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A note on teleparallel Killing vector fields in Bianchi type VIII and IX space–times in teleparallel theory of gravitation

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In this paper we classify Bianchi type VIII and IX space–times according to their teleparallel Killing vector fields in the teleparallel theory of gravitation by using a direct integration technique. It turns out that the dimensions of the teleparallel Killing vector fields are either 4 or 5. From the above study we have shown that the Killing vector fields for Bianchi type VIII and IX space–times in the context of teleparallel theory are different from that in general relativity.

Keywords: Weitzenböck geometry, teleparallel theory of gravitation, conservation laws

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1. Introduction

Conservation laws play a vital role in discovering the hidden realities of our universe. Laws of conservation can be studied alternatively in the presence of curvature or in the presence of torsion in the space–time, the details can be found in Refs. [1]–[12]. The study of conservation laws in the presence of curvature falls in the field of general relativity, while the study of conservation laws in the presence of torsion falls in the field of teleparallel theory of gravitation. Teleparallel theory of gravitation is based on Weitzenböck geometry.[13] In this theory, the torsion plays an important role and the curvature of the space–time is zero. Thus like general relativity, in which gravitation is attributed to the curvature of the space–time having zero torsion, the gravitation in teleparallel theory is attributed to the torsion of the space–time having zero curvature, which plays the role of a force,[14] a detailed description of this theory can be found in Ref. [15]. Working in the field of teleparallel theory, Sharif and Amir[5] introduced the teleparallel version of the Lie derivative for Killing vector fields and used those equations to find the teleparallel Killing vector fields in the Einstein universe. With this definition of teleparallel Lie derivative, an approach to studying symmetries in the teleparallel theory has been initiated. The study of symmetries and their importance in general relativity is quite advantageous in exploring some crucial aspects of the space–time physics, examples can be found in Refs. [16] and [17]. To understand the importance of symmetries in teleparallel theory refer to Ref. [18], in which the author calculated energy, momentum, angular momentum and teleparallel Killing vector fields for the Brane world black hole in teleparallel theory and showed that when the results of energy, momentum and angular momentum are in agreement with that of general relativity, the teleparallel Killing vector fields are also in agreement with the results of general relativity. Keeping this point in mind, it will be interesting to investigate the Killing symmetry in the teleparallel theory, which in turn will help to understand and to estimate the various physical properties of space–time, such as energy, momentum and angular momentum in this alternative description of gravity.

Our current study will help not only to understand the geometrical and the physical properties of space–time but also to explore the effect of torsion on the conservation laws. To compare the results of conservation laws in the teleparallel and the general relativity theories, we investigate Killing vector

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fields in some well known space–times in the context of teleparallel theory. We explore Killing vector fields in Bianchi type I,[4] Bianchi type III, Kantowski–Sachs[5] and cylindrically symmetric static space–times.[6] The dimensions of the Killing vector fields for these space–times are the same as those in general relativity, while the teleparallel Killing vector fields are multiples of the corresponding components of the line elements. In this paper, we focus our investigations on the classification of Bianchi types VIII and IX space–times according to their Killing vector fields in teleparallel theory of gravitation. Throughout this paper, a, b, $c, \ldots (= 0, 1, 2, 3)$ are the tangent space indices and μ, ν, ρ, $\ldots (= 0, 1, 2, 3)$ refer to the space–time indices. In the next section, an overview of the teleparallel theory of gravitation is given.

