Asymptotic solution for the El Niño time delay sea–air oscillator model*

Mo Jia-Qi(莫嘉琪)[a)b)†], Lin Wan-Tao(林万涛)[c)], and Lin Yi-Hua(林一骅)[c)]

[a)] *Department of Mathematics, Anhui Normal University, Wuhu* 241003, *China*

[b)] *Division of Computational Science, E-Institutes of Shanghai Universities at Shanghai Jiaotong University, Shanghai* 200240, *China*

[c)] *Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing* 100029, *China*

(Received 18 August 2010; revised manuscript received 12 February 2011)

A sea–air oscillator model is studied using the time delay theory. The aim is to find an asymptotic solving method for the El Niño-southern oscillation (ENSO) model. Employing the perturbed method, an asymptotic solution of the corresponding problem is obtained. Thus we can obtain the prognoses of the sea surface temperature (SST) anomaly and the related physical quantities.

Keywords: nonlinear, approximate solution, El Niño-Southern oscillator model

PACS: 02.30.Mv **DOI:** 10.1088/1674-1056/20/7/070205

1. Introduction

The El Niño-Southern Oscillation (ENSO) is an interannual phenomenon involved with the tropical Pacific ocean–atmosphere interactions.[1−10] Mo, Lin *et al* have considered a class of problems in atmospheric physics, ocean weather, dynamic system and so on.[11−19] An interdecadal fluctuation which contributes to the irregularity of the southern oscillation, is superimposed on the natural model of the coupled ocean–atmosphere system. In the eastern equatorial tropical Pacific, a shoaling of subsurface isotherms signal the end of the El Niño and the return of colder surface waters. The oscillatory nature of the ENSO requires both positive and negative ocean–atmosphere feedback. This El Niño anomaly reduces the zonal sea surface temperature (SST) gradient and hence reduces the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn drive the ocean circulation and the changes that reinforce SST anomaly. This positive ocean–atmosphere feedback or the coupled ocean–atmosphere instability leads the equatorial Pacific into a never-end warm state.

Recently, many scholars have investigated the nonlinear problem.[20−24] Approximate methods have been developed and refined. In this paper, we consider a time delay sea–air oscillator model using a simple and valid perturbation method.

2. ENSO sea–air oscillator model

The response of the atmosphere to the warming in the tropics includes an intensification of the extratropical westerly, thus leading to colder surface waters in extratropical regions, which will be windows to the equatorial thermocline. It implies a continual, interdecadal climate fluctuation with a period that the water parcels take to travel from the extratropics to the equator.

Now we consider the ENSO time delay sea–air oscillator model of interdecadal climate fluctuations:[2]

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = \alpha\,\gamma\,(T_2 - T_1) + \beta_1 T_2 - \varepsilon\,T_1^3, \tag{1}$$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} = -\gamma(T_2 - T_1) - \delta\,(T_2 - T_1(t-\tau)) + \beta_2 T_2 - \varepsilon\,T_2^3, \tag{2}$$

$$T_1(t) = h(t), \quad -\tau \le t < 0, \quad T_2(0) = A, \tag{3}$$

*Project supported by the National Natural Science Foundation of China (Grant No.40876010), the Natural Science Foundation of Zhejiang Province of China (Grant No. Y6110502), the Natural Science Foundation of the Education Bureau of Anhui Province of China (Grant Nos. KJ2011A135 and KJ2011Z003), the LASG State Key Laboratory Special Fund of China, and the Foundation of E-Institutes of Shanghai Municipal Education Commission, China (Grant No. E03004).

†Corresponding author. E-mail: mojiaqi@mail.ahnu.edu.cn

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where T_1 is the extratropical temperature approximately at region from 25° N to 50° N (and from 25° S to 50° S); T_2 is the tropical temperature approximately at region from 20° S to 20° N; α is the fraction of poleward atmospheric heat transport; β_1 is a negative constant; β_2 is a feedback parameter; coefficients γ, δ, ε are positive constants; the initial function $h(t)$ for the extratropical temperature is infinitely differentiable; A is the initial value for the tropical temperature; τ is the time delay. In this paper, τ and ε are also small parameters and $\tau/\varepsilon = O(1)$.

