Conservation laws for variable coefficient nonlinear wave equations with power nonlinearities*

Huang Ding-Jiang(黄定江)[a)b)c)†], Zhou Shui-Geng(周水庚)[b)c)], and Yang Qin-Min(杨勤民)[a)]

[a)] *Department of Mathematics, East China University of Science and Technology, Shanghai* 200237, *China*

[b)] *School of Computer Science, Fudan University, Shanghai* 200433, *China*

[c)] *Shanghai Key Laboratory of Intelligent Information Processing, Fudan University, Shanghai* 200433, *China*

(Received 3 August 2010; revised manuscript received 4 February 2011)

Conservation laws for a class of variable coefficient nonlinear wave equations with power nonlinearities are investigated. The usual equivalence group and the generalized extended one including transformations which are nonlocal with respect to arbitrary elements are introduced. Then, using the most direct method, we carry out a classification of local conservation laws with characteristics of zero order for the equation under consideration up to equivalence relations generated by the generalized extended equivalence group. The equivalence with respect to this group and the correct choice of gauge coefficients of the equations play the major roles for simple and clear formulation of the final results.

Keywords: nonlinear wave equations, conservation laws, equivalence group, symmetries

PACS: 02.20.Sv, 02.30.Jr **DOI:** 10.1088/1674-1056/20/7/070202

1. Introduction

Conservation laws play important roles in the study of differential equations in mathematical physics.[1−6] They have been widely used in the exploring of integrability, linearization, constants of motion, exact and numerical methods for partial differential equations. In fact, the investigation of conservation laws for the Korteweg-de Vries equation is the beginning of the discovery of a number of techniques, such as Lax pair, inverse scattering transformation, bi-Hamiltonian structures, which can be used to solve nonlinear evolution equations.[7] The existence of a large number of conservation laws for a evolutionary partial differential equation (system) is a strong indication of its integrability. The conservation laws have also been used in the theory of non-classical transformations and in the theory of normal forms and asymptotic integrability.[8−10]

In this paper, we are interested in the classification of local conservation laws for a class of variable coefficient nonlinear wave equations of the form

$$f(x)u_{tt} = (g(x)u^n u_x)_x + h(x)u^m, \qquad (1)$$

where $f = f(x)$, $g = g(x)$ and $h = h(x)$ are arbitrary smooth functions of independent variable x, n and m are arbitrary constants, $nf(x)g(x) \neq 0$. This study continues the series of works on modern group analysis of Eq. (1) started in Ref. [11].

Equation (1) and its special cases describe a variety of nonlinear phenomena in physics and mechanics, such as the flow of one-dimensional gas, the shallow water waves, the longitudinal wave propagation on a moving threadline, the dynamics of a finite nonlinear string and the electromagnetic transmission line (see Refs. [12] and [13]). Conservation laws for some subclasses of Eq. (1) have been investigated in literature.[5] In particular, Anco and Bluman[14] constructed the local conservation laws for a class of one-dimensional wave equations of the form $u_{tt} = c(u)(c(u)u_x)_x$ by using an effective algorithmic method.[15,16] Bluman and Cheviakov[17] obtained the local conservation laws with characteristics of zero order for the nonlinear wave equation of the form $u_{tt} = (c^2(u)u_x)_x$. Recently, Cheviakov[18] reviewed some common methods regarding the flux computation for the situation that the local-conservation-law multipliers have been found and applied the local-

*Project supported by the National Key Basic Research Program of China (Grant No. 2010CB126600), the National Natural Science Foundation of China (Grant No. 60873070), the Shanghai Leading Academic Discipline Project, China (Grant No. B114), the Postdoctoral Science Foundation of China (Grant No. 20090450067), the Shanghai Postdoctoral Sustentation Foundation, China (Grant No. 09R21410600), and the Fundamental Research Funds for the Central Universities, China (Grant No. WM0911004).

†Corresponding author. E-mail: djhuang@fudan.edu.cn; djhuang@ecust.edu.cn

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conservation-law multipliers to classify the nonlinear wave equations.