2. Overview

The teleparallel covariant derivative ∇_ρ of a covariant tensor of rank 2 is defined as[19]

$$\nabla_\rho A_{\mu\nu} = A_{\mu\nu,\rho} - \Gamma^{\theta}_{\ \rho\nu} A_{\mu\theta} - \Gamma^{\theta}_{\ \mu\rho} A_{\nu\theta}, \tag{1}$$

where comma denote the partial derivative and $\Gamma^{\theta}_{\ \rho\nu}$ are Weitzenböck connections defined as[19]

$$\Gamma^{\theta}_{\ \mu\nu} = S_a^{\ \theta} \partial_\nu S^a_{\ \mu}, \tag{2}$$

where $S^a_{\ \mu}$ is the non-trivial tetrad field and its inverse field is denoted by $S_a^{\ \nu}$ and satisfies the relations

$$S^a_{\ \mu} S^a_{\ \nu} = \delta_\mu^{\ \nu}, \quad S^a_{\ \mu} S_b^{\ \mu} = \delta_b^{\ a}. \tag{3}$$

The Remannian metric can be generated from the tetrad field as

$$g_{\mu\nu} = \eta_{ab} S^a_{\ \mu} S^b_{\ \nu}, \tag{4}$$

where η_{ab} is the Minkowski metric given by $\boldsymbol{\eta} = \mathrm{diag}(-1, 1, 1, 1)$. The Weitzenböck and the Levi–Civita connections are related by

$$\Gamma^{\theta}_{\ \mu\nu} = \Gamma^{0\theta}_{\ \ \mu\nu} + K^{\theta}_{\ \ \mu\nu}, \tag{5}$$

where

$$K^{\theta}_{\ \mu\nu} = \frac{1}{2}(T_\mu^{\ \theta}{}_\nu + T_\nu^{\ \theta}{}_\mu - T^{\theta}_{\ \mu\nu}) \tag{6}$$

is a tensor quantity called the contortion tensor and $\Gamma^{0\theta}_{\ \ \mu\nu}$ is the Levi–Civita connection defined as

$$\Gamma^{0\theta}_{\ \ \mu\nu} = \frac{1}{2} g^{\theta\sigma}(g_{\sigma\nu,\mu} + g_{\sigma\mu\nu} - g_{\mu\nu\sigma}). \tag{7}$$

The torsion of the space–time in terms of Weitzenböck connections is defined as

$$T^{\theta}_{\ \mu\nu} = \Gamma^{\theta}_{\ \nu\mu} - \Gamma^{\theta}_{\ \mu\nu}, \tag{8}$$

which is anti-symmetric with respect to its lower indices. The Riemann curvature tensor in terms of Weitzenböck connection in teleparallel theory is given as

$$\begin{aligned} R^{\theta}_{\ \sigma\mu\nu} = {} & \Gamma^{\theta}_{\ \sigma\nu,\mu} - \Gamma^{\theta}_{\ \sigma\mu,\nu} + \Gamma^{\theta}_{\ \lambda\mu}\Gamma^{\lambda}_{\ \sigma\nu} \\ & - \Gamma^{\theta}_{\ \lambda\nu}\Gamma^{\lambda}_{\ \sigma\mu}. \end{aligned} \tag{9}$$

Now substituting Eq. (5) into Eq. (9), we obtain

$$R^{\sigma}_{\ \theta\mu\nu} = R^{0\sigma}_{\ \ \theta\mu\nu} + Q^{\sigma}_{\ \theta\mu\nu} = 0, \tag{10}$$

where $R^{0\sigma}_{\ \ \theta\mu\nu}$ represents Riemann curvature tensor in general relativity and

$$\begin{aligned} Q^{\sigma}_{\ \theta\mu\nu} = {} & \nabla_\mu K^{\sigma}_{\ \theta\nu} - \nabla_\nu K^{\sigma}_{\ \theta\mu} - K^{\lambda}_{\ \theta\nu} K^{\sigma}_{\ \lambda\mu} \\ & + K^{\lambda}_{\ \theta\mu} K^{\sigma}_{\ \lambda\nu} \end{aligned} \tag{11}$$

is the tensor quantity based on Weitzenböck connection only. The teleparallel Killing equation is defined as[5]

$$\begin{aligned} \underset{\boldsymbol{X}}{L}^{\mathrm{T}} g_{\mu\nu} = {} & g_{\mu\nu,\rho} X^\rho + g_{\rho\nu} X^\rho_{\ ,\mu} + g_{\mu\rho} X^\rho_{\ ,\nu} \\ & + X^\rho (g_{\theta\nu} T^{\theta}_{\ \mu\rho} + g_{\mu\theta} T^{\theta}_{\ \nu\rho}) = 0, \end{aligned} \tag{12}$$

where $\underset{\boldsymbol{X}}{L}^{\mathrm{T}}$ denotes the teleparallel Lie derivative with respect to the vector field $\boldsymbol{X}$ and $T^{\theta}_{\ \mu\nu}$ denotes the torsion tensor which is anti-symmetric with respect to its lower two indices.