3. Reduced solution for the time delay model

We first construct the reduced solution for the small time delay sea–air oscillator model. For $\tau = 0$, we have

$$\frac{\mathrm{d}Y_1}{\mathrm{d}t} = \alpha\,\gamma\,(Y_2 - Y_1) + \beta_1 Y_2 - \varepsilon\,Y_1^3, \tag{4}$$

$$\frac{\mathrm{d}Y_2}{\mathrm{d}t} = -\gamma(Y_2 - Y_1) - \delta\,(Y_2 - Y_1) + \beta_2 Y_2 - \varepsilon\,Y_2^3, \tag{5}$$

$$Y_1(0) = h(0), \quad Y_2(0) = A. \tag{6}$$

Let $(Y_{10}(t), Y_{20}(t))$ be the solution for the reduced problem given in Eqs. (4)–(6). And we assume

$$Y_{i0}(t) = \sum_{j=0}^{\infty} Y_{ij0}\varepsilon^j, \quad i = 1,\ 2. \tag{7}$$

Substituting Eq. (7) into Eqs. (4)–(6), developing nonlinear terms in ε and equating coefficients of the same powers of ε (ε^j, $j = 0, 1, \ldots$), we obtain

$$\frac{\mathrm{d}Y_{100}}{\mathrm{d}t} = -\alpha\,\gamma\,Y_{100} + (\alpha\,\gamma + \beta_1)Y_{200}, \tag{8}$$

$$\frac{\mathrm{d}Y_{200}}{\mathrm{d}t} = (\gamma + \delta)Y_{100} - (\gamma - \beta_2 + \delta)Y_{200}, \tag{9}$$

$$Y_{100}(0) = h(0), \quad Y_{200}(0) = A, \tag{10}$$

$$\frac{\mathrm{d}Y_{1j0}}{\mathrm{d}t} = -\alpha\,\gamma\,Y_{1j0} + (\alpha\,\gamma + \beta_1)Y_{2j0} + F_{1j0}, \quad j = 1, 2, \ldots, \tag{11}$$

$$\frac{\mathrm{d}Y_{2j0}}{\mathrm{d}t} = (\gamma + \delta)Y_{1j0} - (\gamma - \beta_2 + \delta)Y_{2j0} + F_{2j0}, \quad j = 1, 2, \ldots, \tag{12}$$

$$Y_{1j0}(0) = 0, \quad Y_{j00}(0) = 0, \quad j = 1, 2, \ldots, \tag{13}$$

where

$$F_{ij0} = \frac{1}{(j-1)!}\left[\frac{\partial^{j-1}}{\partial\varepsilon^{j-1}}\left(\sum_{k=0}^{\infty} Y_{ik0}\varepsilon^k\right)^3\right]_{\varepsilon=0},$$

$$i = 1, 2; \quad j = 1, 2, \ldots \tag{14}$$

The characteristic equation for the system given in Eqs. (8) and (9) is

$$\lambda^2 + [(1+\alpha)\gamma - \beta_2 + \delta]\lambda - (\gamma + \delta)(\alpha\gamma + \beta_1) = 0.$$

And the characteristic roots are

$$\lambda_{1,2} = \frac{1}{2}\big(-[(1+\alpha)\gamma - \beta_2 + \delta] \pm \{[(1+\alpha)\gamma - \beta_2 + \delta]^2 + 4(\gamma+\delta)(\alpha\gamma+\beta_1)\}^{1/2}\big). \tag{15}$$

Thus the solution $(Y_{100}(t),\ Y_{200}(t))$ of the linear problem given in Eqs. (8)–(10) is

$$Y_{100}(t) = \frac{1}{\lambda_1 - \lambda_2}\{[A(\alpha\gamma + \beta_1) - h(0)(\lambda_2 + \alpha\gamma)] \times \exp(\lambda_1 t) - [A(\alpha\gamma + \beta_1) - h(0)(\lambda_1 + \alpha\gamma)]\exp(\lambda_2 t)\}, \tag{16}$$

$$Y_{200}(t) = \frac{1}{(\lambda_1 - \lambda_2)(\alpha\gamma + \beta_1)}\{(\lambda_1 + \alpha\gamma) \times [A(\alpha\gamma + \beta_1) - h(0)(\lambda_2 + \alpha\gamma)] \times \exp(\lambda_1 t) - (\lambda_2 + \alpha\gamma)[A(\alpha\gamma + \beta_1) - h(0)(\lambda_1 + \alpha\gamma)]\exp(\lambda_2 t)\}. \tag{17}$$

