The approximate conserved quantity of the weakly nonholonomic mechanical-electrical system

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We study the approximate conserved quantity of the weakly nonholonomic mechanical-electrical system. By means of the Lagrange–Maxwell equation, the Noether equality of the weakly nonholonomic mechanical-electrical system is obtained. The multiple powers-series expansion of the parameter of the generators of infinitesimal transformations and the gauge function is put into a generalized Noether identity. Using the Noether theorem, we obtain an approximate conserved quantity. An example is provided to prove the existence of the approximate conserved quantity.

Keywords: weakly nonholonomic mechanical-electrical system, Noether theorem, approximate conserved quantity

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1. Introduction

It is significant to study the symmetries and conserved quantities of dynamic systems not only in mathematics but also in physics. There are mainly three kinds of symmetries: the Noether symmetry,[1−3] the Lie symmetry[4−8] and the Mei symmetry.[9−13] Respectively, there are three main kinds of conserved quantities (i.e., the Noether conserved quantities, the Hojman conserved quantities and a new type of conserved quantities) according to these main kinds of symmetries.[14,15]

In 1873, Maxwell studied the problems of the mechanico-electrical system by using the Lagrange's method and obtained the Lagrange–Maxwell equation, which is very famous to date. Recently, Qiu developed the theory about analytical mechanics of mechanico-electrical system by applying the method and theory of analytical mechanics to the multiplied mechanico-electrical system.[16] Mei *et al.* studied the mechanico-electrical system in Ref. [17]. Luo *et al.*[18] summarized a mechanico-electrical system. For the sake of obtaining the essential characteristics of the mechanico-electrical system, researchers began to study the symmetries and conserved quantities of this system and have reached some important results.[19−21] The theory of symmetries and conserved quantities of the mechanico-electrical system shows much more significance. Recently, Ge[22] gave the approximate conserved quantity of a weakly nonholonomic system. In this paper, the small parameter is presented in the nonholonomic bond equation of the system which is called the weakly nonholonomic system. On the basis of Ref. [22], we study the approximate conserved quantity of the weakly nonholonomic mechanico-electrical system.

2. The Lagrange–Maxwell equation of the system

We consider a mechanico-electrical system consisting of m loops, each loop is composed of linear conductors and circuits which are independent of each other, but the electromagnetic processes of the loops are not mutually independent. The current in the k-th loop is i_k $(k = 1, \ldots, m)$ and u_k is the electromotive force in the kth loop. Suppose e_k is the electric charge in the capacitor, the relation between the charge and the current is $\dot{e}_k = i_k$. The resistance and the capacitance of the kth loop are R_k and C_k, respectively, and its motion is subjected to g ideal, linear as well as homogeneous nonholonomic constraints[23]

$$f_\beta = \dot{q}_{\varepsilon+\beta} - \mu B_{\varepsilon+\beta}(t, \boldsymbol{q})\,\dot{q}_\sigma = 0$$
$$(\beta = 1, \ldots, g; \quad \sigma = 1, \ldots, \varepsilon; \quad \varepsilon = n - g), \qquad (1)$$

where μ is a small parameter, $B_{\varepsilon+\beta}(t, \boldsymbol{q})$ is the function of coordinates q_s $(s = 1, 2, \ldots, n)$ and time t. The

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f_β is restricted by g relations, only $\varepsilon = n - g$ is independent, $\sigma = 1, \ldots, \varepsilon$ is independent.

Its configuration is determined by the $n+m$ generalized coordinates e_k $(k = 1, \ldots, m)$, q_s $(s = 1, \ldots, n)$. The Lagrange function of the mechanico-electrical system is

$$L = T(q_s, \dot{q}_s) - V(q_s) + W_{\rm m}(q_s, \dot{e}_k) - W_{\rm e}(q_s, e_k). \quad (2)$$

