Lie–Mei symmetry and conserved quantities of the Rosenberg problem

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The Rosenberg problem is a typical but not too complicated problem of nonholonomic mechanical systems. The Lie–Mei symmetry and the conserved quantities of the Rosenberg problem are studied. For the Rosenberg problem, the Lie and the Mei symmetries for the equation are obtained, the conserved quantities are deduced from them and then the definition and the criterion for the Lie–Mei symmetry of the Rosenberg problem are derived. Finally, the Hojman conserved quantity and the Mei conserved quantity are deduced from the Lie–Mei symmetry.

Keywords: nonholonomic systems, Rosenberg problem, Lie–Mei symmetry, conserved quantity

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1. Introduction

The research on symmetries and conserved quantities of a mechanical system is very important in mathematics, mechanics and physics. Over the past decades, the investigation on symmetry and conserved quantity has achieved rapid progress in the field of analytical mechanics and a series of research results have been obtained. Motion equations appearing in more than ten forms in analytical mechanics were divided into three systems by Mei.[1] Meanwhile, other investigators have also done a lot of work and have obtained fruitful results.[2−6] In 1918, Noether[7] revealed a potential relationship between symmetry and conserved quantity and research on the symmetry and the conserved quantity developed rapidly in the field of analytical mechanics and fruitful results were obtained.[8−20] However, there remains much room to study the Rosenberg's symmetry and the conserved quantity.

In 1966, Rosenberg gave an example of a nonholonomic mechanical system in analytical dynamics of discrete systems[21−22] to explain the fact that the embedded constraint Lagrange equation (Lindelöf equation) may not be applicable to a nonholonomic system, which is later called the Rosenberg problem. In this paper, we study the Lie–Mei symmetry and the conserved quantity of the Rosenberg problem. The definition and the criterion for the Lie–Mei symmetry of the Rosenberg problem are derived. Lie–Mei symmetry, Hojman conserved quantity and Mei conserved quantity are obtained in a broader sense.

2. Lie–Mei symmetry of Rosenberg problem

The Lagrange function of the Rosenberg problem and the nonholonomic constraint are written respectively as

$$L = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right), \tag{1}$$

$$f = \dot{q}_3 - q_2\dot{q}_1 = 0. \tag{2}$$

The system motion satisfies the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda\frac{\partial f}{\partial \dot{q}_s}, \quad (s = 1, \cdots, n), \tag{3}$$

where m is the mass of the particle and the non-potential generalized force is zero. Before integrating the equation, we can calculate the function of $\lambda = \lambda\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right)$ as

$$\Lambda_s = \lambda\frac{\partial f}{\partial \dot{q}_s} = \Lambda_s\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right), \tag{4}$$

which is the non-holonomic constraint force. Form Ref. [23], we have

$$\lambda = \frac{m\dot{q}_1\dot{q}_2}{1+q_2^2}, \quad \ddot{q}_1 = -\frac{q_2\dot{q}_1\dot{q}_2}{1+q_2^2} = \alpha_1,$$

$$\ddot{q}_2 = 0 = \alpha_2, \quad \ddot{q}_3 = -\frac{\dot{q}_1\dot{q}_2}{1+q_2^2} = \alpha_3. \tag{5}$$

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We first introduce the infinitesimal transformation

$$t^* = t + \varepsilon\xi_0\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right),$$
$$q_s^*\left(t^*\right) = q_s\left(t\right) + \varepsilon\xi_s\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right), \tag{6}$$

where ε is the infinitesimal parameter, ξ_0 and ξ_s are the infinitesimal transformations. For the Rosenberg problem, the determining equation of the Lie symmetry is