3. Main results

Consider Bianchi type VIII and IX space–times in usual coordinates (t, x, y, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with the line element[11]

$$\begin{aligned} \mathrm{d}s^2 = {} & -\mathrm{d}t^2 + A(t)\,\mathrm{d}x^2 + B(t)\,\mathrm{d}y^2 + [A(t)\,F^2(y) \\ & + B(t)\,G^2(y)]\,\mathrm{d}z^2 + 2\,A(t)\,F(y)\,\mathrm{d}x\,\mathrm{d}z, \end{aligned} \tag{13}$$

where A and B are no-where-zero functions of t only. The above space–times (13) becomes Bianchi type VIII if $F(y) = \cosh y$ and $G(y) = \sinh y$ and Eq. (13) becomes Bianchi type IX when $F(y) = \cos y$ and $G(y) = \sin y$. The above space–times Eq. (13) admit four linearly independent Killing vector fields in gen-

eral relativity, which are[20]

$$\frac{\partial}{\partial x},\ \frac{\partial}{\partial z},\ \frac{1}{G(y)}\sin z\frac{\partial}{\partial x}+\cos z\frac{\partial}{\partial y}-\frac{F(y)}{G(y)}\sin z\frac{\partial}{\partial z}$$

and

$$-\frac{1}{G(y)}\cos z\frac{\partial}{\partial x}+\sin z\frac{\partial}{\partial y}+\frac{F(y)}{G(y)}\cos z\frac{\partial}{\partial z}.$$

The tetrad components and its inverse can be obtained by using relation (4) as

$$S^a{}_\mu=\begin{pmatrix}1&0&0&0\\0&\sqrt{A(t)}&0&\sqrt{A(t)}\cos y\\0&0&\sqrt{B(t)}&0\\0&0&0&\sqrt{B(t)}\sin y\end{pmatrix}, \tag{14}$$

$$S_a{}^\mu=\begin{pmatrix}1&0&0&0\\0&\dfrac{1}{\sqrt{A(t)}}&0&-\dfrac{\cos y}{\sqrt{B(t)}\sin y}\\0&0&\dfrac{1}{\sqrt{B(t)}}&0\\0&0&0&\dfrac{1}{\sqrt{B(t)}\sin y}\end{pmatrix}. \tag{15}$$

It can be verified easily that Eqs. (3) and (4) between $S^a{}_\mu$ and $S_a{}^\mu$ are satisfied. Using Eq. (2), the corresponding non-vanishing Weitzenböck connections are obtained as

$$\begin{aligned}&\Gamma^1{}_{10}=\frac{\dot A}{2A},\quad \Gamma^2{}_{20}=\frac{\dot B}{2B},\\&\Gamma^3{}_{30}=\frac{\dot B}{2B},\quad \Gamma^3{}_{32}=\frac{F(y)}{G(y)},\\&\Gamma^1{}_{30}=\frac{1}{2}F(y)\left(\frac{\dot A}{A}-\frac{\dot B}{B}\right),\quad \Gamma^1{}_{32}=-\frac{1}{G(y)},\end{aligned} \tag{16}$$

where dot denotes the derivative with respect to t. Thus the non vanishing torsion components by using Eq. (8) are obtained as

$$\begin{aligned}&T^1{}_{01}=\frac{\dot A}{2A},\quad T^1{}_{03}=\frac{1}{2}F(y)\left(\frac{\dot A}{A}-\frac{\dot B}{B}\right),\\&T^2{}_{02}=\frac{\dot B}{2B},\quad T^3{}_{03}=\frac{\dot B}{2B},\\&T^1{}_{23}=-\frac{1}{G(y)},\quad T^3_{23}=\frac{F(y)}{G(y)}.\end{aligned} \tag{17}$$