Up to now, there exist many methods to find the conservation laws for given differential equations, including the direct method,[19] the Noether's theorem,[20] the characteristic method,[21] the variational approach[3,14−16] and so on[8,22−32] (see Ref. [33] for a detailed comparison of nine different approaches to conservation laws for partial differential equations in fluid mechanics). In the present work, the main tool used in our investigation of (local) conservation laws for Eq. (1) is a modified most direct method, which was proposed in Refs. [34] and [35] and has been applied to construct conservation laws for many equations.[36−45] The method is based on using the definition of conservation laws, the notion of equivalence of conservation laws with respect to transformation groups and the classification up to the equivalence group of a class of differential equations. Let us note that other kinds of direct methods,[14,16,22,46] which are based on the characteristic form of the conservation laws[14,16] could also be applied (see Ref. [8] for a comparison of the versions and their realizations in computer algebra programs).

The rest of the paper is organized as follows. In Section 2, some basic results about the usual equivalence group and about the generalized extended one including transformations that are nonlocal with respect to arbitrary elements for Eq. (1) are introduced. Then in Section 3 we carry out a classification of the local conservation laws with characteristics of zero order up to the equivalence relations generated by the equivalence group using the most direct method. Conclusion and discussion are given in Section 4.

2. Equivalence transformations and choice of investigated class

In order to classify the conservation laws for Eq. (1), first we have to investigate the equivalence transformations for this equation.

In the first part of the presented series, we have shown a detailed study of different kinds of equivalence transformations for solving the group classification problem.[11] In particular, we have investigated the equivalence groups (usual and generalized extended ones), discussed their structure and performed a complete group classification for the groups of Eq. (1). We find that the complete usual equivalence group $G^{\sim}$ for Eq. (1) consists of the following transformations,

$$\begin{aligned}&\tilde t=\epsilon_2 t+\epsilon_3,\quad \tilde x=X(x),\quad \tilde u=\epsilon_4 u,\\ &\tilde f=\epsilon_1\epsilon_2^2\epsilon_4^{-1}X_x^{-1}f,\quad \tilde g=\epsilon_1\epsilon_4^{-n-1}X_x g,\\ &\tilde h=\epsilon_1\epsilon_4^{-m}X_x^{-1}h,\quad \tilde n=n,\quad \tilde m=m,\end{aligned}$$

where ϵ_j $(j=1,\ 2,\ 3,\ 4)$ are arbitrary constants, $\epsilon_1\epsilon_2\epsilon_4\neq 0$, X is an arbitrary smooth function of x, $X_x\equiv \mathrm{d}X/\mathrm{d}x\neq 0$.

It appears that Eq. (1) admits further equivalence transformations, which do not belong to $G^{\sim}$ and depend on arbitrary elements in some non-fixed (possibly, nonlocal) way. We have constructed the complete generalized extended equivalence group $\hat G^{\sim}$ for Eq. (1) by using the direct method. The $\hat G^{\sim}$ is formed by the following transformations,

$$\begin{aligned}&\tilde t=\epsilon_2 t+\epsilon_3,\quad \tilde x=X(x),\quad \tilde u=Y(x)u,\\ &\tilde f=\epsilon_1\epsilon_2^2X_x^{-1}Y^{-n-2}f,\quad \tilde g=\epsilon_0 X_x Y^{-2n-2}g,\\ &\tilde h=\epsilon_1 X_x^{-1}Y^{-m-n-1}h,\quad \tilde n=n,\quad \tilde m=m,\end{aligned}$$

where ϵ_j $(j=0,\ 1,\ 2,\ 3)$ are arbitrary constants, $\epsilon_1\epsilon_2\neq 0$, X is an arbitrary smooth function of x, $X_x\neq 0$, the function $Y(x)$ is determined by the following formula,

$$Y(x)=\begin{cases}[1-(n+1)F(x)]^{-1/n+1}, & n\neq -1,\\ \mathrm{e}^{F(x)}, & n=-1,\end{cases}$$

where $F(x)=\epsilon_4\int(\mathrm{d}x/g(x))+\epsilon_5$, ϵ_4 and ϵ_5 are arbitrary constants.