From Eqs. (11)–(13), we obtain

$$Y_{1j0}(t) = \frac{1}{\lambda_1 - \lambda_2}\int_0^t F_{1j0}(\tau)\{[A(\alpha\gamma + \beta_1) - h(0)(\lambda_2 + \alpha\gamma)]\exp\lambda_1(t-\tau) - [A(\alpha\gamma + \beta_1) - h(0)(\lambda_1 + \alpha\gamma)] \times \exp\lambda_2(t-\tau)\}^{-1}\mathrm{d}\tau, \quad j = 1, 2, \ldots, \tag{18}$$

$$Y_{2j0}(t) = \frac{1}{(\lambda_1 - \lambda_2)(\alpha\gamma + \beta_1)} \times \int_0^t F_{2j0}(\tau)\{(\lambda_1 + \alpha\gamma)[A(\alpha\gamma + \beta_1) - h(0)(\lambda_2 + \alpha\gamma)]\exp\lambda_1(t-\tau) - (\lambda_2 + \alpha\gamma)[A(\alpha\gamma + \beta_1) - h(0)(\lambda_1 + \alpha\gamma)] \times \exp\lambda_2(t-\tau)\}^{-1}\mathrm{d}\tau, \quad j = 1, 2, \ldots \tag{19}$$

Substituting Eqs. (16)–(19) into Eq. (7), we can obtain the reduced solution $(Y_{10}(t),\ Y_{20}(t))$.

4. Outer solution

We assume that the model given in Eqs. (1)–(3) possesses the following formal solution:

$$T_i(t,\tau) = Y_i(t,\tau) + \tau\,V_i(\eta,\tau), \quad i = 1, 2, \tag{20}$$

where $(Y_1(t,\tau),\ Y_2(t,\tau))$ is the outer solution of the time delay model given in Eqs. (1)–(3) with

$$Y_i(t,\tau) = \sum_{k=0}^{\infty} Y_{ik}(t)\tau^k$$

$$\equiv \sum_{j,k=0}^{\infty} Y_{ijk}\varepsilon^j\tau^k, \quad i=1,2, \tag{21}$$

and $V_i(\eta,\tau)$ $(i=1,\ 2)$ are the initial-shock layer corrective terms

$$V_i(\eta,\tau) \sim \sum_{k=0}^{\infty} v_{ik}(\eta)\tau^k, \quad i=1,2. \tag{22}$$

The $\eta = t/\tau$ is a stretched variable. The $V_i(\eta,\tau)$ $(i=1,\ 2)$ tend to zero as $\eta \to +\infty$ for each $v_{ik}(\eta,x)$. Developing $T_1(t-\tau)$ in τ, we have

$$T_1(t-\tau) \sim T_1(t) - \frac{\mathrm{d}T_1(t)}{\mathrm{d}t}\tau + \frac{1}{2}\frac{\mathrm{d}^2T_1(t)}{\mathrm{d}t^2}\tau^2 + \cdots . \tag{23}$$

Substituting Eqs. (21)–(23) into Eqs. (1) and (2), developing the nonlinear term in τ, comparing the coefficients of τ^k and noting the Eq. (7), for $k=1,2,\ldots$, we have

$$\frac{\mathrm{d}Y_{1k}}{\mathrm{d}t} = -\alpha\ \gamma\ Y_{1k} + (\alpha\gamma+\beta_1)Y_{2k} + G_{1k}, \tag{24}$$

$$\frac{\mathrm{d}Y_{2k}}{\mathrm{d}t} = (\gamma+\delta)\ Y_{1k} - (\gamma-\beta_2+\delta)Y_{2k} + G_{2k}, \tag{25}$$