A vector field $\boldsymbol{X}$ is said to be teleparallel Killing vector field if it satisfies Eq. (12). We can write Eq. (12) explicitly using Eqs. (13) and (16) as

$$X^0{}_{,0}=0,\ \ X^2{}_{,2}=0, \tag{18}$$

$$X^1{}_{,1}+F(y)\,X^3{}_{,1}=0, \tag{19}$$

$$2B\,X^2{}_{,0}-2\,X^0{}_{,2}+\dot B\,X^2=0, \tag{20}$$

$$\begin{aligned}&B\,X^2{}_{,1}+A\,X^1{}_{,2}+A\,F(y)\,X^3{}_{,2}\\&\quad -A\,G(y)\,X^3=0,\end{aligned} \tag{21}$$

$$\begin{aligned}&A\,X^1{}_{,0}-X^0{}_{,1}+\frac{1}{2}\dot A\,X^1+A\,F(y)\,X^3{}_{,0}\\&\quad+\frac{1}{2}\dot A\,F(y)\,X^3=0,\end{aligned} \tag{22}$$

$$\begin{aligned}&A\,F(y)X^1{}_{,0}+[AF^2(y)+BG^2(y)]X^3{}_{,0}-X^0{}_{,3}\\&\quad+\frac{1}{2}\dot A\ F(y)X^1+\frac{1}{2}[\dot A\,F^2(y)\\&\quad+\dot BG(y)]\,X^3=0,\end{aligned} \tag{23}$$

$$\begin{aligned}&A\,F(y)X^1{}_{,1}+[AF^2(y)+BG^2(y)]X^3{}_{,1}\\&\quad+AX^1{}_{,3}+A\,F(y)X^3{}_{,3}=0,\end{aligned} \tag{24}$$

$$\begin{aligned}&A\,F(y)X^1{}_{,2}+[AF^2(y)+BG^2(y)]X^3{}_{,2}\\&\quad+BX^2{}_{,3}-(A\,-B)\,F(y)\,G(y)X^3=0,\end{aligned} \tag{25}$$

$$A\,F(y)X^1{}_{,3}+[AF^2(y)+BG^2(y)]X^3{}_{,3}=0. \tag{26}$$

Integrating Eq. (18) we yields

$$X^0=E^1(x,y,z),\quad X^2=E^2(t,x,z)\,, \tag{27}$$

where $E^1(x,y,z)$ and $E^2(t,x,z)$ are functions of integration and are to be determined. Now considering Eq. (24) and using Eq. (19), we obtain

$$BG^2(y)X^3{}_{,1}+A[X^1{}_{,3}+\ F(y)]X^3{}_{,3}=0. \tag{28}$$

Now differentiating Eq. (19) with respect to z, we have

$$X^1{}_{,13}+F(y)\,X^3{}_{,13}=0. \tag{29}$$

Differentiating Eq. (28) with respect to x and using Eq. (29), we have $X^3{}_{,11}=0$. Now integrating this equation twice with respect to x yields

$$X^3=x\,E^3(t,y,z)+E^4(t,y,z). \tag{30}$$

Substituting the above value of X^3 to Eq. (19) and integrating the resulting equation, we have

$$X^1=-xF(y)\,E^3(t,y,z)+E^5(t,y,z), \tag{31}$$

where $E^3(t,y,z)$, $E^4(t,y,z)$ and $E^5(t,y,z)$ are functions of integration and are to be determined. Thus we have the following system of equations:

$$\begin{aligned}&X^0=E^1(x,y,z),\\&X^1=-xF(y)\,E^3(t,y,z)+E^5(t,y,z),\\&X^2=E^2(t,x,z)\,,\\&X^3=x\,E^3(t,y,z)+E^4(t,y,z).\end{aligned} \tag{32}$$

Now we need to solve system (32) by using the remaining six equations. To avoid lengthy details here we shall write only the results as follows.

