bar{y}\\
&\quad +\ p\epsilon g(x_n)+(1-p\epsilon)f(\bar{y})|\\
&\le \tilde{k}M+m, \qquad \text{(B2)}\\
&|(\mathrm{F}h)(x_{n+1})-(\mathrm{F}\tilde{h}(x_{n+1})|\\
&=|p\omega(x_n,h(x_n))-p\omega(x_n,\tilde{h}(x_n))\\
&\quad +\ (1-p)f(h(x_n))-(1-p)f(\tilde{h}(x_n)|\\
&\le p\tilde{k}|h(x_n)-\tilde{h}(x_n)|+(1-p)r_1|h(x_n)-\tilde{h}(x_n)|\\
&=[p\tilde{k}+(1-p)r_1]|h(x_n)-\tilde{h}(x_n)|\\
&\le \tilde{k}|h(x_n)-\tilde{h}(x_n)|\\
&\le \tilde{k}\parallel h-\tilde{h}\parallel . \qquad \text{(B3)}
\end{aligned}
$$

Thus,

$$
\parallel \mathrm{F}h-\mathrm{F}\tilde{h}\parallel=\mathrm{Sup}|(\mathrm{F}h)(x_{n+1})-(\mathrm{F}\tilde{h})(x_{n+1})|\le\tilde{k}\parallel h-\tilde{h}\parallel .
$$

We now define Hölder space $C^{0,\alpha}$, for $\sigma, \sigma_1, \sigma_2 \in C^{0,\alpha}$, whose norm and distance are respectively

$$\| \sigma \|_{0,\alpha} = \mathrm{Sup}_{x\in B_x}|\sigma(x)| + \mathrm{Sup}_{x\neq x'}\frac{|\sigma(x)-\sigma(x')|}{|x-x'|^\alpha},$$

and

$$\begin{aligned}&\| \sigma_1 - \sigma_2 \|_{0,\alpha}\\ =\ & \mathrm{Sup}_{x\in B_x}|\sigma_1(x)-\sigma_2(x)|\\ &+ \mathrm{Sup}_{x\neq x'}\frac{|\sigma_1(x)-\sigma_2(x)-[\sigma_1(x')-\sigma_2(x')]|}{|x-x'|^\alpha}.\end{aligned}$$

Then $C^{0,\alpha}$ is a Banach space.

Let the closed ball with a sufficiently large radius R in $C^{0,\alpha}$ be a bounded closed convex set $\hat{M}, h \in \hat{M}$,

$$\begin{aligned}&\| h \|_{0,\alpha}\\ =\ & \mathrm{Sup}_{x\in B_x}|h(x)| + \mathrm{Sup}_{x\neq x'}\frac{|h(x)-h(x')|}{|x-x'|^\alpha}\\ \leq\ & M + L_\alpha\\ \leq\ & R,\end{aligned}$$

$$\begin{aligned}&\| \mathrm{F}h \|_{0,\alpha}\\ =\ & \mathrm{Sup}_{x\in B_x}|\mathrm{F}h(x)| + \mathrm{Sup}_{x\neq x'}\frac{|\mathrm{F}h(x)-\mathrm{F}h(x')|}{|x-x'|^\alpha}\\ \leq\ & \tilde{k}M + m + L_\alpha\\ \leq\ & R.\end{aligned}$$

Secondly, we prove that F is a continuous mapping.

For arbitrary $\varepsilon > 0$, let

$$|h(x)-h'(x)| < \frac{\varepsilon}{3\tilde{k}},\quad \frac{|h(x)-h'(x)|}{|x-x'|^\alpha} < \frac{\varepsilon}{3\tilde{k}},$$

and

$$\frac{|h(x')-h'(x')|}{|x-x'|^\alpha} < \frac{\varepsilon}{3\tilde{k}},$$

in other words, $\| h-h' \|_{0,\alpha} \leq \delta = \varepsilon/\tilde{k}$. From (B3), we have

$$\begin{aligned}|\mathrm{F}(h(x)) - \mathrm{F}(h'(x))| &\leq \tilde{k} \| h-h' \|\\ &\cong \tilde{k}\mathrm{Sup}_{x\in B_x}|h(x)-h'(x)|,\end{aligned}$$

