The $\mathscr{H}_\infty$ synchronization of nonlinear Bloch systems via dynamic feedback control approach*

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We consider an $\mathscr{H}_\infty$ synchronization problem in nonlinear Bloch systems. Based on Lyapunov stability theory and linear matrix inequality formulation, a dynamic feedback controller is designed to guarantee asymptotic stability of the master-slave synchronization. Moreover, this controller reduces the effect of an external disturbance to the $\mathscr{H}_\infty$ norm constraint. A numerical example is given to validate the proposed synchronization scheme.

Keywords: $\mathscr{H}_\infty$ synchronization, Bloch system, dynamic control, linear matrix inequality

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1. Introduction

Chaos is a very interesting nonlinear phenomenon and has been studied extensively in the last few decades.[1−13] In particular, chaos synchronization is attracting great attention because of its various applications, such as in biology, economics, signal generator design and secure communication. Various synchronization schemes appear in the copious literature, such as variable structure control,[14,15] observer-based control,[16] time-delay feedback approach,[17] and back-stepping design technique.[18] These methods target the synchronization problem for the well-known chaotic systems such as Lorenz, Chen, Lü and Rossler systems.

On the other hand, the dynamics of an ensemble of spins which do not exhibit mutual coupling, except for some interactions leading to relaxation, is well described by simple Bloch systems. Abergel[19] investigated the original Bloch systems, in which the nonlinear effects caused by the presence of a feedback field are considered. Ucar *et al.*[20] extended the calculations of Abergel[19] and demonstrated that it is possible to synchronize two of these nonlinear Bloch systems. These studies are based on the exact knowledge of the parameters. However, it is not easy to know the exact system parameters. In this regard, an adaptive control scheme was proposed which works without an exact knowledge of all the parameters.[21]

In the real world, the occurrence of noise or disturbances that causes instability and poor performance is unavoidable. Therefore, it is necessary to reduce the effects of noise or disturbances to a certain acceptable value. In this regard, Hou *et al.*[22] adopted $\mathscr{H}_\infty$ control for the chaotic synchronization problem using a static output feedback controller. Recently, a dynamic feedback controller for $\mathscr{H}_\infty$ chaos synchronization has been proposed in Ref. [23].

In this paper, the problem of $\mathscr{H}_\infty$ synchronization for nonlinear Bloch systems with disturbances is considered. A dynamic feedback controller for synchronization between master and slave systems is reported, with a design based on Lyapunov stability theory and linear matrix inequality (LMI) formulation. The closed-loop error system is asymptotically stable and $\mathscr{H}_\infty$-norm is reduced to a prescribed level.

The rest of this paper is organized as follows: In Section 2, the problem statement is presented and in Section 3, the master-slave synchronization scheme is presented for nonlinear Bloch systems with external disturbances. A numerical example is given to validate the proposed synchronization scheme. Finally, a

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conclusion is given.

2. Problem statement

In dimensionless units, the dynamic model of nonlinear Bloch systems with feedback field[19] is given by

$$\begin{cases} \dot{x}(t) = \delta y(t) + \lambda z(t)[x(t)\sin\psi \\ \qquad - y(t)\cos\psi] - \dfrac{x(t)}{\tau_2}, \\ \dot{y}(t) = -\delta x(t) - z(t) + \lambda z(t)[x(t)\cos\psi \\ \qquad + y(t)\sin\psi] - \dfrac{y(t)}{\tau_2}, \\ \dot{z}(t) = y(t) - \lambda\sin(\psi)[x^2(t) + y^2(t)] - \dfrac{z(t)-1}{\tau_1}, \end{cases} \tag{1}$$

where δ, λ and ψ are the system parameters, τ_1 is the longitudinal relaxation time and τ_2 is the transverse relaxation time. Abergel[19] has investigated the dynamics of system (1) for a fixed subset of the system parameters (δ, λ, τ_1, τ_2) and for a space area range of the radiation damping feedback ψ. Especially, the regions of the parameter ψ that would admit chaotic behaviour were derived. The parameters which make the system chaotic are, for example, $\delta = -0.4\pi$, $\lambda = 35$, $\tau_1 = 5$, $\tau_2 = 2.5$ and $\psi = 0.173$. For details of other dynamic properties of the system, see Refs. [19] and [20].

