Binary nonlinearization of the super classical-Boussinesq hierarchy*

Tao Si-Xing(陶司兴)[a], Wang Hui(王 惠)[b], and Shi Hui(史 会)[c]†

[a] Department of Mathematics, Shangqiu Normal University, Shangqiu 476000, China

[b] Department of Mathematics, Shanghai University, Shanghai 200444, China

[c] Department of Physics and Information Engineering, Shangqiu Normal University, Shangqiu 476000, China

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The symmetry constraint and binary nonlinearization of Lax pairs for the super classical-Boussinesq hierarchy is obtained. Under the obtained symmetry constraint, the n-th flow of the super classical-Boussinesq hierarchy is decomposed into two super finite-dimensional integrable Hamiltonian systems, defined over the super-symmetry manifold with the corresponding dynamical variables x and t_n. The integrals of motion required for Liouville integrability are explicitly given.

Keywords: symmetry constraints, binary nonlinearization, super classical-Boussinesq hierarchy, super finite-dimensional integrable Hamiltonian systems

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1. Introduction

For almost twenty years, much attention has been paid to the construction of new integrable systems as well as their integrable couplings by using Lie algebras[1−5] and to the construction of finite dimensional integrable systems from soliton equations by using symmetry constraints. Either (2+1)-dimensional soliton equations[6,7] or (1+1)-dimensional soliton equations[8,9] can be decomposed into compatible finite-dimensional integrable systems. It is well known that a crucial idea in carrying out symmetry constraints is the nonlinearization of Lax pairs for soliton hierarchies. Symmetry constraints give relations of potentials with eigenfunctions and adjoint eigenfunctions of Lax pairs. The nonlinearization of Lax pairs can be classified into mono-nonlinearization[10,11] and binary nonlinearization.[12,13] The technique of nonlinearization has been successfully applied to many well-known (1+1)-dimensional soliton equations, such as the Ablowitz-Kaup-Newell-Segur (AKNS) system,[8] the Kortweg-de Vries (KdV) hierarchy[9] and the Dirac system.[14]

With the development of soliton theory, super-integrable systems associated with Lie super algebra have been attracting growing attention.[15−18] In 1997, Hu[15] proposed the super-trace identity and applied it to establish the super-Hamiltonian structures of super-integrable systems. Then Ma[16] gave a systematic proof of the super-trace identity and presented the super-Hamiltonian structures of the super AKNS and the super Dirac hierarchies as its application. Wang and Dong[17] presented a new super-Hamiltonian structure of the super AKNS hierarchy. Tao and Xia[18] presented super-Hamiltonian structures of the super classical-Boussinesq system. However there are few results about the nonlinearization of super-integrable systems in the literature. Studies provide many examples of super-symmetry integrable systems with super dependent variables or super independent variables.[19−21] Only very recently, He *et al.* applied the binary nonlinearization method to the super AKNS and the super Dirac systems and obtained their corresponding super finite-dimensional systems.[22−24] In this paper, we consider the binary nonlinearization of the super classical-Boussinesq hierarchy.

This paper is organized as follows. In the next

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†Corresponding author. E-mail: shihui529@126.com

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section, we will review the super classical-Boussinesq hierarchy and its super-Hamiltonian structure. Then in Section 3, we obtain a symmetry constraint for the potential of the super classical-Boussinesq hierarchy. In Section 4, we apply the binary nonlinearization to the super classical-Boussinesq hierarchy and obtain a super finite-dimensional integrable Hamiltonian system on the supersymmetry manifold, whose integrals of motion are explicitly given. Some conclusions are given in Section 5.

2. The super classical-Boussinesq hierarchy

The super classical-Boussinesq spectral problem associated with the Lie super algebra $B(0,1)$ is given by

$$\boldsymbol{\phi}_x = \boldsymbol{U}\boldsymbol{\phi}, \quad \boldsymbol{U} = \begin{pmatrix} -\lambda - \frac{1}{4}q & r & \alpha \\ -1 & \lambda + \frac{1}{4}q & \beta \\ \beta & -\alpha & 0 \end{pmatrix},$$
$$\boldsymbol{u} = (q \;\; r \;\; \alpha \;\; \beta)^{\mathrm{T}}, \quad \boldsymbol{\phi} = (\phi_1 \;\; \phi_2 \;\; \phi_3)^{\mathrm{T}}, \qquad (1)$$

where λ is a spectral parameter, r and q are even variables, and α and β are odd variables .[16] Taking

$$\boldsymbol{V} = \begin{pmatrix} A & B & \rho \\ C & -A & \delta \\ \delta & -\rho & 0 \end{pmatrix},$$

the co-adjoint equation associated with Eq. (1) $\boldsymbol{V}_x = [\boldsymbol{U}, \boldsymbol{V}]$ gives

$$\begin{cases} A_x = B + rC + \beta\rho + \alpha\delta, \\ B_x = -2\lambda B - 2rA - \frac{1}{2}qB - 2\alpha\rho, \\ C_x = 2\lambda C - 2A + \frac{1}{2}qC + 2\beta\delta, \\ \rho_x = -\lambda\rho - \alpha A - \beta B - \frac{1}{4}q\rho + r\delta, \\ \delta_x = \lambda\delta + \beta A - \alpha C - \rho + \frac{1}{4}q\delta. \end{cases} \qquad (2)$$

If we set

$$A = \sum_{i\geq 0} A_i\lambda^{-i}, \quad B = \sum_{i\geq 0} B_i\lambda^{-i}, \quad C = \sum_{i\geq 0} C_i\lambda^{-i},$$
$$\rho = \sum_{i\geq 0} \rho_i\lambda^{-i}, \quad \delta = \sum_{i\geq 0} \delta_i\lambda^{-i}, \qquad (3)$$

then the system expressed by Eq. (2) is equivalent to $(i \geq 0)$

$$\begin{cases} B_{i+1} = -\frac{1}{2}B_{i,x} - rA_i - \frac{1}{4}qB_i - \alpha\rho_i, \\ C_{i+1} = \frac{1}{2}C_{i,x} + A_i - \frac{1}{4}qC_i - \beta\delta_i, \\ \rho_{i+1} = -\rho_{i,x} - \alpha A_i - \beta B_i - \frac{1}{4}q\rho_i + r\delta_i, \\ \delta_{i+1} = \delta_{i,x} - \beta A_i + \alpha C_i + \rho_i - \frac{1}{4}q\delta_i, \\ A_{i+1,x} = B_{i+1} + rC_{i+1} + \beta\rho_{i+1} + \alpha\delta_{i+1}, \end{cases} \qquad (4)$$