Using the gauge equivalence transformation,

$$\tilde t=t,\quad \tilde x=\int\frac{\mathrm{d}x}{g(x)},\quad \tilde u=u, \tag{2}$$

we can reduce Eq. (1) to $\tilde f(\tilde x)\tilde u_{\tilde t\tilde t}=(\tilde u^n\tilde u_{\tilde x})_{\tilde x}+\tilde h(\tilde x)\tilde u^m$, where $\tilde f(\tilde x)=g(x)f(x)$, $\tilde g(\tilde x)=1$ and $\tilde h(\tilde x)=g(x)h(x)$. Similarly, any equation with the form of Eq. (1) can be reduced to the same form with $\tilde f(\tilde x)=1$. That is why without loss of generality we can restrict ourselves to the investigation of the following equation,

$$f(x)u_{tt}=(u^nu_x)_x+h(x)u^m. \tag{3}$$

All results on symmetries, solutions and conservation laws for Eq. (3) can be extended to Eq. (1) with the transformation given in Eq. (2).

The generalized equivalence group $G_1^{\sim}$ for Eq. (3) with $n\neq -1$ has the form

$$\begin{aligned}&\tilde t=\epsilon_1 t+\epsilon_2,\quad \tilde x=\frac{\epsilon_3 x+\epsilon_4}{\epsilon_5 x+\epsilon_6}=:X(x),\\ &\tilde u=\epsilon_7 X_x^{\frac{1}{2n+2}}u,\quad \tilde f=\epsilon_1^2\epsilon_7^n X_x^{-\frac{3n+4}{2n+2}}f,\end{aligned}$$

$$\tilde{h} = \epsilon_7^{-m+n+1} X_x^{-\frac{m+3n+3}{2n+2}} h, \quad \tilde{n} = n, \quad \tilde{m} = m,$$

where ϵ_j $(j = 1, \dots, 7)$ are arbitrary constants, $\epsilon_1\epsilon_7 \neq 0$ and $\epsilon_3\epsilon_6 - \epsilon_4\epsilon_5 = \pm 1$. For $n = -1$, group $G_1^{\sim}$ takes the form

$$\begin{aligned}
&\tilde{t} = \epsilon_1 t + \epsilon_2, \quad \tilde{x} = \epsilon_3 x + \epsilon_4,\\
&\tilde{u} = \epsilon_5 \mathrm{e}^{\epsilon_6 x} u, \quad \tilde{f} = \epsilon_1^2 \epsilon_3^{-2} \epsilon_5^{-1} \mathrm{e}^{-\epsilon_6 x} f,\\
&\tilde{h} = \epsilon_3^{-2} \epsilon_5^{-m} \mathrm{e}^{-m\epsilon_6 x} h, \quad \tilde{n} = n, \quad \tilde{m} = m,
\end{aligned}$$

where ϵ_j $(j = 1, \dots, 6)$ are arbitrary constants, $\epsilon_1\epsilon_3\epsilon_5 \neq 0$.

Using the above results, we have investigated the group classification for groups of Eq. (1) in Ref. [11]. Below we use them to classify the local conservation laws for Eq. (1). In view of the aforementioned results, it is sufficient to only classify the conservation laws for Eq. (3).

3. Conservation laws

In this section, we first give some necessary notions and statements about the conservation laws for differential equations with two independent variables (see Refs. [34], [35] and [39] for more details and general formulations), then perform the classification of the local conservation laws with characteristics of zero order for Eq. (3).

Let $\mathcal{L}$ be a system $L(t, x, u_{(\rho)}) = 0$ of l partial differential equations(PDEs), $L^1 = 0, \dots, L^l = 0$ for m unknown functions $u = (u^1, \dots, u^m)$ of two independent variables t and x. Here $u_{(\rho)}$ denotes the set that includes all partial derivatives of the functions u of order not greater than ρ, including u as the derivatives of the zero order.