Case I In this case we have $A \neq B$, $A \neq$ constant and $B \neq$ constant. The space–time is given in Eq. (13). Solution of Eqs. (18)–(26) is given by

$$\begin{aligned} X^0 &= c_1, \\ X^1 &= -\frac{c_2}{\sqrt{B}} \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)} + \frac{c_3}{\sqrt{A}}, \\ X^2 &= \frac{z c_2}{\sqrt{B}} + \frac{c_4}{\sqrt{B}}, \\ X^3 &= \frac{c_2}{\sqrt{B}} \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)}, \end{aligned} \tag{33}$$

where c_1, c_2, c_3, $c_4 \in \boldsymbol{R}$. Here the space–time (13) admits four linearly independent teleparallel Killing vector fields, which can be written as $\partial/\partial\, t$, $(1/\sqrt{A})(\partial/\partial\, x)$, $(1/\sqrt{B})(\partial/\partial\, y)$ and

$$\begin{aligned} &\frac{1}{\sqrt{B}} \Bigg[- \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)} \frac{\partial}{\partial\, x} + z \frac{\partial}{\partial\, y} \\ &\quad + \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)} \frac{\partial}{\partial\, z} \Bigg]. \end{aligned}$$

Killing vector fields in general relativity are $\partial/\partial\, x$, $\partial/\partial\, z$,

$$\frac{1}{G(y)} \sin z \frac{\partial}{\partial x} + \cos z\, \frac{\partial}{\partial y} - \frac{F(y)}{G(y)} \sin z \frac{\partial}{\partial z}$$

and

$$-\frac{1}{G(y)} \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y} + \frac{F(y)}{G(y)} \cos z \frac{\partial}{\partial z}.$$

We can easily see that the teleparallel Killing vector fields are different from that in general relativity. The number of Killing vector fields in both theories are the same.

Case II In this case, we have $A \neq B$, $A \neq$ constant and $B = \eta$, where $\eta \in \boldsymbol{R}\backslash\{0\}$. In this case the line element for Bianchi type VIII and IX space–times becomes

$$\begin{aligned} \mathrm{d}s^2 \;=\; & -\mathrm{d}t^2 + A(t)\, \mathrm{d}x^2 + \eta\, \mathrm{d}y^2 \\ & + [A(t)\, F^2(y) + \eta\, G^2(y)]\, \mathrm{d}z^2 \\ & + 2\, A(t)\, F(y)\, \mathrm{d}x\, \mathrm{d}z. \end{aligned} \tag{34}$$

Solution of Eqs. (18)–(26) is given by

$$\begin{aligned} X^0 &= c_1, \\ X^1 &= -c_2 \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)} + \frac{c_3}{\sqrt{A}}, \\ X^2 &= z c_2 + c_4, \\ X^3 &= c_2 \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)}, \end{aligned} \tag{35}$$

where c_1, c_2, c_3, $c_4 \in \boldsymbol{R}$. Here space–time (34) admits four linearly independent teleparallel Killing vector fields, which can be written as $\partial/\partial\, t$, $(1/\sqrt{A})(\partial/\partial\, x)$, $\partial/\partial\, y$ and

$$\begin{aligned} &- \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)} \frac{\partial}{\partial\, x} + z \frac{\partial}{\partial\, y} \\ &\quad + \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)} \frac{\partial}{\partial\, z}. \end{aligned}$$

We can easily see that the teleparallel Killing vector fields are different from that in general relativity. The number of Killing vector fields in both theories are the same.