Here, T is the kinetic energy and V is the potential energy of the system,

$$T = \frac{1}{2}\sum_{s=1}^{n}\sum_{l=1}^{n} a_{sl}\dot{q}_s\dot{q}_l, \quad V = V(q_s), \quad (3)$$

where $a_{sl} = a_{sl}(q_s)$ are coefficients, the magnetic field energy $W_{\rm m}$, the electric field energy $W_{\rm e}$ and the electrical dissipation function $\psi_{\rm e}$ are expressed as

$$\begin{aligned} W_{\rm m} &= \frac{1}{2}\sum_{k=1}^{m}\sum_{r=1}^{m} L_{kr} i_k i_r, \\ W_{\rm e} &= \frac{1}{2}\sum_{k=1}^{m}\frac{e_k^2}{C_k}, \\ \psi_{\rm e} &= \frac{1}{2}\sum_{k=1}^{m} R_k i_k^2, \end{aligned} \quad (4)$$

where L_{kr} (with $k \neq r$) is the inductance on circuit k due to circuits r and $L_{kr} = L_{rk}(q_s)$ with L_{kk} being the self-inductance of circuit k.

Introduce the summation of the electrical and mechanical dissipation function

$$\psi = \psi_{\rm e}(i_k) + \psi_{\rm m}(q_s, \dot{q}_s).$$

Then the Lagrange–Maxwell equation of the system is

$$\begin{aligned} &\frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial \dot{e}_k} - \frac{\partial L}{\partial e_k} + \frac{\partial \psi}{\partial \dot{e}_k} = u_k, \quad (k = 1, \ldots, m), \\ &\frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial \psi}{\partial \dot{q}_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \\ &\quad (s = 1, \ldots, n). \end{aligned} \quad (5)$$

Equation (5) consists of $n + m$ second order differential equations with q_s and e_k as the generalized coordinates.

Let the generalized coordinates q_s $(s = 1, \ldots, n, n+1, \ldots, n+m)$ expressing both configuration coordinates and generalized electrical charges, where q_s $(s = 1, \ldots, n)$ are configuration coordinates and q_s $(s = n+1, \ldots, n+m)$ are generalized electrical charges. Then Eq. (5) becomes

$$\frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial \psi}{\partial \dot{q}_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}$$
$$(s = 1, \ldots, n, n+1, \ldots, n+m), \quad (6)$$

where Q_s $(s = 1, \ldots, n)$ are the non-potential generalized force, Q_s $(s = n+1, \ldots, n+m)$ are the electric potential energy.

In general, it is supposed that before integrating Eq. (4), we can express λ_β as a function of t, $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ and μ from Eqs. (1) and (4), while Eq. (5) can be expressed as

$$\begin{aligned} &\frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial \psi}{\partial \dot{q}_s} = Q_s + \Lambda_s \\ &\quad (s = 1, \ldots, n, n+1, \ldots, n+m), \end{aligned} \quad (7)$$

where $\Lambda_s = \lambda_\beta(\partial f_\beta/\partial \dot{q}_s) = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \mu)$ represents the nonholonomic constraint counterforce. Equation (7) is the equation of nonholonomic system in Eqs. (1) and (5). Note that μ only appears in reactive force.

3. Noether symmetry and approximate conserved quantity of the system

For the weakly nonholonomic mechanical-electrical system, the Noether identity can be written as

$$\begin{aligned} &L\frac{\rm d}{{\rm d}t}\xi_0 + X^{(1)}(L) + \left(Q_s + \Lambda_s - \frac{\partial \psi}{\partial \dot{q}_s}\right)(\xi_s - \dot{q}_s\xi_0) \\ &\quad + \frac{\rm d}{{\rm d}t}G_{\rm N} = 0, \end{aligned} \quad (8)$$

where

$$X^{(1)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \left(\frac{\rm d}{{\rm d}t}\xi_s - \dot{q}_s\frac{\rm d}{{\rm d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s},$$

where $G_{\rm N} = G_{\rm N}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are gauge function.