which results in the recurrence relations

$$\begin{cases} (-\frac{1}{2}A_{i+1}, C_{i+1}, 2\delta_{i+1}, -2\rho_{i+1})^{\mathrm{T}} \\ \quad = \mathscr{L}(-\frac{1}{2}A_i, C_i, 2\delta_i, -2\rho_i)^{\mathrm{T}}, \\ A_i = \partial^{-1}(B_i + rC_i + \beta\rho_i + \alpha\delta_i), \end{cases} \qquad (5)$$

where

$$\mathscr{L} = \begin{pmatrix} -\frac{1}{2}\partial - \frac{1}{4}\partial^{-1}q\partial & -\frac{1}{4}\partial^{-1}r\partial - \frac{1}{4}r & -\frac{1}{4}\partial^{-1}\alpha\partial - \frac{1}{8}\alpha & -\frac{1}{4}\partial^{-1}\beta\partial + \frac{1}{8}\beta \\ -2 & \frac{1}{2}\partial - \frac{1}{4}q & -\frac{1}{2}\beta & 0 \\ 4\beta & 2\alpha & \partial - \frac{1}{4}q & -1 \\ -4\alpha - 4\beta\partial & -2r\beta & -r + \alpha\beta & -\partial - \frac{1}{4}q \end{pmatrix}. \qquad (6)$$

Upon choosing the initial conditions

$$B_0 = C_0 = \rho_0 = \delta_0 = 0, \quad A_0 = -1,$$

all other parameters $A_i, B_i, C_i, \rho_i, \delta_i$ $(i \geq 1)$ can be worked out by using the recurrence relations given in Eq. (5). The first few results are

$$A_1 = 0, \quad B_1 = r, \quad C_1 = -1, \quad \rho_1 = \alpha, \quad \delta_1 = \beta,$$
$$A_2 = -\frac{1}{2}r + \alpha\beta, \quad B_2 = -\frac{1}{2}r_x - \frac{1}{4}qr,$$
$$C_2 = \frac{1}{4}q, \quad \rho_2 = -\alpha_x - \frac{1}{4}q\alpha, \quad \delta_2 = \beta_x - \frac{1}{4}q\beta,$$
$$A_3 = \frac{1}{4}r_x + \frac{1}{4}qr + \alpha\beta_x - \alpha_x\beta - \frac{1}{2}q\alpha\beta,$$
$$B_3 = \frac{1}{4}r_{xx} + \frac{1}{8}q_xr + \frac{1}{4}qr_x + \frac{1}{2}r^2 - r\alpha\beta + \frac{1}{16}q^2r + \alpha\alpha_x,$$
$$C_3 = \frac{1}{8}q_x - \frac{1}{2}r + \alpha\beta - \frac{1}{16}q^2 - \beta\beta_x,$$
$$\rho_3 = \alpha_{xx} + \frac{1}{4}q_x\alpha + \frac{1}{2}q\alpha_x + \frac{1}{2}r\alpha$$

$$+\frac{1}{2}r_x\beta + r\beta_x + \frac{1}{16}q^2\alpha,$$
$$\delta_3 = \beta_{xx} - \frac{1}{4}q_x\beta - \frac{1}{2}q\beta_x + \frac{1}{2}r\beta - \alpha_x + \frac{1}{16}q^2\beta.$$

Let us associate the spectral problem given in Eq. (1) with the following auxiliary problem

$$\boldsymbol{\phi}_{t_n} = \boldsymbol{V}^{(n)}\boldsymbol{\phi}, \tag{7}$$

with

$$\boldsymbol{V}^{(n)} = \sum_{i=0}^{n}\begin{pmatrix} A_i & B_i & \rho_i \\ C_i & -A_i & \delta_i \\ \delta_i & -\rho_i & 0 \end{pmatrix}\lambda^{n-i} + \begin{pmatrix} -C_{n+1} & 0 & 0 \\ 0 & C_{n+1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The compatible condition of the spectral problem given in Eq. (1) and the auxiliary problem given in Eq. (7) is

$$\boldsymbol{U}_{t_n} - \boldsymbol{V}_x^{(n)} + [\boldsymbol{U}, \boldsymbol{V}^{(n)}] = 0, \tag{8}$$

which infers the super classical-Boussinesq soliton hierarchy

$$\begin{aligned}\boldsymbol{u}_{t_n} = \boldsymbol{K}_n =\ & (4C_{n+1,x},\ -2A_{n+1,x} + 2\beta\rho_{n+1} + 2\alpha\delta_{n+1},\\ & -\rho_{n+1} - \alpha C_{n+1},\ \delta_{n+1} + \beta C_{n+1})^{\mathrm{T}}.\end{aligned} \tag{9}$$

The $\boldsymbol{u}_{t_n} = \boldsymbol{K}_n$ in Eq. (9) is called the n-th super classical-Boussinesq flow of the hierarchy.

Using the super-trace identity[15,16]

$$\begin{aligned}&\frac{\delta}{\delta u}\int \mathrm{Str}\left(V\frac{\partial U}{\partial\lambda}\right)\mathrm{d}x\\ &= \left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)\mathrm{Str}\left(\frac{\partial U}{\partial u}V\right),\end{aligned} \tag{10}$$

where Str means the super trace, we have

$$\begin{pmatrix} -\frac{1}{2}A_{i+1} \\ C_{i+1} \\ 2\delta_{i+1} \\ -2\rho_{i+1} \end{pmatrix} = \frac{\delta}{\delta\boldsymbol{u}}\boldsymbol{H}_i,\ \boldsymbol{H}_i = \int\frac{2A_{i+2}}{i+1}\mathrm{d}x,\ i \geq 0. \tag{11}$$

Therefore, the super classical-Boussinesq soliton hierarchy given in Eq. (9) can be written in the following super-Hamiltonian form

$$\boldsymbol{u}_{t_n} = \boldsymbol{J}\frac{\delta\boldsymbol{H}_n}{\delta\boldsymbol{u}}, \tag{12}$$

where

$$\boldsymbol{J} = \begin{pmatrix} 0 & 4\partial & 0 & 0 \\ 4\partial & 0 & \alpha & -\beta \\ 0 & -\alpha & 0 & \frac{1}{2} \\ 0 & \beta & \frac{1}{2} & 0 \end{pmatrix}$$

is a super-symplectic operator and $\boldsymbol{H}_n$ is given by Eq. (11).