and

$$Y_{ik}(0) = -v_{i(k-1)}(0), \quad i=1,2, \tag{26}$$

where G_{ik} $(i=1,2,\ k=1,2,\ldots)$ are determined functions, whose construction are omitted. From Eqs. (24) and (25), we have

$$\begin{aligned}
Y_{1k}(t) =\ & \frac{1}{\lambda_1-\lambda_2}\{-[v_{2(k-1)}(0)(\alpha\gamma+\beta_1)+v_{1(k-1)}(0)(\lambda_2+\alpha\gamma)]\exp\lambda_1 t \\
& -[v_{2(k-1)}(0)(\alpha\gamma+\beta_1)+v_{1(k-1)}(0)(\lambda_1+\alpha\gamma)]\exp\lambda_2 t\} \\
& +\frac{1}{\lambda_1-\lambda_2}\int_0^t G_{1k}(\tau_1)\{-[v_{2(k-1)}(0)(\alpha\gamma+\beta_1)+v_{1(k-1)}(0)(\lambda_2+\alpha\gamma)]\exp\lambda_1(t-\tau_1) \\
& -[-v_{2(k-1)}(0)(\alpha\gamma+\beta_1)+v_{1(k-1)}(0)(\lambda_1+\alpha\gamma)]\exp\lambda_2(t-\tau_1)\}^{-1}\mathrm{d}\tau_1, \quad k=1,2,\ldots,
\end{aligned} \tag{27}$$

$$\begin{aligned}
Y_{2k}(t) =\ & \frac{1}{(\lambda_1-\lambda_2)(\alpha\gamma+\beta_1)}\{(\lambda_1+\alpha\gamma)[-v_{2(k-1)}(0)(\alpha\gamma+\beta_1)+v_{1(k-1)}(0)(\lambda_2+\alpha\gamma)]\exp\lambda_1 t \\
& -(\lambda_2+\alpha\gamma)[-v_{2(k-1)}(0)(\alpha\gamma+\beta_1)+v_{1(k-1)}(0)(\lambda_1+\alpha\gamma)]\exp\lambda_2 t\} \\
& +\frac{1}{(\lambda_1-\lambda_2)(\alpha\gamma+\beta_1)}\int_0^t G_{2k}(\tau_1)\{[-v_{2(k-1)}(0)(\alpha\gamma+\beta_1)+v_{1(k-1)}(0)(\lambda_2+\alpha\gamma)]\exp\lambda_1(t-\tau_1) \\
& -[-v_{2(k-1)}(0)(\alpha\gamma+\beta_1)+v_{1(k-1)}(0)(\lambda_1+\alpha\gamma)]\exp\lambda_2(t-\tau_1)\}^{-1}d\tau_1, \quad k=1,2,\ldots,
\end{aligned} \tag{28}$$

where $v_{i2(k-1)}(0)$ $(i=1,2,\ k=1,2,\ldots)$ will be decided below (see Eq. (38)).

5. Initial-shock layer functions

Now we construct the initial-shock layer functions $v_{ik}(\eta)$ $(i=1,2,\ k=0,1,2,\ldots)$. Using the integration by subsections, they are obtained successively on the intervals $p \le \eta \le p+1$, $p \ge 0$. Substituting Eqs. (20)–(22) into Eqs. (1) and (2), we have

$$\begin{aligned}
\frac{\mathrm{d}V_1}{\mathrm{d}\eta} =\ & \alpha\ \gamma\ \tau\ (V_2-V_1) + \beta_1\tau\ V_2 \\
& -\varepsilon\ (Y_1+\tau\ V_1)^3 + \varepsilon\ Y_1^3, \quad 0 \le \eta \le 1,
\end{aligned} \tag{29}$$

$$\begin{aligned}
\frac{\mathrm{d}V_2}{\mathrm{d}\eta} =\ & -\gamma\ \tau(V_2-V_1) - \delta\ \tau\ \{V_2 - h[\tau(\eta-1)]\} \\
& +\beta_2\tau\ V_2 - \varepsilon\ (Y_2+\tau\ V_2)^3 + \varepsilon\ Y_2^3,
\end{aligned} \tag{30}$$