The restrictions of nonholonomic constraints (1) to the generator can be expressed as

$$\begin{aligned} &X^{(1)}(f_\beta) \\ &= \left(\dot{\xi}_{\varepsilon+\beta} - \dot{q}_{\varepsilon+\beta}\dot{\xi}_0\right) - \mu B_{\varepsilon+\beta,\sigma}\left(\dot{\xi}_\sigma - \dot{q}_\sigma\dot{\xi}_0\right) \\ &= \dot{\xi}_{\varepsilon+\beta} - \mu B_{\varepsilon+\beta,\sigma}\dot{\xi}_\sigma - \dot{q}_{\varepsilon+\beta}\dot{\xi}_0 + \mu B_{\varepsilon+\beta,\sigma}\dot{q}_\sigma\dot{\xi}_0 \\ &= \dot{\xi}_{\varepsilon+\beta} - \mu B_{\varepsilon+\beta,\sigma}\dot{\xi}_\sigma - (\dot{q}_{\varepsilon+\beta} - \mu B_{\varepsilon+\beta,\sigma}\dot{q}_\sigma)\dot{\xi}_0 \\ &= 0, \end{aligned}$$

where

$$\xi_{\varepsilon+\beta} - \mu B_{\varepsilon+\beta,\sigma}\xi_\sigma = 0. \quad (9)$$

If there exists a gauge function $G_{\rm N}$ such that the infinitesimal generators ξ_0, ξ_s satisfy Eqs. (8) and (9), then the symmetry of the mechanical-electrical system is Noether symmetry.

For the weakly nonholonomic mechanical-electrical system, the symmetry will lead to the Noether conserved quantity

$$I_{\rm N} = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}\left(\xi_s - \dot{q}_s\xi_0\right) + G_{\rm N} = \text{const.} \tag{10}$$

According to Ref. [24], in order to probe into approximate conserved quantity, ξ_0, ξ_s and $G_{\rm N}$ can be spread out in terms of μ power series expansion as

$$\begin{aligned} \xi_0 &= (\xi_0)_0 + \mu\,(\xi_0)_1 + \cdots, \\ \xi_s &= (\xi_s)_0 + \mu\,(\xi_s)_1 + \cdots, \\ G_{\rm N} &= (G_{\rm N})_0 + \mu\,(G_{\rm N})_1 + \cdots. \end{aligned} \tag{11}$$

Substituting relations (11) into Eq. (8), we obtain

$$\begin{aligned} & L\left[\left(\dot{\xi}_0\right)_0 + \mu\left(\dot{\xi}_0\right)_1 + \cdots\right] \\ & + \frac{\partial L}{\partial t}\left[(\xi_0)_0 + \mu\,(\xi_0)_1 + \cdots\right] \\ & + \frac{\partial L}{\partial q_s}\left[(\xi_s)_0 + \mu\,(\xi_s)_1 + \cdots\right] \\ & + \frac{\partial L}{\partial \dot{q}_s}\Big\{\left[\left(\dot{\xi}_s\right)_0 + \mu\left(\dot{\xi}_s\right)_1 + \cdots\right] \\ & - \dot{q}_s\left[\left(\dot{\xi}_0\right)_0 + \mu\left(\dot{\xi}_0\right)_1 + \cdots\right]\Big\} \\ & + \left(Q_s + (\Lambda_s)_0 + \mu\,(\Lambda_s)_1 + \cdots - \frac{\partial \psi}{\partial \dot{q}_s}\right) \\ & \times \left[(\xi_s)_0 - \dot{q}_s\,(\xi_0)_0 + \mu\,(\xi_s)_1 - \mu\dot{q}_s\,(\xi_0)_1 + \cdots\right] \\ & + \left(\dot{G}_{\rm N}\right)_0 + \mu\left(\dot{G}_{\rm N}\right)_1 + \cdots = 0. \end{aligned} \tag{12}$$

Substituting relations (11) into Eq. (9), we obtain

$$\begin{aligned} & (\xi_{\varepsilon+\beta})_0 + \mu\,(\xi_{\varepsilon+\beta})_1 + \cdots - \mu B_{\varepsilon+\beta,\sigma} \\ & \quad \times \left[(\xi_\sigma)_0 + \mu\,(\xi_\sigma)_1 + \cdots\right] = 0. \end{aligned} \tag{13}$$

If the corresponding generators $(\xi_0)_0$, $(\xi_s)_0$, $(\xi_0)_1$, $(\xi_s)_1$, ...and gauge functions $(G_{\rm N})_0$, $(G_{\rm N})_1$ can be found in Eqs. (12) and (13), the approximate conserved quantity of the weakly nonholonomic mechanical-electrical system can be obtained by means of Eq. (10).