We assume that two identical Bloch systems constitute a master-slave configuration. The slave system with subscript s is driven to the master system with subscript m. For Eq. (1), the master system is

$$\begin{cases} \dot{x}_{\rm m}(t) = \delta y_{\rm m}(t) + \lambda z_{\rm m}(t)[x_{\rm m}(t)\sin\psi \\ \qquad - y_{\rm m}(t)\cos\psi] - \dfrac{x_{\rm m}(t)}{\tau_2}, \\ \dot{y}_{\rm m}(t) = -\delta x_{\rm m}(t) - z_{\rm m}(t) + \lambda z_{\rm m}(t)[x_{\rm m}(t)\cos\psi \\ \qquad + y_{\rm m}(t)\sin\psi] - \dfrac{y_{\rm m}(t)}{\tau_2}, \\ \dot{z}_{\rm m}(t) = y_{\rm m}(t) - \lambda\sin(\psi)[x_{\rm m}^2(t) + y_{\rm m}^2(t)] \\ \qquad - \dfrac{z_{\rm m}(t)-1}{\tau_1}, \end{cases} \tag{2}$$

and the slave system is

$$\begin{cases} \dot{x}_{\rm s}(t) = \delta y_{\rm s}(t) + \lambda z_{\rm s}(t)[x_{\rm s}(t)\sin\psi - y_{\rm s}(t)\cos\psi] \\ \qquad - \dfrac{x_{\rm s}(t)}{\tau_2} + u_1(t) + w_1(t), \\ \dot{y}_{\rm s}(t) = -\delta x_{\rm s}(t) - z_{\rm s}(t) + \lambda z_{\rm s}(t)[x_{\rm s}(t)\cos\psi \\ \qquad + y_{\rm s}(t)\sin\psi] - \dfrac{y_{\rm s}(t)}{\tau_2} + u_2(t) + w_2(t), \\ \dot{z}_{\rm s}(t) = y_{\rm s}(t) - \lambda\sin(\psi)[x_{\rm s}^2(t) + y_{\rm s}^2(t)] \\ \qquad - \dfrac{z_{\rm s}(t)-1}{\tau_1} + u_3(t) + w_3(t), \end{cases} \tag{3}$$

where $u_1(t)$, $u_2(t)$ and $u_3(t) \in \mathscr{R}$ are control inputs; and $w_1(t)$, $w_2(t)$ and $w_3(t) \in \mathscr{R}$ are external disturbances, in which $\mathscr{R}^n$ denotes the n-dimensional Euclidean space.

Defining the synchronization error as

$$\begin{cases} e_x(t) = x_{\rm s}(t) - x_{\rm m}(t), \\ e_y(t) = y_{\rm s}(t) - y_{\rm m}(t), \\ e_z(t) = z_{\rm s}(t) - z_{\rm m}(t), \end{cases} \tag{4}$$

then we can obtain the following error dynamics:

$$\begin{cases} \dot{\boldsymbol{e}}(t) = \boldsymbol{A}\boldsymbol{e}(t) + \boldsymbol{f}(t) + \boldsymbol{u}(t) + \boldsymbol{w}(t), \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{e}(t), \end{cases} \tag{5}$$

where

$$\boldsymbol{e}(t) \triangleq \begin{bmatrix} e_x(t) \\ e_y(t) \\ e_z(t) \end{bmatrix}, \quad \boldsymbol{u}(t) \triangleq \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix},$$

$$\boldsymbol{w}(t) \triangleq \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}, \quad \boldsymbol{A} \triangleq \begin{bmatrix} -\dfrac{1}{\tau_2} & \delta & 0 \\ -\delta & -\dfrac{1}{\tau_2} & -1 \\ 0 & 1 & -\dfrac{1}{\tau_1} \end{bmatrix}, \tag{6}$$

