Glassy behaviour of random field Ising spins on Bethe lattice in external magnetic field

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The thermodynamics and the phase diagram of random field Ising model (RFIM) on Bethe lattice are studied by using a replica trick. This lattice is placed in an external magnetic field (B). A Gaussian distribution of random field ($h_{\rm i}$) with zero mean and variance $\langle h_{\rm i}^2\rangle = H_{\rm RF}^2$ is considered. The free-energy (F), the magnetization (M) and the order parameter (q) are investigated for several values of coordination number (z). The phase diagram shows several interesting behaviours and presents tricritical point at critical temperature $T_{\rm C} = J/k$ and when $H_{\rm RF} = 0$ for finite z. The free-energy (F) values increase as T increases for different intensities of random field ($H_{\rm RF}$) and finite z. The internal energy (U) has a similar behaviour to that obtained from the Monte Carlo simulations. The ground state of magnetization decreases as the intensity of random field $H_{\rm RF}$ increases. The ferromagnetic (FM)–paramagnetic (PM) phase boundary is clearly observed only when $z \to \infty$. While FM–PM-spin glass (SG) phase boundaries are present for finite z. The magnetic susceptibility (χ) shows a sharp cusp at $T_{\rm C}$ in a small random field for finite z and rounded different peaks on increasing $H_{\rm RF}$.

Keywords: random field, replica trick, coordination number, order parameter, susceptibility and phase diagram

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1. Introduction

The magnetic random systems with a frustration in the exchange interactions between the spins exhibit a spin-glass phase at low temperature.[1,2] The Sherrington–Kirkpatrick (SK) model with infinite-ranged interactions was suggested as a mean field model of spin glasses.[3] Parisi found a stable solution using the replica-symmetry-breaking scheme.[4]

The random field Ising model (RFIM) is one of the simplest models of disordered interacting systems. This RFIM describes the behaviour of diluted antiferromagnets in strong magnetic fields.[5] In recent years, there have been many theoretical[6] and experimental indications.[7] The essential difficulties in breaking down the behaviour of the RFIM may be related to the existence of many metastable states and associated glassy phase, similar to that found in spin glass materials.[8]

Standard approaches, such as the Bragg–Williams mean field theory (MFT), when used to examine the RFIM predicted only the existence of the ferromagnetic (FM) and the paramagnetic (PM) phases, but failed to identify a spin-glass (SG) phase.[9] Very recently, by including fluctuations beyond MFT using a $1/N$ approach, Mezard and Young[6] were successful in predicting the SG phase, based on an appropriate replica-symmetry-breaking scheme.

The simplest effect of turning on a weak random field is the resulting depression of the critical temperature for uniform ordering, while for sufficiently strong randomness the ordered phase completely disappears. The more sophisticated theoretical schemes have to be used in order to identify the corresponding instability of the high-temperature paramagnetic phase. Such a theory was formulated by Mezard and Young,[6] who utilized the self-consistent screening (SCS) approach of Bray,[10] and identified the glassy phase by carrying out a replica-symmetry-breaking stability analysis. Similar results were obtained by numerically solving the mean-field equations for a fixed realization of disorder by Lancaster *et al.*,[11] confirming the existence of the glassy phase. Mezard and Monasson[12] and de Dominicis *et al.*[13] presented the arguments that the glass phase should persist even at weak disorder, everywhere and precede the uniform ordering. Pastor *et al.*[14] presented a simplest possible approach that is capable of providing a description of glassy phase. The recent theoretical efforts were made to use the sophisticated numerical

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approaches for studying the behaviours of the order parameter under magnetic field and the temperature for RFIM.[15−17] The phase diagram has been investigated by Monte–Carlo simulation.[18,19] The one-dimensional disordered Ising system is considered to be one of the most important physical problems.[20]

The statistical properties of peculiar responses are studied in glassy systems on the mesoscopic scales based on a class of mean-field spin-glass model which exhibits one step replica-symmetry-breaking.[21] For the RFI model, Belanger[23] and Nattermann[23] discussed experiments and surveyed the theory only about the nearest–neighbour interactions in the absence of an external magnetic field (B). Belanger pointed out that the experiments gave a clear evidence for a finite temperature transition in three dimensions and discussed the observed critical behaviour by noting the difficulties arising from irreversible effects below the critical point. In spin glass models, replica symmetry breaking implies the existence of many pure states of equilibrium. Their hierarchical organization is described by the order parameter (q).[24] In the SK model, q has a continuous form and it is a marginally stable saddle point of the replica free energy surface in the limit $n \to 0$,[25] where n is the number of replicas.