Definition 1 A conservation law for system $\mathcal{L}$ is a divergence expression

$$\mathrm{D}_t F(t, x, u_{(r)}) + \mathrm{D}_x G(t, x, u_{(r)}) = 0, \tag{4}$$

which vanishes for all solutions of $\mathcal{L}$. Here D_t and D_x are the operators of total differentiation with respect to t and x, respectively, namely, $\mathrm{D}_t = \partial_t + u_t\partial_u + \cdots$, $\mathrm{D}_x = \partial_x + u_x\partial_u + \cdots$; F and G are correspondingly called the density and the flux of the conservation law and the tuple (F, G) is a conserved vector of the conservation law.

The crucial notion for the theory of conservation laws is the equivalence and triviality of the conservation laws.

Definition 2 Two conserved vectors (F, G) and (F', G') are equivalent if there exist such functions $\hat{F}$, $\hat{G}$ and H of t, x and derivatives of u that $\hat{F}$ and $\hat{G}$ vanish for all solutions of $\mathcal{L}$, and $F' = F + \hat{F} + \mathrm{D}_x H$, $G' = G + \hat{G} - \mathrm{D}_t H$. A conserved vector is called trivial if it is equivalent to the zeroth vector.

Let the system $\mathcal{L}$ be totally nondegenerate.[3] Then application of the Hadamard lemma to the definition of conservation law and integrating by parts imply that the left hand side of any conservation law of $\mathcal{L}$ can always be presented up to the equivalence relation as a linear combination of the left hand sides of the independent equations of $\mathcal{L}$ with coefficients λ^μ being functions of t, x and derivatives of u:

$$\mathrm{D}_t F + \mathrm{D}_x G = \lambda^1 L^1 + \cdots + \lambda^l L^l. \tag{5}$$

Definition 3 Formula (5) and the l-tuple $\lambda = (\lambda^1, \dots, \lambda^l)$ are called the characteristic form and the characteristic of the conservation law $\mathrm{D}_t F + \mathrm{D}_x G = 0$, respectively.

The characteristic λ is trivial if it vanishes for all solutions of $\mathcal{L}$. Since $\mathcal{L}$ is nondegenerate, the characteristics λ and $\tilde{\lambda}$ are called equivalent if they satisfy Eq. (5) for the same conserved vector (F, G) and $\lambda - \tilde{\lambda}$ is a trivial characteristic.

Lemma 1[34] Any point transformation σ between systems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ induces a linear one-to-one mapping σ_* between the corresponding linear spaces of the conservation laws.

Consider the class $\mathcal{L}|_{\mathcal{S}}$ of systems $\mathcal{L}_\theta$: $L(t, x, u_{(\rho)}, \theta) = 0$ parameterized with the parameter-functions $\theta = \theta(t, x, u_{(\rho)})$, where L is a fixed function of $t, x, u_{(\rho)}$ and θ. The symbol θ denotes the tuple of arbitrary (parametric) differential functions $\theta(t, x, u_{(\rho)}) = (\theta^1(t, x, u_{(\rho)}), \dots, \theta^k(t, x, u_{(\rho)}))$ running through the set $\mathcal{S}$, which consists of the solutions of the system $S(t, x, u_{(\rho)}, \theta_{(q)}(t, x, u_{(\rho)})) = 0$.

Let $P = P(L, S)$ denote the set of pairs, each pair consists of a system $\mathcal{L}_\theta$ from $\mathcal{L}|_{\mathcal{S}}$ and a conservation law for that system. The action of transformations from the equivalence group $G^{\sim}$ for the class $\mathcal{L}|_{\mathcal{S}}$ together with the pure equivalence relation of the conserved vectors naturally generates an equivalence relation on P. The classification of conservation laws with respect to $\mathcal{L}|_{\mathcal{S}}$ will be understood as a classification in P with respect to the above equivalence relation. This problem can be investigated in a way that is similar to the group classification in the systems of differential equations. Specifically, we first construct the conservation laws that are defined for all values of

the arbitrary elements. The corresponding conserved vectors may depend on the arbitrary elements. Then we classify, with respect to the equivalence group, the arbitrary elements for each of the systems that admit additional conservation laws.