Case III In this case we have $A \neq B$, $B \neq$ constant and $A = \lambda$, where $\lambda \in \boldsymbol{R}\backslash\{0\}$. Here the line element for Bianchi type VIII and IX space–times becomes

$$\begin{aligned} \mathrm{d}s^2 \;=\; & -\mathrm{d}t^2 + \lambda \mathrm{d}x^2 + B(t)\, \mathrm{d}y^2 \\ & + [\lambda\, F^2(y) + B(t)\, G^2(y)]\, \mathrm{d}z^2 \\ & + 2\, \lambda\, F(y)\, \mathrm{d}x\, \mathrm{d}z. \end{aligned} \tag{36}$$

Solution of Eqs. (18)–(26) is given by

$$\begin{aligned} X^0 &= c_1, \\ X^1 &= -\frac{c_2}{\sqrt{B}} \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)} + c_3, \\ X^2 &= \frac{z c_2}{\sqrt{B}} + \frac{c_4}{\sqrt{B}}, \\ X^3 &= \frac{c_2}{\sqrt{B}} \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)}, \end{aligned} \tag{37}$$

where c_1, c_2, c_3, $c_4 \in \boldsymbol{R}$. Here space–time (36) admits four linearly independent teleparallel Killing vector fields, which can be written as $\partial/\partial\, t$, $\partial/\partial\, x$, $(1/\sqrt{B})(\partial/\partial\, y)$ and

$$\begin{aligned} &\frac{1}{\sqrt{B}} \Bigg[- \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)} \frac{\partial}{\partial\, x} + z \frac{\partial}{\partial\, y} \\ &\quad + \ln \left| \frac{1}{G(y)} + \frac{F(y)}{G(y)} \right| \frac{F(y)}{G(y)} \frac{\partial}{\partial\, z} \Bigg]. \end{aligned}$$

We can easily see that the teleparallel Killing vector fields are different from that in general relativity. The number of Killing vector fields in both theories are the same.

Case IV In this case, we have $A = B$, $A \neq$ constant and $B \neq$ constant. The line element for the space–times in this case becomes

$$\begin{aligned} \mathrm{d}s^2 \;=\; & -\mathrm{d}t^2 + A(t)[\, \mathrm{d}x^2 + \mathrm{d}y^2] \\ & + A(t)\, [F^2(y) + G^2(y)]\, \mathrm{d}z^2 \\ & + 2\, A(t)\, F(y)\, \mathrm{d}x\, \mathrm{d}z. \end{aligned} \tag{38}$$

Solution of Eqs. (18)–(26) is given by

$$\begin{aligned}
X^0 &= c_1,\\
X^1 &= -\frac{1}{\sqrt{A}}\Big[c_5\ln|G(y)| + c_2\ln\Big|\frac{1}{G(y)}+\frac{F(y)}{G(y)}\Big|\Big]\\
&\quad\times\frac{F(y)}{G(y)} - \frac{y\,c_5}{\sqrt{A}} + \frac{c_3}{\sqrt{A}},\\
X^2 &= \frac{x c_5}{\sqrt{A}} + \frac{z c_2}{\sqrt{A}} + \frac{c_4}{\sqrt{A}},\\
X^3 &= \frac{1}{\sqrt{A}}\Big[c_5\ln|G(y)|\\
&\quad + c_2\ln\Big|\frac{1}{G(y)}+\frac{F(y)}{G(y)}\Big|\Big]\frac{F(y)}{G(y)},
\end{aligned}\tag{39}$$

where $c_1,\ c_2,\ c_3,\ c_4,\ c_5 \in \boldsymbol{R}$. Here space–time (38) admits five linearly independent teleparallel Killing vector fields, which can be written as $\partial/\partial\,t$, $(1/\sqrt{A})(\partial/\partial\,x)$, $(1/\sqrt{A})(\partial/\partial\,y)$,

$$\begin{aligned}
&\frac{1}{\sqrt{A}}\Big[-\ln\Big|\frac{1}{G(y)}+\frac{F(y)}{G(y)}\Big|\frac{F(y)}{G(y)}\frac{\partial}{\partial\,x} + z\frac{\partial}{\partial\,y}\\
&\quad+\ln\Big|\frac{1}{G(y)}+\frac{F(y)}{G(y)}\Big|\frac{F(y)}{G(y)}\frac{\partial}{\partial\,z}\Big]
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{\sqrt{A}}\ln|G(y)| - \frac{y}{\sqrt{A}}\Big]\frac{\partial}{\partial\,x} + \frac{x}{\sqrt{A}}\frac{\partial}{\partial\,y}\\
&\quad+\frac{1}{\sqrt{A}}\ln|G(y)|\frac{\partial}{\partial\,z}.
\end{aligned}$$