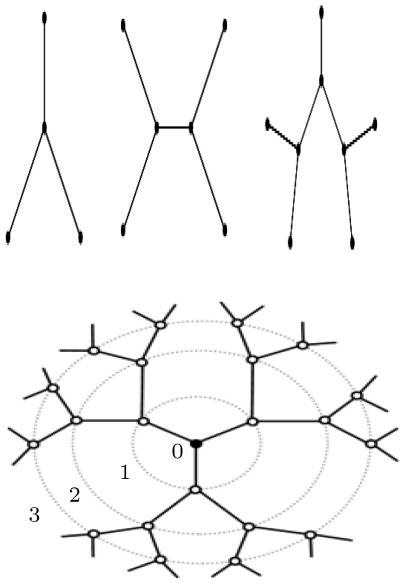


Fig. 1. A Bethe lattice with coordination number $z = 3$.

A Bethe lattice was introduced by Hans Bethe in 1935. It is a connected cycle-free graph, where each node is connected to z neighbours, where $z = 2$, 3,... and is called the coordination number. It can be seen as a tree-like structure emanating from a central node. The central node may be called the "root or origin" of the lattice as shown in Fig. 1. The name "Bethe lattice" originates from the fact that Bethe approximation for the Ising model is exact on this lattice.[26,27] A finite portion of the Bethe lattice is called Cayley tree.

There are two special properties that make the Bethe lattice particularly suited for theoretical investigations. One is its self-similar structure which may lead to recursive solutions and it plays an important role in statistical and condensed-matter physics, because some problems involving disorder and/or interactions can be solved exactly when defined on a Bethe lattice, e.g., Ising models.[26−28] In a recent publication, Bruinsma[29] has obtained a recursion relation for the random-field distribution function in RFIM on a Bethe lattice. He concentrated on the $T = 0$ properties that followed from this recursion relation. Hartztein and Entin-Wohlman[30] explored the finite-temperature phase diagram by analysing the limit of the distribution average. They obtained the first- and second-order transition line and a tricritical point for RFIM on a Bethe lattice. Castelliani *et al.*[31] derived the zero-temperature phase diagram of spin glass models with a generic fraction of ferromagnetic interactions on the Bethe lattice. They used the cavity method at the level of one-step replica-symmetry-breaking (1RSB) and found three phases namely, a replica-symmetric (RS) ferromagnetic one, a magnetized spin glass one (mixed phase) and an unmagnetized spin glass one.[31] Liers *et al.*[32] studied the Ising spin glass on random graph with fixed connectivity z, mean μ and unit variance. They exactly computed ground states by using a sophisticated branch-and-cut method for $z = 4$, 6 and found the spin-glass/ferromagnet phase transition at $\mu = 0.77 \pm 0.02$ ($z = 4$) and $\mu = 0.56 \pm 0.02$ ($z = 6$). Finally, Broges and Silva[33] studied the RFIM with a differential operator method and obtained the phase diagrams for coordination numbers $z = 4$, 6 and showed several interesting behaviours, presenting tricritical points (only for $z = 6$).