The first non-trivial nonlinear equation of the hierarchy given in Eq. (9) is given by the second flow

$$\begin{cases} q_{t_2} = \frac{1}{2}q_{xx} - 2r_x + 4(\alpha\beta)_x - \frac{1}{2}qq_x - 4\beta\beta_{xx}, \\ r_{t_2} = -\frac{1}{2}r_{xx} - \frac{1}{2}(qr)_x - 2\alpha\alpha_x + 2r\beta\beta_x, \\ \alpha_{t_2} = -\alpha_{xx} - \frac{3}{8}q_x\alpha - \frac{1}{2}q\alpha_x - \frac{1}{2}r_x\beta \\ \qquad -r\beta_x + \alpha\beta\beta_x, \\ \beta_{t_2} = \beta_{xx} - \frac{1}{8}q_x\beta - \frac{1}{2}q\beta_x - \alpha_x, \end{cases} \tag{13}$$

which possesses a Lax pair of $\boldsymbol{U}$ defined in Eq. (1) and $\boldsymbol{V}^{(2)}$ defined by

$$\boldsymbol{V}^{(2)} = \begin{pmatrix} -\lambda^2 - \frac{1}{8}q_x + \frac{1}{16}q^2 + \beta\beta_x & r\lambda - \frac{1}{2}r_x - \frac{1}{4}qr & \alpha\lambda - \alpha_x - \frac{1}{4}q\alpha \\ -\lambda + \frac{1}{4}q & \lambda^2 + \frac{1}{8}q_x - \frac{1}{16}q^2 - \beta\beta_x & \beta\lambda + \beta_x - \frac{1}{4}q\beta \\ \beta\lambda + \beta_x - \frac{1}{4}q\beta & -\alpha\lambda + \alpha_x + \frac{1}{4}q\alpha & 0 \end{pmatrix}. \tag{14}$$

3. Symmetry constraint of super classical-Boussinesq hierarchy

In order to compute a symmetry constraint, we consider the following adjoint spectral problem of the spectral problem given in Eq. (1)

$$\boldsymbol{\psi}_x = -\boldsymbol{U}^{\mathrm{St}}\boldsymbol{\psi} = \begin{pmatrix} \lambda + \frac{1}{4}q & 1 & \beta \\ -r & -\lambda - \frac{1}{4}q & -\alpha \\ -\alpha & -\beta & 0 \end{pmatrix}\boldsymbol{\psi},$$

$$\boldsymbol{\psi} = (\psi_1\ \ \psi_2\ \ \psi_3)^{\mathrm{T}}, \tag{15}$$

where St means the super transposition. The following result is a general formula for the variational derivative with respect to the potential $\boldsymbol{u}$ (see Ref. [8] for the classical case).

Lemma 1[22–24] Let $\boldsymbol{U}(\boldsymbol{u}, \lambda)$ be an even matrix of order $m + n$ depending on $\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx}, \ldots$, and parameter λ. Suppose that $\boldsymbol{\phi} = (\boldsymbol{\phi}_{\mathrm{e}}, \boldsymbol{\phi}_{\mathrm{o}})^{\mathrm{T}}$ and $\boldsymbol{\psi} = (\boldsymbol{\psi}_{\mathrm{e}}, \boldsymbol{\psi}_{\mathrm{o}})^{\mathrm{T}}$ satisfy the spectral problem and the

adjoint spectral problem

$$\boldsymbol{\phi}_x = \boldsymbol{U}(\boldsymbol{u},\lambda)\boldsymbol{\phi}, \quad \boldsymbol{\psi}_x = -\boldsymbol{U}^{\mathrm{St}}\boldsymbol{\psi}, \tag{16}$$

where $\boldsymbol{\phi}_{\mathrm{e}} = (\phi_1,\ldots,\phi_m)$ and $\boldsymbol{\psi}_{\mathrm{e}} = (\psi_1,\ldots,\psi_m)$ are even functions and $\boldsymbol{\phi}_{\mathrm{o}} = (\phi_{m+1},\ldots,\phi_{m+n})$ and $\boldsymbol{\psi}_{\mathrm{o}} = (\psi_{m+1},\ldots,\psi_{m+n})$ are odd eigenfunctions. Then the variational derivative of the spectral parameter λ with respect to the potential $\boldsymbol{u}$ is given by

$$\frac{\delta\lambda}{\delta\boldsymbol{u}} = \frac{(\boldsymbol{\psi}_{\mathrm{e}},(-1)^{p(\boldsymbol{u})}\boldsymbol{\psi}_{\mathrm{o}})(\partial\boldsymbol{U}/\partial\boldsymbol{u})\boldsymbol{\phi}}{-\int\boldsymbol{\psi}^{\mathrm{T}}(\partial\boldsymbol{U}/\partial\lambda)\boldsymbol{\phi}\mathrm{d}x}, \tag{17}$$

where we denote

$$p(v) = \begin{cases} 0, & v \text{ is even variable,} \\ 1, & v \text{ is odd variable.} \end{cases} \tag{18}$$

By using Lemma 1, it is not difficult to find that

$$\frac{\delta\lambda}{\delta\boldsymbol{u}} = \frac{1}{E}\begin{pmatrix} \frac{1}{4}(\psi_2\phi_2 - \psi_1\phi_1) \\ \psi_1\phi_2 \\ \psi_1\phi_3 + \psi_3\phi_2 \\ \psi_2\phi_3 - \psi_3\phi_1 \end{pmatrix}, \tag{19}$$

where $E = \int(\phi_1\psi_1 - \phi_1\psi_1)\mathrm{d}x$. If we consider zero boundary conditions, $\lim_{|x|\to\infty}\boldsymbol{\phi} = \lim_{|x|\to\infty}\boldsymbol{\psi} = 0$, we can obtain a characteristic property (a recurrence relation) for the variational derivative of λ,

$$\mathscr{L}\frac{\delta\lambda}{\delta\boldsymbol{u}} = \lambda\frac{\delta\lambda}{\delta\boldsymbol{u}}, \tag{20}$$

where $\mathscr{L}$ and $\delta\lambda/\delta\boldsymbol{u}$ are given by Eqs. (5) and (19), respectively.