4. An example

Suppose that the mechanico-electrical system has a capacitor, the voltage of the electrical source is U. The active plank is described by generalized coordinate q and the electro-dynamical part is described by generalized electric quantity e. We denote the kinetic energy of the system as $T = (1/2)\left(\dot{q}_1^2 + \dot{q}_2^2\right)$, by the magnetic energy of capacitor $W_{\rm m} = (1/2)\dot{e}^2$, by the potential of spring $V = q_2$ and by the electric energy of the capacitor $W_{\rm e} = (1/2)Ae^2$. Then the Lagrange function of the system is $L = (1/2)\left(\dot{q}_1^2 + \dot{q}_2^2\right) + (1/2)\dot{e}^2 - q_2 - (1/2)Ae^2$, where A is a constant. The electrical dissipation function is $\psi = (1/2)R\dot{e}^2$ and the capacitor voltage drop is $u = R\dot{e}$, where R is the resistance. The nonnholonomic constraints and the generalized force are

$$f = \dot{q}_2 - \mu t\dot{q}_1 = 0, \quad Q_1 = 0, \quad Q_2 = 0. \tag{14}$$

The differential equations of motion of the system are

$$\ddot{q}_1 = -\lambda\mu t, \quad \ddot{q}_2 + 1 = \lambda, \quad \ddot{e} = U - Ae. \tag{15}$$

We have

$$\lambda = \frac{1 + \mu\dot{q}_1}{1 + \mu^2 t^2}, \tag{16}$$

where

$$\begin{aligned} \Lambda_1 &= -\lambda\mu t = -\mu t - \mu^2 t\dot{q}_1 + \cdots, \\ \Lambda_2 &= \lambda = 1 + \mu\dot{q}_1 - \mu^2 t^2 + \cdots. \end{aligned} \tag{17}$$

Using Eq. (12), we can also obtain

$$\begin{aligned} & L\left[\left(\dot{\xi}_0\right)_0 + \mu\left(\dot{\xi}_0\right)_1 + \cdots\right] - \left[(\xi_2)_0 + \mu\,(\xi_2)_1 + \cdots\right] + \dot{q}_1\left[\left(\dot{\xi}_1\right)_0 - \dot{q}_1\left(\dot{\xi}_0\right)_0 + \mu\left(\dot{\xi}_1\right)_1 - \mu\dot{q}_1\left(\dot{\xi}_0\right)_1 + \cdots\right] \\ & + \dot{q}_2\left[\left(\dot{\xi}_2\right)_0 - \dot{q}_2\left(\dot{\xi}_0\right)_0 + \mu\left(\dot{\xi}_2\right)_1 - \mu\dot{q}_2\left(\dot{\xi}_0\right)_1 + \cdots\right] + \left(-\mu t - \mu^2 t\dot{q}_1 - \cdots\right) \\ & \times \left[(\xi_1)_0 - \dot{q}_1\,(\xi_0)_0 + \mu\,(\xi_1)_1 - \mu\dot{q}_1\,(\xi_0)_1 + \cdots\right] + \left[1 + \mu\dot{q}_1 + \cdots\right] \\ & \times \left[(\xi_2)_0 - \dot{q}_2\,(\xi_0)_0 + \mu\,(\xi_2)_1 - \mu\dot{q}_2\,(\xi_0)_1 + \cdots\right] - U \times \left[(\xi_1)_0 - \dot{q}_1\,(\xi_0)_0 + \mu\,(\xi_1)_1 - \mu\dot{q}_1\,(\xi_0)_1 + \cdots\right] \\ & - U \times \left[(\xi_2)_0 - \dot{q}_2\,(\xi_0)_0 + \mu\,(\xi_2)_1 - \mu\dot{q}_2\,(\xi_0)_1 + \cdots\right] + \left(\dot{G}_{\rm N}\right)_0 + \mu\left(\dot{G}_{\rm N}\right)_1 + \cdots = 0. \end{aligned} \tag{18}$$