$$\boldsymbol{f}(t) \triangleq \begin{bmatrix} \lambda z_{\rm s}(t)[x_{\rm s}(t)\sin\psi - y_{\rm s}(t)\cos\psi] \\ -\lambda z_{\rm m}(t)[x_{\rm m}(t)\sin\psi - y_{\rm d}(t)\cos\psi] \\ \lambda z_{\rm s}(t)[x_{\rm s}(t)\cos\psi + y_{\rm s}(t)\sin\psi] \\ -\lambda z_{\rm m}(t)[x_{\rm m}(t)\cos\psi + y_{\rm d}(t)\sin\psi] \\ -\lambda\sin\psi[x_{\rm s}^2(t) + y_{\rm s}^2(t)] \\ +\lambda\sin\psi[x_{\rm m}^2(t) + y_{\rm m}^2(t)] \end{bmatrix}, \tag{7}$$

where $\boldsymbol{C} \in \mathscr{R}^3$ is a constant matrix and $\boldsymbol{y}(t)$ is the controlled output error.

Next, in order to achieve $\mathscr{H}_\infty$ synchronization, the following dynamical feedback controller is considered:

$$\begin{cases} \dot{\boldsymbol{\zeta}}(t) = \boldsymbol{A}_{\rm c}\boldsymbol{\zeta}(t) + \boldsymbol{B}_{\rm c}\boldsymbol{y}(t), \\ \boldsymbol{u}(t) = \boldsymbol{C}_{\rm c}\boldsymbol{\zeta}(t) - \boldsymbol{f}(t), \end{cases} \tag{8}$$

where $\boldsymbol{\zeta}(t) \in \mathscr{R}^3$ is the state vector of the controller, $\boldsymbol{A}_{\rm c} \in \mathscr{R}^{3\times3}$, $\boldsymbol{B}_{\rm c} \in \mathscr{R}^{3\times3}$ and $\boldsymbol{C}_{\rm c} \in \mathscr{R}^{3\times3}$ are constant matrices, in which $\mathscr{R}^{m\times n}$ denotes the set of an $m \times n$ real matrix.

By applying controller (8) to system (5), we have the following closed-loop system:

$$\dot{\bar{x}}(t) = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{C}_{\rm c} \\ \boldsymbol{B}_{\rm c}\boldsymbol{C} & \boldsymbol{A}_{\rm c} \end{bmatrix} \bar{\boldsymbol{x}}(t) + \begin{bmatrix} \boldsymbol{w}(t) \\ \boldsymbol{0} \end{bmatrix} \tag{9}$$

$$\triangleq \bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{w}}(t), \tag{10}$$

where $\bar{\boldsymbol{x}} \triangleq \begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{\zeta}(t) \end{bmatrix}$ is the augmented vector.

Definition 1 ($\mathscr{H}_\infty$ synchronization[24]) Synchronization error system (5) is the $\mathscr{H}_\infty$ synchronization with the disturbance attenuation γ if the following conditions are satisfied:

1) With zero disturbance, synchronization error system (5) with control input $\boldsymbol{u}(t)$ is exponentially stable.

2) With zero initial condition and a given constant $\gamma > 0$, the following condition holds:

$$J = \int_0^\infty [\boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{y}(t) - \gamma^2 \boldsymbol{w}^{\mathrm{T}}(t)\boldsymbol{w}(t)]\,\mathrm{d}t \le 0 \quad \left(\text{i.e.} \sup_{w\neq 0, w\in L_2[0,\infty]} \frac{\|\boldsymbol{y}(t)\|_2}{\|\boldsymbol{w}(t)\|_2} \le \gamma\right), \tag{11}$$

where the superscript T means the transpose of a given vector or matrix.

In the next section, we show that controller (8) guarantees the $\mathscr{H}_\infty$ synchronization with the disturbance attenuation γ. The γ is called the $\mathscr{H}_\infty$-norm bound of the controller.

3. Main results

First of all, for symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, the notation $\boldsymbol{X} > \boldsymbol{Y}$ (respectively, $\boldsymbol{X} \ge \boldsymbol{Y}$) means that the matrix $\boldsymbol{X} - \boldsymbol{Y}$ is positive definite (respectively, nonnegative).

Now, the design procedure of the controller for achieving the $\mathscr{H}_\infty$ synchronization is presented in the following theorem.