In the quantum magnetic dynamics systems, Xie and Liu[34] analysed the properties of the superfluid–Mottinsulator transition of dipolar bosons in an optical lattice by the Bogoliubov transformation and the mean-field decoupling approximation scheme in Ref. [35]. The mean-field theory in a phase transition problem is subject to an approximation that the order parameter is treated, as a constant. It is a useful description if fluctuations are not important. It becomes an exact theory only when the range of interactions

becomes infinite. It nevertheless makes quantitatively correct predictions (e.g., the phase diagram) about some aspects of phase transition in high spatial dimensions and makes qualitatively correct predictions in physical dimensions. In low dimensions (e.g., one dimension), the application of a mean-field calculation could be questionable because of the possibly important role of fluctuations. If the fluctuations are incorporated, the mean-field theory will also give some useful predictions but a definite proof of this requires further study.[35] The mean-field theory has an advantage of being mathematically simple and it is almost invariably the first approach taken to predict phase diagrams and properties of new experimental systems. The results show that the superfluid–Mott-insulator phase transition of dipolar bosons in an optical lattice can be tuned by the dipole–dipole interaction as well as the interatomic repulsion and the hopping term. So dipolar boson system in an optical lattice gives us an additional tool to study the quantum phase transition. Ji *et al.*[36] described an optical system that allowed for a direct experimental observation of the quantum magnetic correlated dynamics of polarized light. By adjusting the Zeeman and the Raman fields, they could realize a ferromagnetic phase, supercounter-fluidity phase and antiferromagnetic phase of polarized light, which are of interest for studying spin-dependent photon–photon interactions. They also designed an experimental protocol for the observation of these phases. Moreover, the technique of controlling photospin correlation may be used for building quantum information devices. Liu *et al.*[37] showed that the Riemann–Hilbert approach is the sufficiently more simple and elegant way to construct the multidomain wall states of an anisotropic (easy-plane) ferromagnet. The important point is that these solutions are quite nontrivial, for example, includes moving domain walls.

In this work, we present the thermodynamic properties for the RFIM on Bethe lattice by using the replica trick for different coordination numbers (z). Gaussian distribution of the local random field is considered. The phase diagram, the critical temperature and the thermodynamic magnetic parameters (magnetization, order parameter and susceptibility) are investigated. The mean-field theory (MFT) enables us to identify the ferromagnetic (FM), paramagnetic (PM) and spin-glass (SG) phases and study its properties for high disorder. We also study the effect of the static external magnetic field (B) on the same thermodynamic parameters for the RFIM. The remainder of the present paper is organized as follows. In Section 2, we describe the details of the model and use the replica trick to obtain the free-energy (F). We calculate in Section 3 magnetization (M) and order parameter (q) self-consistently. We deduce the equation of the critical temperature at which the phase transition occurs. In Section 4, we analyse our results for RFIM on Bethe lattice in the presence of an external magnetic field. We discuss and compare our results with other numerical simulations obtained before. Finally, in Section 5 we summarize some conclusions from the present study.

2. Model

We consider a Bethe lattice of N Ising spins $S_i = \pm 1$ which are coupled by finite constant interactions. The Hamiltonian of the random field Ising model (RFIM) is given by

$$\mathscr{H} = -\sum_{ij} J_{ij} S_i S_j - \sum_{i=1}^{N} H_i S_i - B \sum_{i=1}^{N} S_i, \qquad (1)$$

where $J_{ij} = j/z$ is uniform ferromagnetic interaction between nearest-neighbour sites, rescaled with the coordination number (z) and B is the external magnetic field applied to our system. The random field variables $\{H_i\}$ are Gaussian-distributed, with zero mean and variance $\left[H_i^2\right]_{\rm av} = H_{\rm RF}^2$ and we take $H_{\rm RF}$ as initial constant values.

Thermodynamic properties of the model can be calculated from the ensemble averaged free energy per spin

$$F(\beta) = -\frac{1}{\beta N} \lim_{N\to\infty} \ln Z(\beta) \qquad (2)$$

with the partition function

$$Z = \sum_{\{S_i\}} \exp\left(-\beta \mathscr{H}(S_i)\right), \qquad (3)$$

where $\sum_{\{S_i\}}$ means the sum over all states of configurations, $\beta = 1/kT$, with k being the Boltzmann constant and T the absolute temperature. We calculate the average quenched free energy. It is very difficult to compute $[\ln Z]_{\rm av}$ directly. In order to overcome this difficulty, we use the replica trick[1] since the summation runs over the disorder system