Let us now discuss a spatial system and a temporal system

$$\begin{aligned}
\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}_x &= \boldsymbol{U}(u,\lambda_j)\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix} = \begin{pmatrix}-\lambda_j - \frac{1}{4}q & r & \alpha\\ -1 & \lambda_j + \frac{1}{4}q & \beta\\ \beta & -\alpha & 0\end{pmatrix}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}, \\
\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\end{pmatrix}_x &= -\boldsymbol{U}^{\mathrm{St}}(u,\lambda_j)\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\end{pmatrix} = \begin{pmatrix}\lambda_j + \frac{1}{4}q & 1 & \beta\\ -r & -\lambda_j - \frac{1}{4}q & -\alpha\\ -\alpha & -\beta & 0\end{pmatrix}\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\end{pmatrix},
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}_{t_n} &= \begin{pmatrix}\sum_{i=0}^{n}A_i\lambda_j^{n-i} - C_{n+1} & \sum_{i=0}^{n}B_i\lambda_j^{n-i} & \sum_{i=0}^{n}\rho_i\lambda_j^{n-i}\\ \sum_{i=0}^{n}C_i\lambda_j^{n-i} & -\sum_{i=0}^{n}A_i\lambda_j^{n-i} + C_{n+1} & \sum_{i=0}^{n}\delta_i\lambda_j^{n-i}\\ \sum_{i=0}^{n}\delta_i\lambda_j^{n-i} & -\sum_{i=0}^{n}\rho_i\lambda_j^{n-i} & 0\end{pmatrix}\begin{pmatrix}\phi_{1j}\\ \phi_{2j}\\ \phi_{3j}\end{pmatrix}, \\
\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\end{pmatrix}_{t_n} &= \begin{pmatrix}-\sum_{i=0}^{n}A_i\lambda_j^{n-i} + C_{n+1} & -\sum_{i=0}^{n}C_i\lambda_j^{n-i} & \sum_{i=0}^{n}\delta_i\lambda_j^{n-i}\\ -\sum_{i=0}^{n}B_i\lambda_j^{n-i} & \sum_{i=0}^{n}A_i\lambda_j^{n-i} - C_{n+1} & -\sum_{i=0}^{n}\rho_i\lambda_j^{n-i}\\ -\sum_{i=0}^{n}\rho_i\lambda_j^{n-i} & -\sum_{i=0}^{n}\delta_i\rho_j^{n-i} & 0\end{pmatrix}\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\\ \psi_{3j}\end{pmatrix}.
\end{aligned} \tag{22}$$

For the two systems, we have the following symmetry constraint

$$\frac{\delta}{\delta\boldsymbol{u}}\boldsymbol{H}_k = \begin{pmatrix}-\frac{1}{2}A_{k+1}\\ C_{k+1}\\ 2\delta_{k+1}\\ -2\rho_{k+1}\end{pmatrix} = \sum_{j=1}^{N}\gamma_j\frac{\delta\lambda_j}{\delta\boldsymbol{u}}, \quad k \geq 0. \tag{23}$$

If we choose $k = 1$ and $\gamma_j = E_j = \int(\phi_{1j}\psi_{1j} - \phi_{2j}\psi_{2j})\mathrm{d}x$ in Eq. (23), then we obtain the following symmetry constraint

$$\begin{cases} r - 2\alpha\beta = -\langle\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1\rangle + \langle\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2\rangle, \\ \frac{1}{4}q = \langle\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2\rangle, \\ 2\alpha_x + \frac{1}{2}q\alpha = \langle\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_3\rangle - \langle\boldsymbol{\Psi}_3,\boldsymbol{\Phi}_1\rangle, \\ 2\beta_x - \frac{1}{2}q\beta = \langle\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_3\rangle + \langle\boldsymbol{\Psi}_3,\boldsymbol{\Phi}_2\rangle, \end{cases} \tag{24}$$

where we use the following notation

$$\begin{aligned} \boldsymbol{\Phi}_i &= (\phi_{i1},\ldots,\phi_{iN})^{\mathrm{T}}, \\ \boldsymbol{\Psi}_i &= (\psi_{i1},\ldots,\psi_{iN})^{\mathrm{T}}, \quad i = 1,2,3, \end{aligned}$$

and $\langle\cdot,\cdot\rangle$ denotes the standard inner product of the Euclidean space $\mathbb{R}^N$.

4. Nonlinearization of super classical-Boussinesq system under symmetry constraint

Now we are in a position to discuss the nonlinearization of the super classical-Boussinesq system under the symmetry constraint given in Eq. (24). To this aim, we first introduce the following new variables

$$\begin{aligned}\phi_{N+1} &= \alpha,\\ \psi_{N+1} &= 2\beta,\end{aligned} \tag{25}$$

Considering the new variables of Eq. (25) and substituting the constraints given in Eq. (24) into the system expressed by Eq. (21), we obtain the following finite-dimensional super system

$$\begin{cases}\phi_{1j,x} = (-\lambda_j - \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle)\phi_{1j} - (\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle - \phi_{N+1}\psi_{N+1})\phi_{2j} + \phi_{N+1}\phi_{3j},\\ \phi_{2j,x} = -\phi_{1j} + (\lambda_j + \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle)\phi_{2j} + \frac{1}{2}\psi_{N+1}\phi_{3j},\\ \phi_{3j,x} = \frac{1}{2}\psi_{N+1}\phi_{1j} - \phi_{N+1}\phi_{2j},\\ \phi_{N+1,x} = \frac{1}{2}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle) - \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\phi_{N+1},\\ \psi_{1j,x} = (\lambda_j + \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle)\psi_{1j} + \psi_{2j} + \frac{1}{2}\psi_{N+1}\psi_{3j},\\ \psi_{2j,x} = (\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle - \phi_{N+1}\psi_{N+1})\psi_{1j} + (-\lambda_j - \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle)\psi_{2j} - \phi_{N+1}\psi_{3j},\\ \psi_{3j,x} = -\phi_{N+1}\psi_{1j} - \frac{1}{2}\psi_{N+1}\psi_{2j},\\ \psi_{N+1,x} = (\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle) + \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\psi_{N+1}.\end{cases} \tag{26}$$