Theorem 1 For given η and γ, there exists a dynamic feedback controller (8) for error system (5), if there exist any matrices $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$, $\hat{\boldsymbol{C}}$, $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$ satisfying the following LMIs:

$$\begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{I} & \bar{\boldsymbol{X}}\boldsymbol{C}^{\mathrm{T}} \\ \star & \boldsymbol{\Sigma}_{22} & \boldsymbol{X} & \boldsymbol{C}^{\mathrm{T}} \\ \star & \star & -\gamma^2\boldsymbol{I} & \boldsymbol{0} \\ \star & \star & \star & -\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}, \quad \begin{bmatrix} \bar{\boldsymbol{X}} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{X} \end{bmatrix} > \boldsymbol{0}, \tag{12}$$

where $\star$ represents the elements below the main diagonal of a symmetric matrix and $\boldsymbol{I}$ denotes the identity matrix with appropriate dimension, and

$$\begin{aligned} \boldsymbol{\Sigma}_{11} &= \boldsymbol{A}\bar{\boldsymbol{X}} + \bar{\boldsymbol{X}}\boldsymbol{A}^{\mathrm{T}} + \hat{\boldsymbol{C}} + \hat{\boldsymbol{C}}^{\mathrm{T}} + \eta\bar{\boldsymbol{X}}, \\ \boldsymbol{\Sigma}_{12} &= \boldsymbol{A} + \hat{\boldsymbol{A}}^{\mathrm{T}} + \eta\boldsymbol{I}, \\ \boldsymbol{\Sigma}_{22} &= \boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{X} + \hat{\boldsymbol{B}}\boldsymbol{C} + \boldsymbol{C}^{\mathrm{T}}\hat{\boldsymbol{B}}^{\mathrm{T}} + \eta\boldsymbol{X}, \end{aligned} \tag{13}$$

then the $\mathscr{H}_\infty$ synchronization with disturbance attenuation γ is obtained by the controller (8).

Proof Define a Lyapunov candidate as

$$V = \bar{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{P}\bar{\boldsymbol{x}}(t), \tag{14}$$

then the time derivative of the Lyapunov candidate will be

$$\begin{aligned} \dot{V} &= \bar{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{P}(\bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{w}}(t)) + (\bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{w}}(t))^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{x}}(t) \\ &= \bar{\boldsymbol{x}}^{\mathrm{T}}(t)(\boldsymbol{P}\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P})\bar{\boldsymbol{x}}(t) + 2\bar{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{P}\bar{\boldsymbol{w}}(t). \end{aligned} \tag{15}$$

Defining $\boldsymbol{P} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \\ \boldsymbol{Y}^{\mathrm{T}} & \boldsymbol{Z} \end{bmatrix}$, we have

$$\begin{aligned} \dot{V} &= \bar{\boldsymbol{x}}^{\mathrm{T}}(t)(\boldsymbol{P}\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P})\bar{\boldsymbol{x}}(t) + 2(\boldsymbol{X}\boldsymbol{e}(t) \\ &\quad + \boldsymbol{Y}\boldsymbol{\zeta}(t))^{\mathrm{T}}\boldsymbol{w}(t) \\ &= \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{P}\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P} & \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y}^{\mathrm{T}} \end{bmatrix} \\ \star & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix}. \end{aligned} \tag{16}$$

Thus, the inequality (16) is negative definite, we have $\dot{V} \le 0$.

Next, a cost function is defined as

$$J(\bar{\boldsymbol{x}}(t), \boldsymbol{w}(t)) = \dot{V} + \boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{y}(t) - \gamma^2\boldsymbol{w}^{\mathrm{T}}(t)\boldsymbol{w}(t). \tag{17}$$

If $\dot{V}$ in cost function (17) is substituted to Eq. (16), then

$$\begin{aligned} J(\bar{\boldsymbol{x}}(t), \boldsymbol{w}(t)) &= \dot{V} + \boldsymbol{y}^{\mathrm{T}}(t)\boldsymbol{y}(t) - \gamma^2\boldsymbol{w}^{\mathrm{T}}(t)\boldsymbol{w}(t) \\ &\le \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix}^{\mathrm{T}} \boldsymbol{\Omega} \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} \boldsymbol{\Omega} &= \begin{bmatrix} \boldsymbol{P}\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P} + \bar{\boldsymbol{C}} & \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y}^{\mathrm{T}} \end{bmatrix} \\ \star & -\gamma^2\boldsymbol{I} \end{bmatrix}, \\ \bar{\boldsymbol{C}} &= \begin{bmatrix} \boldsymbol{C}^{\mathrm{T}}\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}. \end{aligned} \tag{19}$$