For more detail and rigorous proof of the above definitions and statements, see Refs. [34], [35] and [39]. Using the most direct method described in Ref. [35], we prove the following theorem.

Theorem 1 A complete list of $G_1^{\sim}$-inequivalent equations (3) having nontrivial conservation laws with characteristics of zero order is exhausted by the following ones:

1. $m=n+1$,

$n\neq -1$:

$$\left(\alpha^i f(tu_t-u),\ -\alpha^i t u^n u_x+\alpha^i_x\frac{tu^{n+1}}{n+1}\right);$$

$$\left(\beta^i f u_t,\ -\beta^i u^n u_x+\beta^i_x\frac{1}{n+1}u^{n+1}\right);\quad i=1,2.$$

$n=-1$:

$$\big(f(tu_t-u),\ -tu^{-1}u_x\big);$$

$$\big(xf(tu_t-u),\ -xtu^{-1}u_x+t\ln u\big);$$

$$\big(fu_t,\ -u^{-1}u_x\big);$$

$$\big(xfu_t,\ -xu^{-1}u_x+\ln u\big).$$

2. $m=1,\quad h=\mu f,\quad \mu>0$,

$n\neq -1$:

$$\Big(\mathrm{e}^{\sqrt{\mu}t}xf(u_t-\sqrt{\mu}u),$$
$$-\mathrm{e}^{\sqrt{\mu}t}\left(xu^nu_x-\frac{u^{n+1}}{n+1}\right)\Big);$$

$$\Big(\mathrm{e}^{-\sqrt{\mu}t}xf(u_t+\sqrt{\mu}u),$$
$$-\mathrm{e}^{-\sqrt{\mu}t}\left(xu^nu_x-\frac{u^{n+1}}{n+1}\right)\Big);$$

$$\big(\mathrm{e}^{\sqrt{\mu}t}f(u_t-\sqrt{\mu}u),\ -\mathrm{e}^{\sqrt{\mu}t}u^nu_x\big);$$

$$\big(\mathrm{e}^{-\sqrt{\mu}t}f(u_t+\sqrt{\mu}u),\ -\mathrm{e}^{-\sqrt{\mu}t}u^nu_x\big).$$

$n=-1$:

$$\big(\mathrm{e}^{\sqrt{\mu}t}xf(u_t-\sqrt{\mu}u),$$
$$-\mathrm{e}^{\sqrt{\mu}t}(xu^{-1}u_x-\ln u)\big);$$

$$\big(\mathrm{e}^{-\sqrt{\mu}t}xf(u_t+\sqrt{\mu}u),$$
$$-\mathrm{e}^{-\sqrt{\mu}t}(xu^{-1}u_x-\ln u)\big);$$

$$\big(\mathrm{e}^{\sqrt{\mu}t}f(u_t-\sqrt{\mu}u),\ -\mathrm{e}^{\sqrt{\mu}t}u^{-1}u_x\big);$$

$$\big(\mathrm{e}^{-\sqrt{\mu}t}f(u_t+\sqrt{\mu}u),\ -\mathrm{e}^{-\sqrt{\mu}t}u^{-1}u_x\big).$$

3. $m=1,\quad h=\mu f,\quad \mu<0$,

$n\neq -1$:

$$\Big(xf(\sin(\sqrt{-\mu}t)u_t-\sqrt{-\mu}\cos(\sqrt{-\mu}t)u),$$
$$-\sin(\sqrt{-\mu}t)\left(xu^nu_x-\frac{1}{n+1}u^{n+1}\right)\Big);$$

$$\Big(xf(\cos(\sqrt{-\mu}t)u_t+\sqrt{-\mu}\sin(\sqrt{-\mu}t)u),$$
$$-\cos(\sqrt{-\mu}t)\left(xu^nu_x-\frac{1}{n+1}u^{n+1}\right)\Big);$$

$$\big(f(\sin(\sqrt{-\mu}t)u_t-\sqrt{-\mu}\cos(\sqrt{-\mu}t)u),$$
$$-\sin(\sqrt{-\mu}t)u^nu_x\big);$$

$$\big(f(\cos(\sqrt{-\mu}t)u_t+\sqrt{-\mu}\sin(\sqrt{-\mu}t)u),$$
$$-\cos(\sqrt{-\mu}t)u^nu_x\big).$$