Obviously, Eq. (26) can be written in the following super-Hamiltonian form:

$$\begin{cases}\boldsymbol{\Phi}_{1,x} = \dfrac{\partial H_1}{\partial \boldsymbol{\Psi}_1}, \quad \boldsymbol{\Phi}_{2,x} = \dfrac{\partial H_1}{\partial \boldsymbol{\Psi}_2}, \quad \boldsymbol{\Phi}_{3,x} = \dfrac{\partial H_1}{\partial \boldsymbol{\Psi}_3}, \quad \phi_{N+1,x} = \dfrac{\partial H_1}{\partial \psi_{N+1}},\\ \boldsymbol{\Psi}_{1,x} = -\dfrac{\partial H_1}{\partial \boldsymbol{\Phi}_1}, \quad \boldsymbol{\Psi}_{2,x} = -\dfrac{\partial H_1}{\partial \boldsymbol{\Phi}_2}, \quad \boldsymbol{\Psi}_{3,x} = \dfrac{\partial H_1}{\partial \boldsymbol{\Phi}_3}, \quad \psi_{N+1,x} = \dfrac{\partial H_1}{\partial \phi_{N+1}},\end{cases} \tag{27}$$

where the super-Hamiltonian function H_1 is given by

$$\begin{aligned}H_1 =& -\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle + \langle \Lambda\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle - \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle) - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle + \phi_{N+1}\psi_{N+1}\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\\ &+ \phi_{N+1}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle) + \frac{1}{2}\psi_{N+1}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle).\end{aligned}$$

That is to say, the nonlinearized finite-dimensional super system given in Eq. (26) is a super-Hamiltonian system.

When $n = 1$, the system given in Eq. (21) is exactly the system given in Eq. (22) with $t_1 = x$. When $n = 2$, the system given in Eq. (22) becomes the following system

$$\begin{cases}\phi_{1j,t_2} = \Big(-\lambda_j^2 - \frac{1}{8}\tilde{q}_x + \frac{1}{16}\tilde{q}^2 + \tilde{\beta}\tilde{\beta}_x\Big)\phi_{1j} + \Big(\tilde{r}\lambda_j - \frac{1}{2}\tilde{r}_x - \frac{1}{4}\tilde{q}\tilde{r}\Big)\phi_{2j} + \Big(\tilde{\alpha}\lambda_j - \tilde{\alpha}_x - \frac{1}{4}\tilde{q}\tilde{\alpha}\Big)\phi_{3j},\\ \phi_{2j,t_2} = \Big(-\lambda_j + \frac{1}{4}\tilde{q}\Big)\phi_{1j} + \Big(\lambda_j^2 + \frac{1}{8}\tilde{q}_x - \frac{1}{16}\tilde{q}^2 - \tilde{\beta}\tilde{\beta}_x\Big)\phi_{2j} + \Big(\tilde{\beta}\lambda_j + \tilde{\beta}_x - \frac{1}{4}\tilde{q}\tilde{\beta}\Big)\phi_{3j}\\ \phi_{3j,t_2} = \Big(\tilde{\beta}\lambda_j + \tilde{\beta}_x - \frac{1}{4}\tilde{q}\tilde{\beta}\Big)\phi_{1j} + \Big(-\tilde{\alpha}\lambda_j + \tilde{\alpha}_x + \frac{1}{4}\tilde{q}\tilde{\alpha}\Big)\phi_{2j},\\ \psi_{1j,t_2} = \Big(\lambda_j^2 + \frac{1}{8}\tilde{q}_x - \frac{1}{16}\tilde{q}^2 - \tilde{\beta}\tilde{\beta}_x\Big)\psi_{1j} + \Big(\lambda_j - \frac{1}{4}\tilde{q}\Big)\psi_{2j} + \Big(\tilde{\beta}\lambda_j + \tilde{\beta}_x - \frac{1}{4}\tilde{q}\tilde{\beta}\Big)\psi_{3j},\\ \psi_{2j,t_2} = \Big(-\tilde{r}\lambda_j + \frac{1}{2}\tilde{r}_x + \frac{1}{4}\tilde{q}\tilde{r}\Big)\psi_{1j} - \Big(\lambda_j^2 + \frac{1}{8}\tilde{q}_x - \frac{1}{16}\tilde{q}^2 - \tilde{\beta}\tilde{\beta}_x\Big)\psi_{2j} + \Big(-\tilde{\alpha}\lambda_j + \tilde{\alpha}_x + \frac{1}{4}\tilde{q}\tilde{\alpha}\Big)\psi_{3j},\\ \psi_{3j,t_2} = \Big(-\tilde{\alpha}\lambda_j + \tilde{\alpha}_x + \frac{1}{4}\tilde{q}\tilde{\alpha}\Big)\psi_{1j} + \Big(-\tilde{\beta}\lambda_j - \tilde{\beta}_x + \frac{1}{4}\tilde{q}\tilde{\beta}\Big)\psi_{2j},\end{cases} \tag{28}$$

where $\tilde{q}, \tilde{r}, \tilde{\alpha}, \tilde{\beta}$ denote the functions q, r, α, β defined by the symmetry constraint given in Eq. (24) and $\tilde{q}_x, \tilde{r}_x, \tilde{\alpha}_x, \tilde{\beta}_x$

are given by

$$\begin{cases} \tilde{q}_x = 8\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle + 8\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle^2 - 4(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle) + 2\psi_{N+1}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle), \\ \tilde{r}_x = -2\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle + 2(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle)\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle - 2\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\phi_{N+1}\psi_{N+1}, \\ \tilde{\alpha}_x = \dfrac{1}{2}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle) - \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\phi_{N+1}, \\ \tilde{\beta}_x = \dfrac{1}{2}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle) + \dfrac{1}{2}\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\psi_{N+1}. \end{cases} \tag{29}$$

which are computed through using the spatial constrained flow Eq. (26). Considering the new variables in Eq. (25) and the symmetry constraint given in Eq. (24), the above finite-dimensional super system given in Eq. (28) becomes the following nonlinearized super system