If the matrix $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > \boldsymbol{0}$ and there exists a positive constant η satisfying the following condition:

$$\begin{aligned} \boldsymbol{\Xi} &\triangleq \boldsymbol{\Omega} + \begin{bmatrix} \eta\boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{P}\bar{\boldsymbol{A}} + \bar{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P} + \bar{\boldsymbol{C}} + \eta\boldsymbol{P} & \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{Y}^{\mathrm{T}} \end{bmatrix} \\ \star & -\gamma^2\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}, \end{aligned} \tag{20}$$

then we have

$$J(\bar{\boldsymbol{x}}(t), \boldsymbol{w}(t)) \le \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix}^{\mathrm{T}} \boldsymbol{\Omega} \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix}^{\mathrm{T}}$$

$$\times \left(\bar{\boldsymbol{\Xi}} - \begin{bmatrix} \eta \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \right) \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix} < - \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \eta \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{x}}(t) \\ \boldsymbol{w}(t) \end{bmatrix}. \quad (21)$$

Consequently, we have

$$\dot{V}|_{\boldsymbol{w}(t)=0} < -\eta \lambda_{\min}(\boldsymbol{P}) \|\bar{\boldsymbol{x}}(t)\|^2, \quad \forall \bar{\boldsymbol{x}}(t) \neq 0, \quad (22)$$

where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a given matrix. Eq. (22) implies that the error dynamics (5) with dynamic feedback controller (8) is exponentially stable for $\boldsymbol{w}(t) = 0$.

Moreover, to obtain the $\mathscr{H}_\infty$ synchronization, inequality (21) is integrated from 0 to ∞, giving the following result:

$$V(\infty) - V(0) + \int_0^\infty (\|\boldsymbol{y}(t)\|_2^2 - \gamma^2 \|\boldsymbol{w}(t)\|_2^2) \mathrm{d}t \leq 0. \quad (23)$$

Obviously, inequality (23) yields

$$\int_0^\infty (\|\boldsymbol{y}(t)\|_2^2 - \gamma^2 \|\boldsymbol{w}(t)\|_2^2) \mathrm{d}t \leq 0, \quad (24)$$

with an initial condition of zero. Thus, the $\mathscr{H}_\infty$ synchronization with the disturbance attenuation γ is obtained by Definition 1.

Unfortunately, the derived stability condition (20) is not in a standard LMI form, so that it is not easy to solve and find the controller parameters $\boldsymbol{A}_{\mathrm{c}}$, $\boldsymbol{B}_{\mathrm{c}}$ and $\boldsymbol{C}_{\mathrm{c}}$. Through some variable changes, the stability condition (20) can be solved as convex optimization algorithms. First, let $\boldsymbol{P}^{-1} = \begin{bmatrix} \bar{\boldsymbol{X}} & \bar{\boldsymbol{Y}} \\ \bar{\boldsymbol{Y}}^{\mathrm{T}} & \bar{\boldsymbol{Z}} \end{bmatrix}$, then the equality $\boldsymbol{P}^{-1}\boldsymbol{P} = \boldsymbol{I}$ will yield

$$\bar{\boldsymbol{Y}}\boldsymbol{Y}^{\mathrm{T}} = \boldsymbol{I} - \bar{\boldsymbol{X}}\boldsymbol{X}. \quad (25)$$

Define

$$\boldsymbol{\Gamma}_1 = \begin{bmatrix} \bar{\boldsymbol{X}} & \boldsymbol{I} \\ \bar{\boldsymbol{Y}}^{\mathrm{T}} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{\Gamma}_2 = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{X} \\ \boldsymbol{0} & \boldsymbol{Y}^{\mathrm{T}} \end{bmatrix}, \quad (26)$$

then it will follow that

$$\boldsymbol{P}\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_2, \quad {\boldsymbol{\Gamma}_1}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{\Gamma}_1 = {\boldsymbol{\Gamma}_1}^{\mathrm{T}} \boldsymbol{\Gamma}_2 = \begin{bmatrix} \bar{\boldsymbol{X}} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{X} \end{bmatrix}. \quad (27)$$

If the inequality $\begin{bmatrix} \bar{\boldsymbol{X}} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{X} \end{bmatrix} > \boldsymbol{0}$ is satisfied, then positiveness of the matrix $\boldsymbol{P}$ is guaranteed.