We can easily see that the teleparallel Killing vector fields are different from that in general relativity. It is important to note that in this case the space–times admit one more teleparallel Killing vector field.

Case V In this case, we have $A=\lambda$ and $B=\eta$, where $\lambda,\eta\in\boldsymbol{R}\backslash\{0\}$. The line element for the space–times in this case becomes

$$\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t^2 + \lambda\,\mathrm{d}x^2 + \eta\,\mathrm{d}y^2\\
&\quad+[\lambda F^2(y) + \eta G^2(y)]\,\mathrm{d}z^2\\
&\quad+2\lambda F(y)\,\mathrm{d}x\,\mathrm{d}z.
\end{aligned}\tag{40}$$

Solution of Eqs. (18)–(26) is given by

$$\begin{aligned}
X^0 &= c_1,\\
X^1 &= -\Big[-\frac{\lambda}{\eta}c_5\ln|G(y)|\\
&\quad+c_2\ln\Big|\frac{1}{G(y)}+\frac{F(y)}{G(y)}\Big|\Big]\frac{F(y)}{G(y)} + y\,c_5 + c_3,\\
X^2 &= -\frac{\lambda}{\eta}x\,c_5 + z c_2 + c_4,\\
X^3 &= \Big[-\frac{\lambda}{\eta}c_5\ln|G(y)|\\
&\quad+c_2\ln\Big|\frac{1}{G(y)}+\frac{F(y)}{G(y)}\Big|\Big]\frac{1}{G(y)},
\end{aligned}\tag{41}$$

where $c_1,\ c_2,\ c_3,\ c_4,\ c_5\in\boldsymbol{R}$. Here space–time (40) admits five linearly independent teleparallel Killing vector fields, which can be written as $\partial/\partial\,t$, $\partial/\partial\,x$, $\partial/\partial\,y$,

$$\begin{aligned}
&-\ln\Big|\frac{1}{G(y)}+\frac{F(y)}{G(y)}\Big|\frac{F(y)}{G(y)}\frac{\partial}{\partial\,x} + z\frac{\partial}{\partial\,y}\\
&\quad+\ln\Big|\frac{1}{G(y)}+\frac{F(y)}{G(y)}\Big|\frac{1}{G(y)}\frac{\partial}{\partial\,z}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\lambda}{\eta}\ln|G(y)| + y\Big]\frac{\partial}{\partial\,x} - \frac{\lambda}{\eta}x\frac{\partial}{\partial\,y}\\
&\quad+-\frac{\lambda}{\eta}\ln|G(y)|\frac{1}{G(y)}\frac{\partial}{\partial\,z}.
\end{aligned}$$

By comparison with the Killing vector fields in general relativity, we can see that two Killing vector fields $\partial/\partial\,t$ and $\partial/\partial\,x$ are the same in both theories and all others are different.

4. Conclusion

In this paper, we classify the teleparallel Killing vector fields for Bianchi type VIII and IX space–times and show that the above space–times admit either 4 or 5 teleparallel Killing vector fields, which are different from the Killing vector fields in general relativity. It also turns out that the teleparallel Killing vector fields are multiples of some specific functions of t (these functions involve the components of the metric tensor). These functions appear in teleparallel Killing vector fields because of the non-vanishing torsion components. From the above discussion, it is clear that the presence of the torsion in Bianchi type VIII and IX space–times changes the number of the conservation laws. It is also important to remember that the above space–time admits one more conservation law in teleparallel theory in case IV.

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