$$\ln Z = \lim_{n\to 0} \left(\frac{Z^n - 1}{n}\right), \qquad (4)$$

where Z^n is the partition function of the $\alpha^{\rm th}$ replica, $\alpha = 1, \ldots, n$, with n being the number of replicas. We calculate Z^n by replicating the system n times. Thus we introduce n identical replicas of the original

system. By expanding Z^n in power of the interaction J/z, hence, one can obtain the partition function as follows:

$$Z^n = \mathrm{Tr}_{\{S_i^\alpha\}} \exp\left[\frac{\beta J}{z}\sum_{ij}\sum_{\alpha} S_i^\alpha S_j^\alpha + \left(\frac{\beta^2 J^2}{z^2} + \frac{1}{2}(\beta H_{\mathrm{RF}})^2\right) \times \sum_{ij}\sum_{\alpha\beta} S_i^\alpha S_i^\beta S_j^\alpha S_j^\beta + B\sum_{i,\alpha} S_i^\alpha\right], \quad (5)$$

where α, $\beta = 1, \ldots, n$, with the trace extending over all states of a single replicated spin $\{S_i^\alpha\}$.

By using the following inequalities (the replicas mean-field theory:[1])

$$\sum_{ij} s_i^\alpha s_j^\alpha \cong z\sum_i \left(s_i M - \frac{M^2}{2}\right),$$
$$M_i^\alpha = \langle s_i^\alpha\rangle = M\forall i,$$
$$\sum_{(ij)} s_i^\alpha s_j^\alpha s_i^\beta s_j^\beta \cong z\sum_{ij}\left(q s_i^\alpha s_i^\beta - \frac{q^2}{2}\right),$$
$$q_i^{\alpha\beta} = \left\langle s_i^\alpha s_i^\beta\right\rangle = q\forall i.$$

We can formally average over the disorder and express the free-energy, in thermodynamic limit, as

$$\beta F = -\lim_{N\to\infty}\lim_{n\to 0}(Nn)^{-1} \times\left\{\mathrm{Tr}\exp\left[\frac{\beta J}{2z}\sum_{\alpha}\left(\sum_i S_i^\alpha\right)^2 + \left(\frac{\beta^2 J^2}{2z^2} + \frac{1}{4}(\beta H_{\mathrm{RF}})^2\right)\sum_{(\alpha\beta)}\left(\sum_i S_i^\alpha S_i^\beta\right)^2 + \beta B\sum_\alpha\sum_i S_i^\alpha\right] - 1\right\}, \quad (6)$$

where $(\alpha\beta)$ refers to the combination between α and β with $\alpha \neq \beta$. Note that the exchange terms in the exponent are in the form $\lambda\left(\sum_i O_i\right)^2$, where O_i is local intensive operator, which leads to physical thermodynamic consequence only if $\lambda \propto N^{-1}$. By applying the Hubbard–Stratononvitch identity to the squared terms which is given by

$$\exp(\lambda a^2) = \left(\frac{N}{2\pi}\right)^{1/2}\int \mathrm{d}x \times \exp\left(-\frac{Nx^2}{2} + (2\lambda N)^{1/2} a x\right), \quad (7)$$

and using the single-spin property,

$$\mathrm{Tr}_n \exp\left(g\sum_i S_i^\alpha\right) = \exp(N\ln \mathrm{Tr}_n \exp g(S^\alpha)). \quad (8)$$

On the assumption that the limit $n \to 0$ and the thermodynamic limit $N \to \infty$ can be interchanged, we can drop terms which vanish in the thermodynamic limit. For $N \to \infty$, the integral in Eq. (7) can be done by the steepest descents method, in which the integral is dominated by the region of maximum integrand. Since the replicas are indistinguishable. Then we can obtain the final form of free energy as follows:

$$\beta F = -\left[\frac{1}{2}\beta J M^2 + \frac{1}{2z}(\beta J)^2\left(q - M^2 - \frac{q^2}{2}\right) - \frac{1}{\sqrt{2\pi}}\int \exp\left(-\frac{x^2}{2}\right)\ln\left(2\cosh\left(\beta B + \beta J M + \left((\beta H_{\mathrm{RF}})^2 + \frac{\beta^2 J^2}{z}(q - M^2)\right)^{1/2} x\right)\right)\mathrm{d}x\right]. \quad (9)$$

3. Thermodynamic parameters

The magnetization (M), the order parameter (q), the susceptibility (χ) and the internal energy (U) are calculated as functions of temperature (T) and random field (H_{RF}).