In order to use the concept of congruence transformation, pre- and post-multiplying the stability condition (20) by $\begin{bmatrix} \boldsymbol{\Gamma}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$, then the result is equivalent to

$$\begin{bmatrix} {\boldsymbol{\Gamma}_2}^{\mathrm{T}} \bar{\boldsymbol{A}} \boldsymbol{\Gamma}_1 + {\boldsymbol{\Gamma}_1}^{\mathrm{T}} \bar{\boldsymbol{A}} \boldsymbol{\Gamma}_2 + {\boldsymbol{\Gamma}_1}^{\mathrm{T}} \bar{\boldsymbol{C}} \boldsymbol{\Gamma}_1 + \eta {\boldsymbol{\Gamma}_2}^{\mathrm{T}} \boldsymbol{\Gamma}_1 & \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{X} \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{I} & \boldsymbol{X} \end{bmatrix} & -\gamma^2 \boldsymbol{I} \end{bmatrix} < \boldsymbol{0}. \quad (28)$$

Inequality (28) is then rewritten as

$$\begin{bmatrix} \boldsymbol{A}\bar{\boldsymbol{X}} + \bar{\boldsymbol{X}}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{C}_{\mathrm{c}}\bar{\boldsymbol{Y}}^{\mathrm{T}} + \bar{\boldsymbol{Y}}{\boldsymbol{C}_{\mathrm{c}}}^{\mathrm{T}} + \bar{\boldsymbol{X}}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}\bar{\boldsymbol{X}} + \eta\bar{\boldsymbol{X}} & \boldsymbol{A} + (\boldsymbol{X}\boldsymbol{A}\bar{\boldsymbol{X}} + \boldsymbol{Y}\boldsymbol{B}_{\mathrm{c}}\boldsymbol{C}\bar{\boldsymbol{X}} + \boldsymbol{X}\boldsymbol{C}_{\mathrm{c}}\bar{\boldsymbol{Y}}^{\mathrm{T}} + \boldsymbol{Y}\boldsymbol{A}_{\mathrm{c}}\bar{\boldsymbol{Y}}^{\mathrm{T}})^{\mathrm{T}} + \bar{\boldsymbol{X}}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C} + \eta\boldsymbol{I} & \boldsymbol{I} \\ \star & \boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{X} + \boldsymbol{Y}\boldsymbol{B}_{\mathrm{c}}\boldsymbol{C} + \boldsymbol{C}^{\mathrm{T}}{\boldsymbol{B}_{\mathrm{c}}}^{\mathrm{T}}\boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{C} + \eta\boldsymbol{X} & \boldsymbol{X} \\ \star & \star & -\gamma^2\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}. \quad (29)$$

Applying Schur complements[25] to Eq. (29) leads to

$$\begin{bmatrix} \boldsymbol{A}\bar{\boldsymbol{X}} + \bar{\boldsymbol{X}}\boldsymbol{A}^{\mathrm{T}} + \hat{\boldsymbol{C}} + \hat{\boldsymbol{C}}^{\mathrm{T}} + \eta\bar{\boldsymbol{X}} & \boldsymbol{A} + \hat{\boldsymbol{A}}^{\mathrm{T}} + \eta\boldsymbol{I} & \boldsymbol{I} & \bar{\boldsymbol{X}}\boldsymbol{C}^{\mathrm{T}} \\ \star & \boldsymbol{X}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{X} + \hat{\boldsymbol{B}}\boldsymbol{C} + \boldsymbol{C}^{\mathrm{T}}\hat{\boldsymbol{B}}^{\mathrm{T}} + \eta\boldsymbol{X} & \boldsymbol{X} & \boldsymbol{C}^{\mathrm{T}} \\ \star & \star & -\gamma^2\boldsymbol{I} & \boldsymbol{0} \\ \star & \star & \star & -\boldsymbol{I} \end{bmatrix} < \boldsymbol{0}, \quad (30)$$