From Eq. (13), we have

$$(\xi_2)_0 + \mu\,(\xi_2)_1 + \cdots - \mu t\left[(\xi_1)_0 + \mu\,(\xi_1)_1 + \cdots\right] = 0. \tag{19}$$

Using Eqs. (18) and (19), we can obtain

$$\begin{aligned}&(\xi_0)_0 = 0,\ (\xi_1)_0 = 1,\ (\xi_2)_0 = 0,\\&(G_{\rm N})_0 = Ut,\ (\xi_0)_1 = 0,\ (\xi_1)_1 = 0,\\&(\xi_2)_1 = t,\ (G_{\rm N})_1 = \frac{1}{2}t^2\,(1+U) - q_2,\\&\eta_1 = 0,\ \eta_2 = 0.\end{aligned} \tag{20}$$

Substituting relations (20) into Eq. (10), we obtain

$$I_{\rm N} = \dot{q}_1 + Ut + \mu\left[\dot{q}_2 t + \frac{1}{2}t^2\,(1+U) - q_2\right] + \cdots . \tag{21}$$

Equation (21) in virtue of the Eq. (15) produces the time derivative

$$\frac{{\rm d}I_{\rm N}}{{\rm d}t} = \mu^2\,\{\cdots\} + \cdots . \tag{22}$$

Therefore, Eq. (22) cannot be regarded as a conserved quantity in the strict sense, which is just an approximate conserved quantity.

References

[1] Noether E 1918 *Nacher. Akad. Wiss. Gottingen Math. Phys.* **KI II** 235

[2] Lucky M 1979 *J. Phys.* A: *Math. Gen.* **19** 105

[3] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120

[4] Mei F X 2001 *Chin. Phys.* **10** 177

[5] Li Z P 1993 *Classical and Dynamics of Constrained Systems and Their Symmetrical Properties* (Beijing: Beijing Polytechnic University press) (in Chinese)

[6] Mei F X 1999 *Application of Lie Group and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)

[7] Bihar L Y and Kearny H G 1987 *Int. J. Non-Linear Mech.* **22** 125

[8] Mei F X 2000 *Acta Mech. Sin.* **141** 135

[9] Mei F X 2003 *Acta Phys. Sin.* **52** 1048 (in Chinese)

[10] Zhang Y 2003 *Acta Phys. Sin.* **52** 1832 (in Chinese)

[11] Wang S Y and Mei F X 2001 *Chin. Phys.* **10** 373

[12] Lou Z M 2004 *Acta Phys. Sin.* **53** 2046 (in Chinese)

[13] Lou S K, Guo Y X and Mei F X 2004 *Acta Phys. Sin.* **53** 2418 (in Chinese)

[14] Hojman S A 1992 *J. Phys.* A: *Math. Gen.* **25** L291

[15] Xu X J, Mei F X and Qin M C 2004 *Chin. Phys.* **13** 1999

[16] Qiu J J 1992 *Analysis Mechanics of Mechanico-Electrical system* (Beijing: Science Press) p. 308 (in Chinese)

[17] Mei F X, Liu D and Luo Y 1991 *Advanced Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) p. 349 (in Chinese)

[18] Luo S K, Zhang Y F, *et al.* 2008 *Advances in the Study of Dynamics of Constrained Mechanics Systems* (Beijing: Science Press) (in Chinese)

[19] Zheng S W, Fu J L and Li X H 2005 *Acta Phys. Sin.* **54** 5511 (in Chinese)

[20] Li Y C, Xia L L, Liu B, Jiao Z Y and Wang X M 2008 *Chin. Phys.* B **17** 1545

[21] Fu J L, Wang X J and Xie F P 2008 *Chin. Phys. Lett.* **25** 2413

[22] Ge W K 2009 *Acta Phys. Sin.* **58** 6729 (in Chinese)

[23] Mei F X 1985 *Mechanics of Nonholonomic Systems* (Beijing: Beijing Institute of Technology Press) p. 196 (in Chinese)

[24] Santilli R M 1983 *Foundations of Theoretical Mechanics* II (New York: Springer-Verlag)