By differentiating Eq. (9) with respect to B we find that the magnetization

$$M(T) = \frac{1}{\sqrt{2\pi}}\int \exp\left(-\frac{x^2}{2}\right)\tanh\left(\beta B + \beta J M + \left((\beta H_{\mathrm{RF}})^2 + \frac{\beta^2 J^2}{z}(q - M^2)\right)^{1/2} x\right)\mathrm{d}x. \quad (10)$$

It is a self-consistent equation which can be solved with the saddle-point conditions. The mean-square of frozen moment (q) per site is given by

$$q(T) = \frac{1}{\sqrt{2\pi}}\int \exp\left(-\frac{x^2}{2}\right)\tanh^2\left(\beta B + \beta J M + \left((\beta H_{\mathrm{RF}})^2 + \frac{\beta^2 J^2}{z}(q - M^2)\right)^{1/2} x\right)\mathrm{d}x. \quad (11)$$

In the $z \to \infty$ limit and for $B = 0$, one can obtain the straightforward generalization equation of the well-known Bragg–Williams's condition to include the effect of random fields. This is sufficient to determine the ferromagnetic (FM) phase boundary which is determined by setting M=0.

We can find the zero-temperature magnetization (the ground-state magnetization M $(T = 0)$) to be

$$M(0) = \mathrm{erf}\left[\frac{1}{\sqrt{2}}\left(\frac{J}{H_{\mathrm{RF}}}M(0) + \frac{B}{H_{\mathrm{RF}}}\right)\right]. \quad (12)$$

The critical temperature where the magnetization vanishes is easily computed by differentiating the term on the right-hand side of Eq. (10) with respect to M by putting both M and B to be zero. The FM phase boundary is obtained by being located at a given value of H_{RF} where the magnetization M vanishes. The temperature phase diagram is given by

$$\frac{kT}{J} = \frac{1}{\sqrt{2\pi}}\int \exp\left(\frac{-x^2}{2}\right) \times \mathrm{sech}^2\left(\left((\beta H_{\mathrm{RF}})^2 + \frac{\beta^2 J^2}{z}q\right)^{1/2} x\right)\mathrm{d}x. \quad (13)$$

From this equation, we can plot the phase diagram by differentiating the term on the right-hand side of Eq. (13) with respect to temperature at $T = T_{\mathrm{C}} = 0$. The zero solution occurs when $H_{\mathrm{RF}}/J \geq \sqrt{2/\pi}$ which means that the system becomes SG for all T values. The identification of the SG phase is more difficult. Similarly as for the spin glass models in a uniform external field, in our case the Edward–Anderson order parameter (q) is nonzero for any temperature and thus cannot be used to determine the phase boundary. In our case, the random magnetic field plays a role of a source conjugate to the order parameter, locally breaking the up-down symmetry. The situation is similar to that in spin-glass model in a uniform external field,[24] where the replica symmetric order parameter q remains nonzero for any temperature and thus cannot be used to identify glassy freezing. We follow Mezard and Young[6] and look for instability to replica symmetry breaking (RSB) within the paramagnetic phase. To this, we can set $M = 0$ and note that remaining equation for q is in fact identical to that describing the Sherrington–Kirkpatrick model in the presence of magnetic field. This model is also described with the first two terms of the Hamiltonian of Eq. (1), but this time with J. J_{ij}'s are Gaussian random variables with zero mean and variance $\langle J_{ij}^2 \rangle = J^2/zN$.

By differentiating Eq. (10) with respect to external magnetic field (B) and taking the limit $B \to \infty$, the susceptibility (χ) is given by

$$\chi(T) = \frac{1-q}{kT - J(1-q)}. \quad (14)$$

We use the adaptive Simpson quadrature method to compute the integral in Eq. (14) with MATLAB package. We are able to obtain numerical results for the internal energy (U) by differentiating Eq. (9) with respect to β with using central difference method.