where

$$\hat{\boldsymbol{A}} \triangleq \boldsymbol{X}\boldsymbol{A}\bar{\boldsymbol{X}} + \boldsymbol{Y}\boldsymbol{B}_{\rm c}\boldsymbol{C}\bar{\boldsymbol{X}} + \boldsymbol{X}\boldsymbol{C}_{\rm c}\bar{\boldsymbol{Y}}^{\rm T} + \boldsymbol{Y}\boldsymbol{A}_{\rm c}\bar{\boldsymbol{Y}}^{\rm T},$$
$$\hat{\boldsymbol{B}} \triangleq \boldsymbol{Y}\boldsymbol{B}_{\rm c}, \quad \hat{\boldsymbol{C}} \triangleq \boldsymbol{C}_{\rm c}\bar{\boldsymbol{Y}}^{\rm T}. \tag{31}$$

Inequalities (30) and (27) are the stability conditions and this completes the proof.

Remark 1 For any given solution in Theorem 1, a corresponding dynamic controller in the form of expression (8) can be obtained as follows:

1) compute the invertible matrices $\boldsymbol{Y}$ and $\bar{\boldsymbol{Y}}$ satisfying (25) using matrix algebra.

2) Using the matrices $\boldsymbol{Y}$ and $\bar{\boldsymbol{Y}}$ obtained above, solve Eq. (31) for $\boldsymbol{A}_{\rm c}$, $\boldsymbol{B}_{\rm c}$ and $\boldsymbol{C}_{\rm c}$.

4. Numerical example and simulation

In this section, in order to verify and demonstrate the effectiveness of the proposed method, we discuss the simulation result for Bloch systems. In the simulations, the fourth-order Runge–Kutta method is used to solve the systems with a time step size of 0.001. For the simulation, the following initial values are used:

$$\boldsymbol{x}_{\rm m}(0) = \begin{bmatrix} 0.5 & -0.5 & 0 \end{bmatrix},$$
$$\boldsymbol{x}_{\rm s}(0) = \begin{bmatrix} -0.5 & 0.5 & 0.3 \end{bmatrix}. \tag{32}$$

The system parameters are fixed to be $\delta = -0.4\pi$, $\lambda = 35$, $\tau_1 = 5$, $\tau_2 = 2.5$, and $\psi = 0.173$ which make the Bloch equation exhibit a chaotic behaviour and the matrix $\boldsymbol{C}$ is given by $\boldsymbol{I}$. The disturbance attenuation γ is given by $\gamma = 0.1$ and the constant η is given by $\eta = 0.1$. By using the MATLAB LMI toolbox,[25] we obtain a feasible solution set of the LMIs given in Theorem 1 as follows:

$$\boldsymbol{X} = \mathrm{diag}\{0.5550, 0.5550, 0.5550\},$$
$$\bar{\boldsymbol{X}} = \mathrm{diag}\{5.1725, 5.1725, 5.1725\},$$
$$\hat{\boldsymbol{A}} = \begin{bmatrix} -52.9512 & -1.2566 & 0 \\ 1.2566 & -52.9512 & -1 \\ 0 & 1 & -53.1513 \end{bmatrix},$$
$$\hat{\boldsymbol{B}} = \mathrm{diag}\{-31.0218, -31.0218, -31.1328\},$$
$$\hat{\boldsymbol{C}} = \mathrm{diag}\{-71.4488, -71.4488, -72.4833\},$$

where $\mathrm{diag}\{\cdots\}$ denotes the block diagonal matrix.