$$\begin{aligned}
\frac{\mathrm{d}V_1}{\mathrm{d}\eta} =\ & \alpha\ \gamma\ \tau\ (V_2-V_1) + \beta_1\tau\ V_2 \\
& -\varepsilon\ (Y_1+\tau\ V_1)^3 + \varepsilon\ Y_1^3, \quad \eta \ge 1,
\end{aligned} \tag{31}$$

$$\begin{aligned}
\frac{\mathrm{d}V_2}{\mathrm{d}\eta} =\ & -\gamma\ \tau(V_2-V_1) - \delta\ \tau\ \{V_2 - V_1[\tau(\eta-1)]\} \\
& +\beta_2\tau\ V_2 - \varepsilon\ (Y_2+\tau\ V_2)^3 \\
& +\varepsilon\ Y_2^3, \quad \eta \ge 1.
\end{aligned} \tag{32}$$

From Eqs. (29), (30) and (3), for $\tau=0$, $v_{10}(\eta)$ and $v_{20}(\eta)$ satisfy

$$\frac{\partial v_{i0}}{\partial\eta}(\eta) = 0, \quad \eta \ge 0, \ \ i=1,2, \tag{33}$$

and

$$v_{i0}(0) = 0, \quad i=1,2. \tag{34}$$

Thus, from Eqs. (33) and (34), we can obtain $v_{i0}(\eta) = 0$ $(i=1,2)$.

We can also obtain $Y_{i1}(0) = -v_{i0}(0)$ $(i=1,2)$, i.e., $Y_{i1}(0) = 0$ $(i=1,2)$. From Eqs. (27) and (28), for $k=1$, we have

$$Y_{i1}(t) = 0, \quad i=1,2. \tag{35}$$

Noting Eq. (23), comparing the coefficients of τ^k $(k = 2, 3, \ldots)$ in Eqs. (29)–(32), we have

$$\frac{\partial v_{ik}}{\partial \eta}(\eta) = G_{ip}^{ik}(\eta), \quad p \le \eta \le p+1, \quad i = 1, 2, \quad k = 2, 3, \ldots, \tag{36}$$

where G_{ip}^k, $p = 0, 1, \ldots$, are determined functions. Therefore from Eq. (36), we can obtain $v_{ik}(\eta)$ $(i = 1, 2,\ k = 2, 3, \ldots)$,

$$v_{ik}(\eta) = v_{ik}(0) + \sum_{l=0}^{p-1} \int_l^{l+1} G_{il}^k(s)\mathrm{d}s + \int_p^{\eta} G_{ip}^k(s)\mathrm{d}s, \quad p \le \eta \le p+1, \quad i = 1, 2. \tag{37}$$

Thus we have

$$Y_{i(k+1)}(0) = -v_{ik}(0) = \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{ip}^k(s)\mathrm{d}s \right). \tag{38}$$

From Eqs. (27), (28) and (37), we have

$$\begin{aligned} Y_{1k}(t) = {} & \frac{1}{\lambda_1 - \lambda_2} \Bigg\{ \Bigg[\sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{2p}^{k-1}(s)\mathrm{d}s \right)(\alpha\gamma + \beta_1) + \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{1p}^{k-1}(s)\mathrm{d}s \right)(\lambda_2 + \alpha\gamma) \Bigg] \exp \lambda_1 t \\ & + \Bigg[\sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{2p}^{k-1}(s)\mathrm{d}s \right)(\alpha\gamma + \beta_1) - \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{1p}^{k-1}(s)\mathrm{d}s \right)(\lambda_1 + \alpha\gamma) \Bigg] \exp \lambda_2 t \Bigg\} \\ & + \frac{1}{\lambda_1 - \lambda_2} \int_0^t G_{1k}(\tau_1) \Bigg\{ \Bigg[\sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{2p}^{k-1}(s)\mathrm{d}s \right)(\alpha\gamma + \beta_1) \\ & + \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{1p}^{k-1}(s)\mathrm{d}s \right)(\lambda_2 + \alpha\gamma) \Bigg] \exp \lambda_1 (t - \tau_1) - \Bigg[\sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{2p}^{k-1}(s)\mathrm{d}s \right)(\alpha\gamma + \beta_1) \\ & - \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{1p}^{k-1}(s)\mathrm{d}s \right)(\lambda_1 + \alpha\gamma) \Bigg] \exp \lambda_2 (t - \tau_1) \Bigg\}^{-1} \mathrm{d}\tau_1, \quad k = 1, 2, \ldots, \end{aligned} \tag{39}$$