$$\begin{cases} \phi_{1,t_2} = \Big[-\lambda_j^2 + \dfrac{1}{2}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle) - \langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\Big]\phi_1 + ((\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle - \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle + \phi_{N+1}\psi_{N+1})\lambda_j \\ \qquad + \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle)\phi_2 + \Big[\phi_{N+1}\lambda_j - \dfrac{1}{2}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle)\Big]\phi_3, \\ \phi_{2,t_2} = (-\lambda_j + \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle)\phi_1 + \Big[\lambda_j^2 - \dfrac{1}{2}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle) + \langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\Big]\phi_2 \\ \qquad + \Big[\dfrac{1}{2}\psi_{N+1}\lambda_j + \dfrac{1}{2}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle)\Big]\phi_3, \\ \phi_{3,t_2} = \Big[\dfrac{1}{2}\psi_{N+1}\lambda_j + \dfrac{1}{2}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle)\Big]\phi_1 - \Big[\phi_{N+1}\lambda_j - \dfrac{1}{2}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle)\Big]\phi_2, \\ \phi_{N+1,t_2} = \dfrac{1}{2}(\langle \Lambda\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \Lambda\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle) - \phi_{N+1}\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle, \\ \psi_{1,t_2} = \Big[\lambda_j^2 - \dfrac{1}{2}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle) + \langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\Big]\psi_1 \\ \qquad + (\lambda_j - \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle)\psi_2 + \Big[\dfrac{1}{2}\psi_{N+1}\lambda_j + \dfrac{1}{2}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle)\Big]\psi_3, \\ \psi_{2,t_2} = -\Big[(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle - \langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle + \phi_{N+1}\psi_{N+1})\lambda_j + \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle\Big]\psi_1 - \Big[\lambda_j^2 - \dfrac{1}{2}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle) \\ \qquad + \langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle\Big)\psi_2 - \Big[\phi_{N+1}\lambda_j - \dfrac{1}{2}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle)\Big]\psi_3, \\ \psi_{3,t_2} = -\Big[\phi_{N+1}\lambda_j - \dfrac{1}{2}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle)\Big]\psi_1 - \Big[\dfrac{1}{2}\psi_{N+1}\lambda_j + \dfrac{1}{2}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle)\Big]\psi_2 \\ \psi_{N+1,t_2} = (\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \Lambda\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle) + \psi_{N+1}\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle, \end{cases} \tag{30}$$

where $1 \le j \le N$. It is a direct but tedious check that the nonlinearized super system given in Eq. (30) can be written in the following super-Hamiltonian form

$$\begin{cases} \boldsymbol{\Phi}_{1,t_2} = \dfrac{\partial H_2}{\partial \boldsymbol{\Psi}_1}, \quad \boldsymbol{\Phi}_{2,t_2} = \dfrac{\partial H_2}{\partial \boldsymbol{\Psi}_2}, \quad \boldsymbol{\Phi}_{3,t_2} = \dfrac{\partial H_2}{\partial \boldsymbol{\Psi}_3}, \quad \phi_{N+1,t_2} = \dfrac{\partial H_2}{\partial \psi_{N+1}}, \\ \boldsymbol{\Psi}_{1,t_2} = -\dfrac{\partial H_2}{\partial \boldsymbol{\Phi}_1}, \quad \boldsymbol{\Psi}_{2,t_2} = -\dfrac{\partial H_2}{\partial \boldsymbol{\Phi}_2}, \quad \boldsymbol{\Psi}_{3,t_2} = \dfrac{\partial H_2}{\partial \boldsymbol{\Phi}_3}, \quad \psi_{N+1,t_2} = \dfrac{\partial H_2}{\partial \phi_{N+1}}, \end{cases} \tag{31}$$

where the super-Hamiltonian function H_2 is given by

$$\begin{aligned} H_2 = &-\langle \Lambda^2\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle + \langle \Lambda^2\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle + \frac{1}{4}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle)^2 - \langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle) \\ &- \langle \Lambda\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle + \phi_{N+1}\psi_{N+1}\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle + \frac{1}{2}\psi_{N+1}(\langle \Lambda\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \Lambda\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle) + \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle \\ &- \frac{1}{2}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle)(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle) + \phi_{N+1}(\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \Lambda\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle). \end{aligned}$$

That is to say, for t_2-part, the nonlinearized super system given in Eq. (30) is a finite-dimensional super-Hamiltonian system.

In the following, we prove that for any $n \geq 2$, the super system given in Eq. (22) can be nonlinearized and furthermore, the obtained nonlinearized system is a finite-dimensional super-Hamiltonian system. Therefore, making use of Eq. (20), we obtain the constrained A_i, B_i, C_i, ρ_i, δ_i $(1 \leq i \leq N)$ for the system given in Eq. (22). Only for differentiation, $\tilde{P}(\boldsymbol{U})$ denotes the new expression generated from $P(\boldsymbol{U})$ by using the nonlinear constraint (Eq. (24)), i.e. $(i \geq 1)$

$$\begin{cases} \tilde{A}_{i+1} = \dfrac{1}{2}(\langle \Lambda^{i-1}\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \Lambda^{i-1}\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle), \\ \tilde{B}_{i+1} = \langle \Lambda^{i-1}\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle, \\ \tilde{C}_{i+1} = \langle \Lambda^{i-1}\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle, \\ \tilde{\rho}_{i+1} = -\dfrac{1}{2}(\langle \Lambda^{i-1}\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \Lambda^{i-1}\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle), \\ \tilde{\delta}_{i+1} = \dfrac{1}{2}(\langle \Lambda^{i-1}\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \Lambda^{i-1}\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle). \end{cases} \tag{32}$$