$n=-1$:

$$\big(xf(\sin(\sqrt{-\mu}t)u_t-\sqrt{-\mu}\cos(\sqrt{-\mu}t)u),$$
$$-\sin(\sqrt{-\mu}t)(xu^{-1}u_x-\ln u)\big);$$

$$\big(xf(\cos(\sqrt{-\mu}t)u_t+\sqrt{-\mu}\sin(\sqrt{-\mu}t)u),$$
$$-\cos(\sqrt{-\mu}t)(xu^{-1}u_x-\ln u)\big);$$

$$\big(f(\sin(\sqrt{-\mu}t)u_t-\sqrt{-\mu}\cos(\sqrt{-\mu}t)u),$$
$$-\sin(\sqrt{-\mu}t)u^{-1}u_x\big);$$

$$\big(f(\cos(\sqrt{-\mu}t)u_t+\sqrt{-\mu}\sin(\sqrt{-\mu}t)u),$$
$$-\cos(\sqrt{-\mu}t)u^{-1}u_x\big).$$

Here the functions $\alpha^i=\alpha^i(x)$, $\beta^i=\beta^i(x)$ $(i=1,2)$ are the fundamental solutions for the second-order linear ordinary differential equations $\alpha_{xx}+(n+1)h\alpha=0$ and $\beta_{xx}+(n+1)h\beta=0$, respectively; μ is an arbitrary constant.

Proof We search the first-order conservation laws for Eq. (3) in the form

$$\mathrm{D}_tF(t,x,u,u_t,u_x)+\mathrm{D}_xG(t,x,u,u_t,u_x)=0. \quad (6)$$

Substituting the expression for u_{tt} deduced from Eq. (3) into Eq. (6) and decomposing the obtained equation with respect to u_{xt} and u_{xx}, we obtain

$$\begin{aligned}&F_{u_x}+G_{u_t}=0,\quad F_{u_t}\frac{1}{f}u^n+G_{u_x}=0,\\&F_{u_t}\frac{n}{f}u^{n-1}u_x^2+G_uu_x+F_uu_t\\&\quad+F_{u_t}\frac{h}{f}u^m+F_t+G_x=0.\end{aligned}\quad (7)$$

Up to conserved vector equivalence for the first equation of Eqs. (7), we can assume $F_{u_x}=G_{u_t}=0$, which together with the second equation implies

$$\begin{aligned}&F=F^3(t,x,u)u_t+F^2(t,x,u),\\&G=-F^3(t,x,u)\frac{1}{f}u^nu_x+G^1(t,x,u).\end{aligned}\quad (8)$$

Substituting these expressions into the last equation of Eqs. (7) and splitting it with respect to the powers

of u_x and u_t, we obtain the PDEs for the functions F^3, F^2 and G^1 in the form

$$F^3_u = 0, \quad F^3\frac{f_x}{f^2}u^n + G^1_u - \frac{1}{f}F^3_x u^n = 0,$$
$$F^2_u + F^3_t = 0, \quad F^2_t + F^3\frac{h}{f}u^m + G^1_x = 0. \quad (9)$$

Solving the first three equations yields

$$F^3 = F^1(t,x), \quad F^2 = -F^1_t u + F^0(t,x),$$
$$G^1 = \left(\frac{F^1}{f}\right)_x \int u^n + G^0(t,x). \quad (10)$$

The case $n = -1$ is special with respect to the integration of the function u^n. So

$$G^1 = \begin{cases} \left(\dfrac{F^1}{f}\right)_x \dfrac{u^{n+1}}{n+1} + G^0(t,x), & n \neq -1, \\ \left(\dfrac{F^1}{f}\right)_x \ln u + G^0(t,x), & n = -1. \end{cases}$$