$$\begin{aligned} Y_{2k}(t) = {} & \frac{1}{(\lambda_1 - \lambda_2)(\alpha\gamma + \beta_1)} \Bigg\{ (\lambda_1 + \alpha\gamma) \Bigg[\sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{2p}^{k-1}(s)\mathrm{d}s \right)(\alpha\gamma + \beta_1) \\ & + \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{1p}^{k-1}(s)\mathrm{d}s \right)(\lambda_2 + \alpha\gamma) \Bigg) \exp \lambda_1 t - (\lambda_2 + \alpha\gamma) \\ & \times \Bigg[\sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{2p}^{k-1}(s)\mathrm{d}s \right)(\alpha\gamma + \beta_1) + \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{1p}^{k-1}(s)\mathrm{d}s \right)(\lambda_1 + \alpha\gamma) \Bigg] \exp \lambda_2 t \Bigg\} \\ & + \frac{1}{(\lambda_1 - \lambda_2)(\alpha\gamma + \beta_1)} \int_0^t G_{2k}(\tau_1) \Bigg\{ \Bigg[\sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{2p}^{k-1}(s)\mathrm{d}s \right)(\alpha\gamma + \beta_1) \\ & + \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{1p}^{k-1}(s)\mathrm{d}s \right)(\lambda_2 + \alpha\gamma) \Bigg] \exp \lambda_1 (t - \tau_1) - \Bigg[\sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{2p}^{k-1}(s)\mathrm{d}s \right)(\alpha\gamma + \beta_1) \\ & + \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{1p}^{k-1}(s)\mathrm{d}s \right)(\lambda_1 + \alpha\gamma) \Bigg] \exp \lambda_2 (t - \tau_1) \Bigg\}^{-1} \mathrm{d}\tau_1, \quad k = 1, 2, \ldots, \end{aligned} \tag{40}$$

$$v_{ik}(\eta) = -\sum_{l=0}^{\infty} \int_l^{l+1} G_{ip}^k(s)\mathrm{d}s + \sum_{l=0}^{p-1} \int_l^{l+1} G_{il}^k(s)\mathrm{d}s + \int_p^{\eta} G_{ip}^k(s)\mathrm{d}s, \quad p \le \eta \le p+1, \quad i = 1, 2. \tag{41}$$

Using above iteration method, we can obtain $Y_{ik}(t)$, $v_{ik-1)}(\eta)$ $(i = 1, 2,\ k = 1, 2, \ldots)$ successively.

6. Asymptotic solution of the ENSO sea–air oscillator model

Substituting the determined functions into Eqs. (20)–(22), from Eqs. (7), (27), (28) and (41), we obtain uniformly valid asymptotic expansion solution for the problem given in Eqs. (1)–(3),[25]

$$T_1(t, \tau) = \sum_{k=0}^{\infty} \Bigg(\frac{1}{\lambda_1 - \lambda_2} \Bigg[\sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{2p}^{k-1}(s)\mathrm{d}s \right)(\alpha\gamma + \beta_1) + \sum_{l=0}^{\infty} \left(\int_l^{l+1} G_{1p}^{k-1}(s)\mathrm{d}s \right)(\lambda_2 + \alpha\gamma) \Bigg] \exp \lambda_1 t$$