Inserting Eq. (32) into Eq. (22), we have

$$\begin{cases} \phi_{1j,t_n} = \Big(\sum\limits_{i=0}^{n} \tilde{A}_i\lambda_j^{n-i} - \tilde{C}_{n+1}\Big)\phi_{1j} + \sum\limits_{i=0}^{n} \tilde{B}_i\lambda_j^{n-i}\phi_{2j} + \sum\limits_{i=0}^{n} \tilde{\rho}_i\lambda_j^{n-i}\phi_{3j}, \\ \phi_{2j,t_n} = \sum\limits_{i=0}^{n} \tilde{C}_i\lambda_j^{n-i}\phi_{1j} + \Big(-\sum\limits_{i=0}^{n} \tilde{A}_i\lambda_j^{n-i} + \tilde{C}_{n+1}\Big)\phi_{2j} + \sum\limits_{i=0}^{n} \tilde{\delta}_i\lambda_j^{n-i}\phi_{3j}, \\ \phi_{3j,t_n} = \sum\limits_{i=0}^{n} \tilde{\delta}_i\lambda_j^{n-i}\phi_{1j} - \sum\limits_{i=0}^{n} \rho_i\lambda_j^{n-i}\phi_{2j}, \\ \phi_{N+1,t_n} = \dfrac{1}{2}(\langle \Lambda^{n-1}\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \Lambda^{n-1}\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle) - \phi_{N+1}\langle \Lambda^{n-1}\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle, \\ \psi_{1j,t_n} = \Big(-\sum\limits_{i=0}^{n} \tilde{A}_i\lambda_j^{n-i} + \tilde{C}_{n+1}\Big)\psi_{1j} - \sum\limits_{i=0}^{n} \tilde{C}_i\lambda_j^{n-i}\psi_{2j} + \sum\limits_{i=0}^{n} \tilde{\delta}_i\lambda_j^{n-i}\psi_{3j}, \\ \psi_{2j,t_n} = -\sum\limits_{i=0}^{n} \tilde{B}_i\lambda_j^{n-i}\psi_{1j} + \Big(\sum\limits_{i=0}^{n} \tilde{A}_i\lambda_j^{n-i} - \tilde{C}_{n+1}\Big)\psi_{2j} - \sum\limits_{i=0}^{n} \tilde{\rho}_i\lambda_j^{n-i}\psi_{3j}, \\ \psi_{3j,t_n} = -\sum\limits_{i=0}^{n} \tilde{\rho}_i\lambda_j^{n-i}\psi_{1j} - \sum\limits_{i=0}^{n} \tilde{\delta}_i\lambda_j^{n-i}\psi_{3j}, \\ \psi_{N+1,t_n} = (\langle \Lambda^{n-1}\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \Lambda^{n-1}\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle) + \psi_{N+1}\langle \Lambda^{n-1}\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle. \end{cases} \tag{33}$$

Next, we show that the nonlinearized super system given in Eq. (33) is a finite-dimensional super-Hamiltonian system. From Eq. (32), we know that the constrained co-adjoint representation equation $\boldsymbol{V}_x = [\boldsymbol{U}, \boldsymbol{V}]$ is still satisfied, the equality $\boldsymbol{V}_x^2 = [\boldsymbol{U}, \boldsymbol{V}^2]$ is also satisfied. Therefore, let

$$F = \frac{1}{2}\mathrm{Str}\boldsymbol{V}^2 = \tilde{A}^2 + \tilde{B}\tilde{C} + 2\tilde{\rho}\tilde{\delta}. \tag{34}$$

It is not difficult to find that $F_x = 0$, which means that F is a generating function of integrals of motion for the nonlinearized spatial system given in Eq. (21). Let $F = \sum\limits_{n\geq 0} F_n\lambda^{-n}$, substituting $\tilde{A}_0 = -1$, $\tilde{B}_0 = \tilde{C}_0 = \tilde{\rho}_0 = \tilde{\delta}_0 = \tilde{A}_1 = 0, \tilde{B}_1 = -(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle - \phi_{N+1}\psi_{N+1})$, $\tilde{C}_1 = -1$, $\tilde{\rho}_1 = \phi_{N+1}$, $\tilde{\delta}_1 = \dfrac{1}{2}\psi_{N+1}$ and Eq. (24) into Eq. (34), we can obtain

$$\begin{aligned} F_0 &= 1, \quad F_1 = 0, \quad F_2 = 0, \\ F_3 &= -(\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \Lambda\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle) - (\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle + \phi_{N+1}\psi_{N+1})\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle \\ &\quad + \phi_{N+1}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle) - \frac{1}{2}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle)\psi_{N+1}, \\ F_4 &= -(\langle \Lambda^2\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \Lambda^2\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle) - (\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle - \phi_{N+1}\psi_{N+1})\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle + \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_2\rangle \\ &\quad - \langle \Lambda\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_1\rangle + \phi_{N+1}(\langle \Lambda\boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \Lambda\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle) - \frac{1}{2}(\langle \Lambda\boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \Lambda\boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle)\psi_{N+1} \\ &\quad - \frac{1}{2}(\langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_3\rangle - \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_1\rangle)(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_3\rangle + \langle \boldsymbol{\Psi}_3, \boldsymbol{\Phi}_2\rangle) + \frac{1}{4}(\langle \boldsymbol{\Psi}_1, \boldsymbol{\Phi}_1\rangle - \langle \boldsymbol{\Psi}_2, \boldsymbol{\Phi}_2\rangle)^2, \end{aligned}$$

$$
\begin{aligned}
F_n =& \sum_{i=2}^{n-2}\Big[\frac{1}{4}(\langle\Lambda^{i-2}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1\rangle-\langle\Lambda^{i-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2\rangle)(\langle\Lambda^{n-i-2}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1\rangle-\langle\Lambda^{n-i-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2\rangle)\\
&-\frac{1}{2}(\langle\Lambda^{i-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_3\rangle-\langle\Lambda^{i-2}\boldsymbol{\Psi}_3,\boldsymbol{\Phi}_1\rangle)(\langle\Lambda^{n-i-2}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_3\rangle+\langle\Lambda^{n-i-2}\boldsymbol{\Psi}_3,\boldsymbol{\Phi}_2\rangle)\\
&+\langle\Lambda^{i-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1\rangle\langle\Lambda^{n-i-2}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2\rangle\Big]-(\langle\Lambda^{n-2}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1\rangle-\langle\Lambda^{n-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2\rangle)\\
&+\phi_{N+1}(\langle\Lambda^{n-3}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_3\rangle+\langle\Lambda^{n-3}\boldsymbol{\Psi}_3,\boldsymbol{\Phi}_2\rangle)-\frac{1}{2}(\langle\Lambda^{n-3}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_3\rangle-\langle\Lambda^{n-3}\boldsymbol{\Psi}_3,\boldsymbol{\Phi}_1\rangle)\psi_{N+1}\\
&-(\langle\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1\rangle-\langle\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2\rangle-\phi_{N+1}\psi_{N+1})\langle\Lambda^{n-3}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2\rangle-\langle\Lambda^{n-3}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1\rangle,\quad n\geq 5. \qquad (35)
\end{aligned}
$$