Substituting the expressions for F^3, F^2 and G^1 into the last equation of Eqs. (9), we find that the major role for the classification is played by the differential consequence

$$\left(\frac{F^1}{f}\right)_{xx} u^n + m\frac{F^1}{f}hu^{m-1} - F^1_{tt} = 0. \quad (11)$$

Indeed, it is the unique classification condition for this problem. In all classification cases, we obtain the equation $F^0_t + G^0_x = 0$. Therefore, up to conserved vector equivalence, we can assume $F^0 = G^0 = 0$ and additionally $F^1 \neq 0$ for the conservation laws to be non-trivial. Thus, taking into account Eqs. (8) and (10), the conservation density and the conservation flux can be rewritten as

$$F = F^1 u_t - F^1_t u,$$
$$G = \begin{cases} -F^1\dfrac{1}{f}u^n u_x + \left(\dfrac{F^1}{f}\right)_x \dfrac{u^n}{n+1}, & n \neq -1, \\ -F^1\dfrac{1}{f}u^{-1}u_x + \left(\dfrac{F^1}{f}\right)_x \ln u, & n = -1. \end{cases} \quad (12)$$

It follows from Eq. (11) that Eq. (3) possesses non-trivial conserved vectors only if the parameters n and m take the following special values,

$$(n,\ m) \in \{(0,0),(0,1),(n',n'+1),(n',1), n' \in \mathbf{R}\}.$$

We exclude the cases $(0,0)$ and $(0,1)$, since they correspond to the linear case of Eq. (3).

1. $m = n+1$. Equation (11) is split into two equations $F^1_{tt} = 0$, $(F^1/f)_{xx} + (n+1)(F^1/f)h = 0$, which implies $F^1 = [\alpha(x)t + \beta(x)]f$, where α and β satisfy the homogeneous linear second-order ordinary differential equations $\alpha_{xx} + (n+1)h\alpha = 0$ and $\beta_{xx} + (n+1)h\beta = 0$, respectively. For $n = -1$, $F^1 = (c_1 + c_2 x)t + c_3 + c_4 x$.

2. $m = 1$. The decomposition of (11) results in the equations $(F^1/f)_{xx} = 0$ and $(F^1/f)h - F^1_{tt} = 0$. The first equation gives $F^1 = [\alpha(t)x + \beta(t)]f$. Substituting F^1 to the second equation yields

$$\alpha_{tt}xf + \beta_{tt}f - \alpha xh - \beta h = 0.$$

In view of the condition $F^1 \neq 0$, it follows from the latter equation that $h = \mu f$ for arbitrary constant μ and α and β satisfy the equations $\alpha_{tt} - \mu\alpha = 0$ and $\beta_{tt} - \mu\beta = 0$, respectively, which implies $F^1 = [c_1 \mathrm{e}^{\sqrt{\mu}t} + c_2 \mathrm{e}^{-\sqrt{\mu}t}]x + c_3 \mathrm{e}^{\sqrt{\mu}t} + c_4 \mathrm{e}^{-\sqrt{\mu}t}$ if $\mu > 0$ and $F^1 = (c_1 \sin(\sqrt{-\mu}t) + c_2 \cos(\sqrt{-\mu}t))x + c_3 \sin(\sqrt{-\mu}t) + c_4 \cos(\sqrt{-\mu}t)$ if $\mu < 0$.

4. Conclusion and discussion

In summary, we have presented a classification of local conservation laws with characteristics of zero order for the variable coefficient nonlinear wave equations (3) by using the most direct method. The application of equivalence group and the correct choice of gauge coefficients of the equations allow us to formulate the results in explicit closed form, i.e., theorem 1, which is useful in the analysis of nonlinear phenomena of the corresponding nonlinear wave models contained in Eq. (3).

The method used in this paper can also be applied to other equations in mathematical physics for classification of local conservation laws. Furthermore, as an application of the obtained results in theorem 1, we can use it to perform a complete classification of the potential conservation laws and present an exhaustive list of the locally inequivalent potential systems for the equations under consideration. We may also investigate different kinds of symmetries, say Lie, nonclassical and potential ones for the constructed potential systems and then use them to construct exact solutions for the equations under consideration.

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