$$
\begin{aligned}
&+\Big[\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{2p}^{k-1}(s)\,\mathrm{d}s\Big)(\alpha\gamma+\beta_1)-\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{1p}^{k-1}(s)\,\mathrm{d}s\Big)(\lambda_1+\alpha\gamma)\Big]\exp\lambda_2 t\\
&+\frac{1}{\lambda_1-\lambda_2}\int_0^t G_{1k}(\tau_1)\Big\{\Big[\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{2p}^{k-1}(s)\,\mathrm{d}s\Big)(\alpha\gamma+\beta_1)\\
&+\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{1p}^{k-1}(s)\,\mathrm{d}s\Big)(\lambda_2+\alpha\gamma)\Big]\exp\lambda_1(t-\tau_1)\\
&-\Big[\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{2p}^{k-1}(s)\,\mathrm{d}s\Big)(\alpha\gamma+\beta_1)-\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{1p}^{k-1}(s)\,\mathrm{d}s\Big)(\lambda_1+\alpha\gamma)\Big]\exp\lambda_2(t-\tau_1)\Big\}^{-1}\mathrm{d}\tau_1\Big)\tau^k\\
&-\sum_{k=0}^{\infty}\Big[\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{1p}^{k}(s)\,\mathrm{d}s\Big)+\sum_{l=0}^{p-1}\int_l^{l+1} G_{1l}^{k}(s)\,\mathrm{d}s+\int_p^{\eta} G_{1p}^{k}(s)\,\mathrm{d}s\Big]\tau^{k+1}+O(\tau^{N+1}),\\
&p\le\eta\le p+1,\quad 0\le t\le T_0,\quad 0\le\tau\ll 1,
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
T_2(t,\tau)=&\sum_{k=0}^{\infty}\Bigg(\frac{1}{(\lambda_1-\lambda_2)(\alpha\gamma+\beta_1)}\Big\{(\lambda_1+\alpha\gamma)\Big[\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{2p}^{k-1}(s)\,\mathrm{d}s\Big)(\alpha\gamma+\beta_1)\\
&+\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{1p}^{k-1}(s)\,\mathrm{d}s\Big)(\lambda_2+\alpha\gamma)\Big]\exp\lambda_1 t\\
&-(\lambda_2+\alpha\gamma)\Big[\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{2p}^{k-1}(s)\,\mathrm{d}s\Big)(\alpha\gamma+\beta_1)+\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{1p}^{k-1}(s)\,\mathrm{d}s\Big)(\lambda_1+\alpha\gamma)\Big]\exp\lambda_2 t\\
&+\frac{1}{(\lambda_1-\lambda_2)(\alpha\gamma+\beta_1)}\int_0^t G_{2k}(\tau_1)\Big\{\Big[\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{2p}^{k-1}(s)\,\mathrm{d}s\Big)(\alpha\gamma+\beta_1)\\
&+\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{1p}^{k-1}(s)\,\mathrm{d}s\Big)(\lambda_2+\alpha\gamma)\Big]\exp\lambda_1(t-\tau_1)-\Big[\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{2p}^{k-1}(s)\,\mathrm{d}s\Big)(\alpha\gamma+\beta_1)\\
&+\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{1p}^{k-1}(s)\,\mathrm{d}s\Big)(\lambda_1+\alpha\gamma)\Big]\exp\lambda_2(t-\tau_1)\Big\}^{-1}\mathrm{d}\tau_1\Big)\tau^k\\
&-\sum_{k=0}^{\infty}\Big[\sum_{l=0}^{\infty}\Big(\int_l^{l+1} G_{2p}^{k}(s)\,\mathrm{d}s\Big)+\sum_{l=0}^{p-1}\int_l^{l+1} G_{2l}^{k}(s)\,\mathrm{d}s+\int_p^{\eta} G_{2p}^{k}(s)\,\mathrm{d}s\Big]\tau^{k+1}+O(\tau^{N+1}),\\
&p\le\eta\le p+1,\quad 0\le t\le T_0,\quad 0\le\tau\ll 1.
\end{aligned}
\tag{43}
$$

The T_0 is a large enough constant.

7. Conclusion

The ENSO is a complicated natural phenomenon. Hence we need to reduce basic models for the ENSO sea–air oscillator and solve them by using the approximate methods. The perturbation method is a valid method. The perturbation method is an asymptotic method, which differs from the general numerical method. The expansions of disturbed solution through the perturbation method can be kept in the analytic operation. From Eqs. (42) and (43), we can further study the fixed quality and quantitative behaviour of the temperature anomaly in the equatorial eastern Pacific. We obtained approximate solution $(T_1(t,\tau),\ T_2(t,\tau))$ by using the perturbation method. We can obtain the prognoses of SST and related physical quantities.

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