After further calculations using matrix algebra, $\boldsymbol{Y}$ and $\bar{\boldsymbol{Y}}$ are obtained as follows:

$$\boldsymbol{Y} = \mathrm{diag}\{-1.8706, -1.8706, -1.8706\}, \quad \bar{\boldsymbol{Y}} = \boldsymbol{I}.$$

Using $\boldsymbol{Y}$ and $\bar{\boldsymbol{Y}}$, dynamic feedback controller (8) is obtained as follows:

$$\boldsymbol{A}_{\rm c} = \begin{bmatrix} -79.2830 & -1.2566 & 0 \\ 1.2566 & -79.2830 & -1 \\ 0 & 1 & -79.4830 \end{bmatrix},$$
$$\boldsymbol{B}_{\rm c} = \mathrm{diag}\{16, 5835, 16.5835, 16.6428\},$$
$$\boldsymbol{C}_{\rm c} = \mathrm{diag}\{-71.4488, -71.4488, -72.4833\}.$$

The result of numerical simulation without an external disturbance is shown first. By applying parameters obtained above to dynamical feedback controller (8), the synchronization error between the master system and the slave system is shown in Fig. 1. This figure shows that the synchronization error converges to zero exponentially if there exists no disturbance. In this case, the control input $\boldsymbol{u}(t)$ is shown in Fig. 2.

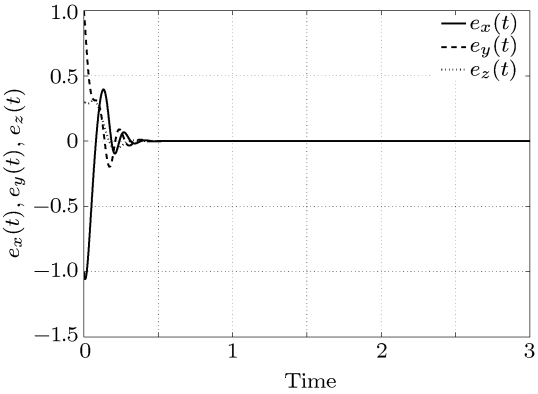


Fig. 1. Synchronization errors without external disturbances.

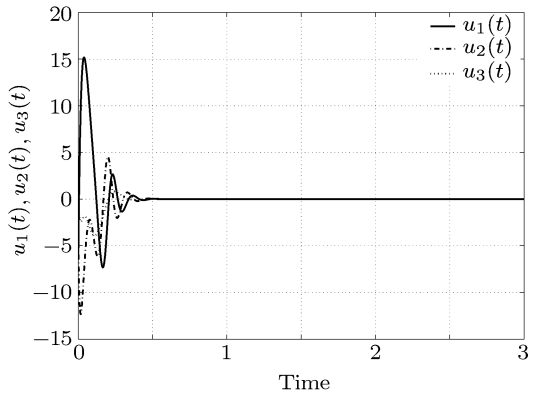


Fig. 2. Dynamic control inputs $\boldsymbol{u}(t)$ without external disturbances.

Next, to observe the $\mathscr{H}_\infty$ performance with disturbance attenuation, Gaussian noise with a mean value of 0 and a variance of 1 is imposed on the slave system with zero initial state condition. The controlled error output $\boldsymbol{y}(t)$ is depicted in Fig. 3 which shows that the $\mathscr{H}_\infty$ controller reduces the effect of the disturbance to a prescribed level $\gamma = 0.1$.

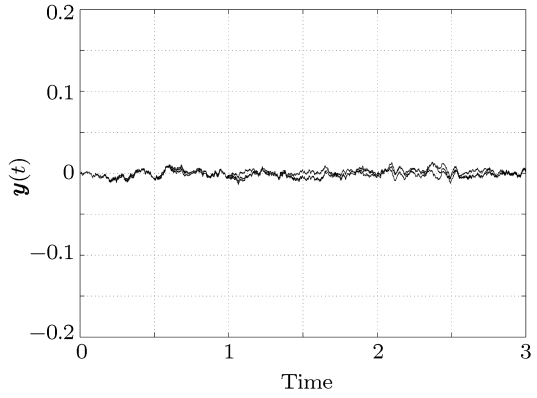


Fig. 3. The response of the output error $\boldsymbol{y}(t)$ with disturbance signal $\boldsymbol{w}(t)$ and zero initial conditions.

5. Conclusion

The $\mathscr{H}_\infty$ synchronization problem for nonlinear Bloch systems is considered. The synchronization between master and slave systems is achieved via a dynamic feedback controller. The dynamic feedback controller is designed based on Lyapunov stability theory and LMI formulation. This scheme reduces the $\mathscr{H}_\infty$-norm from the disturbance to the output error within a prescribed level.

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