$$\begin{aligned}
&\left(\ddot{\xi}_1 - 2\ddot{q}_1\dot{\xi}_0 - \dot{q}_1\ddot{\xi}_0\right)m + \left(\ddot{\xi}_2 - 2\ddot{q}_2\dot{\xi}_0 - \dot{q}_2\ddot{\xi}_0\right)m \\
&+ \left(\ddot{\xi}_3 - 2\ddot{q}_3\dot{\xi}_0 - \dot{q}_3\ddot{\xi}_0\right)m \\
&= m\dot{q}_1\dot{q}_2\xi_2\frac{q_2^2-1}{\left(1+q_2^2\right)^2} - \left(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0\right)\left(\frac{mq_2\dot{q}_2}{1+q_2^2}\right) \\
&- \left(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0\right)\left(\frac{mq_2\dot{q}_1}{1+q_2^2}\right) - m\dot{q}_1\dot{q}_2\xi_2\frac{2q_2}{\left(1+q_2^2\right)^2} \\
&+ \left(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0\right)\left(\frac{m\dot{q}_2}{1+q_2^2}\right) \\
&+ \left(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0\right)\left(\frac{m\dot{q}_1}{1+q_2^2}\right).
\end{aligned} \tag{7}$$

The criterion equation of the Mei symmetry is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\xi}_1 - 2\dot{q}_1 m\dot{\xi}_0\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\xi}_2 - 2\dot{q}_2 m\dot{\xi}_0\right) \\
&+ \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\xi}_3 - 2\dot{q}_3 m\dot{\xi}_0\right) \\
&= m\dot{q}_1\dot{q}_2\xi_2\frac{q_2^2-1}{\left(1+q_2^2\right)^2} - \left(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0\right)\left(\frac{mq_2\dot{q}_2}{1+q_2^2}\right) \\
&- \left(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0\right)\left(\frac{mq_2\dot{q}_1}{1+q_2^2}\right) - m\dot{q}_1\dot{q}_2\xi_2\frac{2q_2}{\left(1+q_2^2\right)^2} \\
&+ \left(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0\right)\left(\frac{m\dot{q}_2}{1+q_2^2}\right) \\
&+ \left(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0\right)\left(\frac{m\dot{q}_1}{1+q_2^2}\right).
\end{aligned} \tag{8}$$

Definition If the symmetry of the Rosenberg problem is a Lie one as well as a Mei one, then the symmetry is called Lie–Mei symmetry of the Rosenberg problem.

Criterion For the Rosenberg problem, if there exist such infinitesimal generators ξ_0 and ξ_s that

$$\begin{aligned}
&\left\{\mathrm{X}^{(2)}\left[\mathrm{E_s}\left(L\right)\right] - \mathrm{X}^{(1)}\left(\Lambda_s\right)\right\}^2 \\
&+ \left\{E_s[\mathrm{X}^{(1)}(L)] - \mathrm{X}^{(1)}(\Lambda_s)\right\}^2 = 0,
\end{aligned} \tag{9}$$

where

$$\mathrm{E}_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s} - \frac{\partial}{\partial q_s},$$
$$\mathrm{X}^{(1)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right)\frac{\partial}{\partial\dot{q}_s},$$
$$\mathrm{X}^{(2)} = \mathrm{X}^{(1)} + \left(\ddot{\xi}_s - 2\ddot{q}_s\dot{\xi}_0 - \dot{q}_s\ddot{\xi}_0\right)\frac{\partial}{\partial\ddot{q}_s},$$

then the symmetry of the Rosenberg problem is the Lie–Mei symmetry.

3. Conserved quantities deduced from Lie–Mei symmetry

The Lie–Mei symmetry of the Rosenberg problem can lead to the Hojman conserved quantity and the Mei conserved quantity under certain conditions.