we obtain

$$\begin{aligned}&\| \mathrm{F}(h) - \mathrm{F}(h') \|_{0,\alpha}\\ =\ & \mathrm{Sup}_{x\in B_x}|\mathrm{F}(h(x)) - \mathrm{F}(h(x'))| + \mathrm{Sup}_{x\neq x'}\frac{|\mathrm{F}(h(x)) - \mathrm{F}(h'(x)) - [\mathrm{F}(h(x')) - \mathrm{F}(h'(x'))]|}{|x-x'|^\alpha}\\ <\ & \tilde{k}\mathrm{Sup}_{x\in B_x}|h(x)-h'(x)| + \tilde{k}\mathrm{Sup}_{x\in B_x}\frac{|h(x)-h'(x)|}{|x-x'|^\alpha} + \tilde{k}\mathrm{Sup}_{x\in B_x}\frac{|h(x')-h'(x')|}{|x-x'|^\alpha}\\ <\ & \varepsilon.\end{aligned}$$

Therefore, F is a continuous mapping.

Thirdly, we prove that F maps the bounded sets of $C^{0,\alpha/2}\cap \mathrm{F}_{L\alpha}$ onto the relative compact set of $C^{0,\alpha/2}\cap \mathrm{F}_{L\alpha}$.

Assume that $\{\mathrm{F}(h(x))\}$ is a family of function with parameter x and uniformly bounded.

By (B1), $|\mathrm{F}(h)(x) - \mathrm{F}(h)(x')| \leq \tilde{L}_{\alpha/2}|x-x'|^{\alpha/2}$, while $|x-x'| < \delta = (\varepsilon/\tilde{L}_{\alpha/2})^{2/\alpha}$, we have $|\mathrm{F}(h)(x) - \mathrm{F}(h)(x')| \leq \varepsilon$.

From the Arzela theorem, we can choose a function sequence $\{\mathrm{F}(h_n(x))\}$ from $\{\mathrm{F}(h(x))\}$ which is convergent in $C^0 \cap \mathrm{F}_{L\alpha}$. Now we show that $\{\mathrm{F}(h_n(x))\}$ is convergent in $C^{0,\alpha/2} \cap \mathrm{F}_{L\alpha}$.

Denoting $g_{mn}(x) = \mathrm{F}(h_m(x)) - \mathrm{F}(h_n(x))$, we have

$$\frac{|g_{mn}(x)-g_{mn}(x')|}{|x-x'|^{\alpha/2}} = \left\{\frac{|\mathrm{F}(h_m(x)) - \mathrm{F}(h_m(x)) - \mathrm{F}(h_m(x')) + \mathrm{F}(h_n(x'))|}{|x-x'|^\alpha}\right\}^{\frac{1}{2}} \times |g_{mn}(x) - g_{mn}(x')|^{1-\frac{1}{2}}. \quad \text{(B4)}$$

Since $C^{0,\alpha} \cap \mathrm{F}_{L\alpha}$ is bounded set, then $\{.\}$ of (B4) is also bounded and let the boundary be K_1, then

$$\begin{aligned}\| g_{mn} \|_{0,\alpha/2} &= \mathrm{Sup}_{x\in R^m}|g_{mn}(x)| + \mathrm{Sup}_{x\neq x'}\frac{|g_{mn}(x)-g_{mn}(x')|}{|x-x'|^{\alpha/2}}\\ &\leq\ \mathrm{Sup}_{x\in R^m}|g_{mn}(x)| + K_1\mathrm{Sup}_{x\neq x'}|g_{mn}(x)-g_{mn}(x')|^{1/2}\\ &\leq \| g_{mn} \|_0 + 2K_1 \| g_{mn} \|_0^{1/2}.\end{aligned}$$

This implies that if $\{\mathrm{F}(h_n(x))\}$ is a fundamental sequence of C^0, then $\{\mathrm{F}(h_n(x))\}$ is a fundamental sequence of $C^{0,\alpha/2}$. Thus, $\{\mathrm{F}(h_n(x))\}$ is convergent in $C^{0,\alpha/2} \cap F_{L\alpha}$ due to the fact that $\{\mathrm{F}(h_n(x))\}$ is convergent in

$C^0 \cap \mathrm{F}_{L\alpha}$. By Schauder fixed point principle, there is a fixed point $h = \mathrm{F}(h)$ of F in $C^{0,\alpha/2} \cap \mathrm{F}_{L\alpha}$. Therefore, there is a Hölder continuous GS manifold between systems (1) and (2).

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