After a direct calculation, we have

$$
\begin{cases}
\boldsymbol{\Phi}_{1,t_n}=\dfrac{\partial F_{n+2}}{\partial\boldsymbol{\Psi}_1},\quad \boldsymbol{\Phi}_{2,t_n}=\dfrac{\partial F_{n+2}}{\partial\boldsymbol{\Psi}_2},\quad \boldsymbol{\Phi}_{3,t_n}=\dfrac{\partial F_{n+2}}{\partial\boldsymbol{\Psi}_3},\quad \phi_{N+1,t_n}=\dfrac{\partial F_{N+2}}{\partial\psi_{N+1}},\\
\boldsymbol{\Psi}_{1,t_n}=-\dfrac{\partial F_{n+2}}{\partial\boldsymbol{\Phi}_1},\quad \boldsymbol{\Psi}_{2,t_n}=-\dfrac{\partial F_{n+2}}{\partial\boldsymbol{\Phi}_2},\quad \boldsymbol{\Psi}_{3,t_n}=\dfrac{\partial F_{n+2}}{\partial\boldsymbol{\Phi}_3},\quad \psi_{N+1,t_n}=\dfrac{\partial F_{N+2}}{\partial\phi_{N+1}},
\end{cases} \qquad (36)
$$

which means that the nonlinearized temporal system given in Eq. (33) is a super-Hamiltonian system. In conclusion, for any n ($n\geq 1$), the nonlinearized system given in Eq. (33) is a finite-dimensional super-Hamiltonian system. This can be checked pretty easily. For example, we can show one equality in the above system as follows:

$$
\begin{aligned}
\boldsymbol{\Psi}_{2,t_n} =& -\sum_{i=0}^{n}B_i\Lambda^{n-i}\boldsymbol{\Psi}_1+\sum_{i=0}^{n}A_i\Lambda^{n-i}\boldsymbol{\Psi}_2\\
&-C_{n+1}\boldsymbol{\Psi}_2-\sum_{i=0}^{n}\rho_i\Lambda^{n-i}\boldsymbol{\Psi}_3\\
=&(\langle\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1\rangle-\langle\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2\rangle-\phi_{N+1}\psi_{N+1})\Lambda^{n-1}\boldsymbol{\Psi}_1\\
&-\sum_{i=2}^{n}\langle\Lambda^{i-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_1\rangle\Lambda^{n-i}\boldsymbol{\Psi}_1-\Lambda^n\boldsymbol{\Psi}_2\\
&+\frac{1}{2}\sum_{i=2}^{n}(\langle\Lambda^{i-2}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_1\rangle\\
&-\langle\Lambda^{i-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_2\rangle)\Lambda^{n-i}\boldsymbol{\Psi}_2-\langle\Lambda^{n-1}\boldsymbol{\Psi}_1,\boldsymbol{\Phi}_2\rangle\boldsymbol{\Psi}_2\\
&+\frac{1}{2}\sum_{i=2}^{n}(\langle\Lambda^{i-2}\boldsymbol{\Psi}_2,\boldsymbol{\Phi}_3\rangle-\langle\Lambda^{i-2}\boldsymbol{\Psi}_3,\boldsymbol{\Phi}_1\rangle)\Lambda^{n-i}\boldsymbol{\Psi}_3\\
&-\phi_{N+1}\Lambda^{n-1}\boldsymbol{\Psi}_3,\\
=&-\frac{\partial F_{n+2}}{\partial\boldsymbol{\Phi}_2}. \qquad (37)
\end{aligned}
$$

We then prove that the nonlinearized system given in Eq. (33) is completely integrable in the Liouville sense. With the help of the results of the nonlinearization for classical integrable systems,[8,25] it is natural for us to set

$$
f_k=\psi_{1k}\phi_{1k}+\psi_{2k}\phi_{2k}+\psi_{3k}\phi_{3k},\quad 1\leq k\leq N, \qquad (38)
$$

and verify that they are also integrals of motion of the constrained spatial system given in Eq. (21) and temporal system given in Eq. (22). So Eq. (38) is in involution in pair. For the nonlinearized spatial system given in Eq. (21) and the nonlinearized temporal system given in Eq. (22), we choose $3N+1$ integrals of motion

$$
f_1,\ldots,f_N,F_3,\ldots,F_{2N+3}, \qquad (39)
$$

and choose the following Poisson bracket

$$
\begin{aligned}
&\{F,G\}\\
&=\sum_{i=1}^{3}\sum_{j=1}^{N}\left(\frac{\partial F}{\partial\phi_{ij}}\frac{\partial G}{\partial\psi_{ij}}-(-1)^{p(\phi_{ij})p(\psi_{ij})}\frac{\partial F}{\partial\psi_{ij}}\frac{\partial G}{\partial\phi_{ij}}\right)\\
&\quad+\left(\frac{\partial F}{\partial\phi_{N+1}}\frac{\partial G}{\partial\psi_{N+1}}+\frac{\partial F}{\partial\psi_{N+1}}\frac{\partial G}{\partial\phi_{N+1}}\right). \qquad (40)
\end{aligned}
$$

At this time, we still have the equality $\tilde{\boldsymbol{V}}_{t_n}=[\tilde{\boldsymbol{V}}^{(n)},\tilde{\boldsymbol{V}}]$ and after a similar discussion, we know that F is also a generating function of integrals of motion for Eq. (22). Hence F_m ($m\geq 0$) are integrals of motion for the system given in Eq. (22), which implies

$$
\{F_m,F_{n+2}\}=\frac{\partial}{\partial t_n}F_m=0,\quad m,n\geq 0. \qquad (41)
$$

The above equality (Eq. (41)) shows that $\{F_m\}_{m=3}^{2N+3}$ are in involution in pair under the Poisson bracket (Eq. (40)). Similar to the methods in Refs. [22], [25] and [26], we can verify that the $3N+1$ functions in expression (39) are functionally independent at least over some region of the supersymmetric manifold $\mathbb{R}^{4N|2N+2}$. Taking into account the above procedure, it is not difficult to draw the following conclusion.

Theorem 1 The constrained systems given in (Eqs. (21) and (22) are super-Hamiltonian systems,

whose $3N + 1$ integrals of motion (Eq. (39)) are in involution in pair and functionally independent over the supersymmetric manifold $\mathbb{R}^{4N|2N+2}$.

5. Conclusion

In this paper, we present a new finite-dimensional integrable super-Hamiltonian system of the super classical-Boussinesq hierarchy. The symmetry constraint of the super classical-Boussinesq hierarchy for even variables can be explicitly given and the symmetry constraint of the super classical-Boussinesq hierarchy for odd variables can be implicitly given. It provides a new and systematic way to construct a finite-dimensional super Hamiltonian system. The methods in this study can be applied to other super integrable systems. Under constraints of this form, the nonlinearization of the other super systems will be studied in our future work.

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