Proposition 1 For the Rosenberg problem, if there exists a function $\mu = \mu\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right)$ such that

$$\frac{\partial\alpha_s}{\partial\dot{q}_s} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\mu = 0, \tag{10}$$

then the Lie–Mei symmetry will lead to a Hojman conserved quantity

$$\begin{aligned}
I_{\mathrm{H}} =\ & \frac{1}{\mu}\frac{\partial\left(\mu\xi_0\right)}{\partial t} \\
&+ \frac{1}{\mu}\frac{\partial\left(\mu\xi_s\right)}{\partial q_s} + \frac{1}{\mu}\frac{\partial}{\partial\dot{q}_s}\left[\mu\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right)\right] - \dot{\xi}_0 \\
=\ & \text{const.}
\end{aligned} \tag{11}$$

Proposition 2 For the Rosenberg problem, if there is a function $G_{\mathrm{M}} = G_{\mathrm{M}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\begin{aligned}
&\mathrm{X}^{(1)}(L)\dot{\xi}_0 + \mathrm{X}^{(1)}\left[\mathrm{X}^{(1)}(L)\right] \\
&+\mathrm{X}^{(1)}(\Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \dot{G}_{\mathrm{M}} = 0,
\end{aligned} \tag{12}$$

then the Lie–Mei symmetry will lead to the Mei conserved quantity

$$\begin{aligned}
I_{\mathrm{M}} =\ & \mathrm{X}^{(1)}(L)\xi_0 + \frac{\partial\mathrm{X}^{(1)}(L)}{\partial\dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{M}} \\
=\ & \text{const.}
\end{aligned} \tag{13}$$

For the Rosenberg problem, the generators are as follows:

$$\xi_0 = 1, \quad \xi_1 = 0, \quad \xi_2 = 0, \quad \xi_3 = 0. \tag{14}$$

If Eq. (9) is satisfied, then the symmetry is the Lie–Mei symmetry of the Rosenberg problem. Substituting Eq. (14) into Eq. (10), we obtain the following functions:

$$\mu = \left(1+q_2^2\right)^{1/2}, \quad \mu = \left(1+q_2^2\right)^{1/2}\left(q_2 - \dot{q}_2 t\right). \tag{15}$$

Form proposition 1, we have

$$I_{\mathrm{H}} = \frac{q_2 - \dot{q}_2}{q_2 - \dot{q}_2 t} = \text{const}, \tag{16}$$

which is a Hojman conserved quantity deduced from the Lie–Mei symmetry. Substituting Eq. (14) into Eq. (12), we obtain the gauge function as

$$G_{\rm M} = 0. \tag{17}$$

Form proposition 2, we have

$$I_{\rm M} = 0, \tag{18}$$

which is the Mei conserved quantity deduced from the Lie–Mei symmetry.

The generators are as follows:

$$\xi_0 = -1, \quad \xi_1 = -1, \quad \xi_2 = 0, \quad \xi_3 = t. \tag{19}$$

If Eq. (9) is satisfied, then the symmetry is the Lie–Mei symmetry of the Rosenberg problem. Substituting Eq. (19) into Eq. (10), we obtain the functions as follows:

$$\mu = \left(1 + q_2^2\right)^{1/2}, \quad \mu = \left(1 + q_2^2\right)^{1/2} \left(q_2 - \dot{q}_2 t\right). \tag{20}$$

Form proposition 1, we have

$$I_{\rm H} = \frac{\dot{q}_2 - q_2}{q_2 - \dot{q}_2 t} = \text{const.} \tag{21}$$

which is a Hojman conserved quantity deduced from the Lie–Mei symmetry. Substituting Eq. (19) into Eq. (12), we obtain the gauge function as

$$G_{\rm M} = -mt. \tag{22}$$

Form Proposition 2, we have

$$I_{\rm M} = -m\dot{q}_3, \tag{23}$$

which is the Mei conserved quantity deduced from the Lie–Mei symmetry.

4. Conclusion

In order to find more conserved quantities, the investigation on joint symmetry of the mechanical system is conducted. The Lie–Mei symmetry in the Rosenberg problem is studied and the relevant definitions and the criteria are also obtained. Under certain conditions, we are able to find Hojman conserved quantity and Mei conserved quantity. Besides, the results obtained in this paper are of general significance, which can probably be extended to restraint systems of another type.

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