4. Results and discussion

The thermodynamic parameters mentioned above are calculated by considering the Gaussian distribution of the local random field. In the absence of external magnetic field (B), we find that the free-energy (F) is constant at low temperature and then linearly increases with temperature increasing, whatever the random field and the coordination number (z) are as shown in Fig. 2. This result accords with that obtained in Ref. [38].

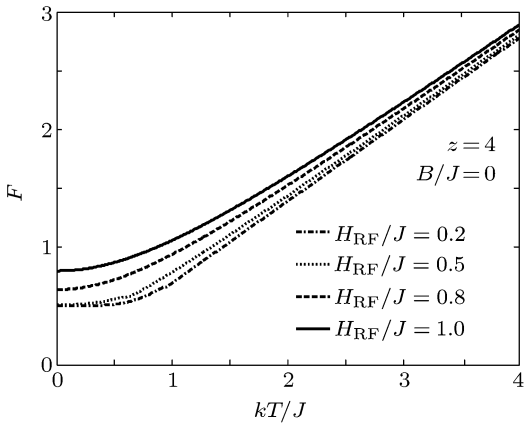


Fig. 2. Curves for free energy (F) versus kT/J at different strengths of random field (H_{RF}/J) and at $z = 4$.

It is clear that equations (10) and (11) are self-consistent. For this reason, we solve them numerically by using the Newton–Raphson method (NRM). The values of M and q are obtained as functions of temperature (T) and random field (H_{RF}) at different values of z and B.

When M and q each are a function of random field (H_{RF}/J). We note that M and $q^{1/2}$ curves are almost coincident for a weak random field but on increasing H_{RF}. Their difference increases until the critical temperature (T_{C}) at which the system becomes SG (i.e. $M = 0$, $q \neq 0$) whatever coordination number (z)

is as shown in Fig. 3. We also find that the curves of both M and $q^{1/2}$ are both separated. We notice that $q^{1/2}$ curve is constant at T=0. On increasing the temperature the curve of $q^{1/2}$ decreases until the critical temperature and then increases on increasing $H_{\rm RF}$ and to converges to one. But at $kT/J = 1$, the system becomes SG for all values of $H_{\rm RF}$. In the presence of B, the effect of the weak external magnetic field is observed as shown in Fig. 3(b). The system has a slow phase transition from FM to SG. But at $B/J = 1$, we note that M and $q^{1/2}$ curves are constant at low $H_{\rm RF}$ and decreases on increasing $H_{\rm RF}$. The difference between M and $q^{1/2}$ curves decreases until the system becomes purly ferromagnet at $B/J = 1$ as shown in Fig. 3(d). The behaviours of M and $q^{1/2}$ curves are the same whatever value of z is. Furthermore, the magnetization increases on increasing the coordination number (z).

When M and q each are a function of kT/J, we note that M and $q^{1/2}$ curves are almost constant at low temperature. In the absence of B, M and $q^{1/2}$ decrease on increasing temperature until the critical temperature ($T_{\rm C}$) at which the system becomes SG (i.e. $M = 0$ and $q \neq 0$), whatever coordination number (z), as indicated in Fig. 4(a). In the presence of B, we find that M and $q^{1/2}$ linearly increase with the increase of external magnetic field (B). Also, we find that $M = q^{1/2}$ as $B/J = 1$. These results accord with those obtained in Refs. [39] and [40]. Because they found that a finite external field removes sharp phase transitions by allowing M and $q^{1/2}$ to be nonzero at all temperatures. On increasing $H_{\rm RF}$, the curve of M and $q^{1/2}$ are separated. This separation decreases with B increasing. The M and $q^{1/2}$ curves are almost coincident until the system becomes purely ferromagnet at $B/J = 1$ as shown in Fig. 4(b). The separation between M- and $q^{1/2}$-curves increases at $H_{\rm RF} \geq \sqrt{2/\pi}J$ as shown in Figs. 4(c) and 4(d). The behaviour of M and $q^{1/2}$ curves is the same whatever the value of z is. Furthermore, the magnetization increases on increasing the coordination number (z).

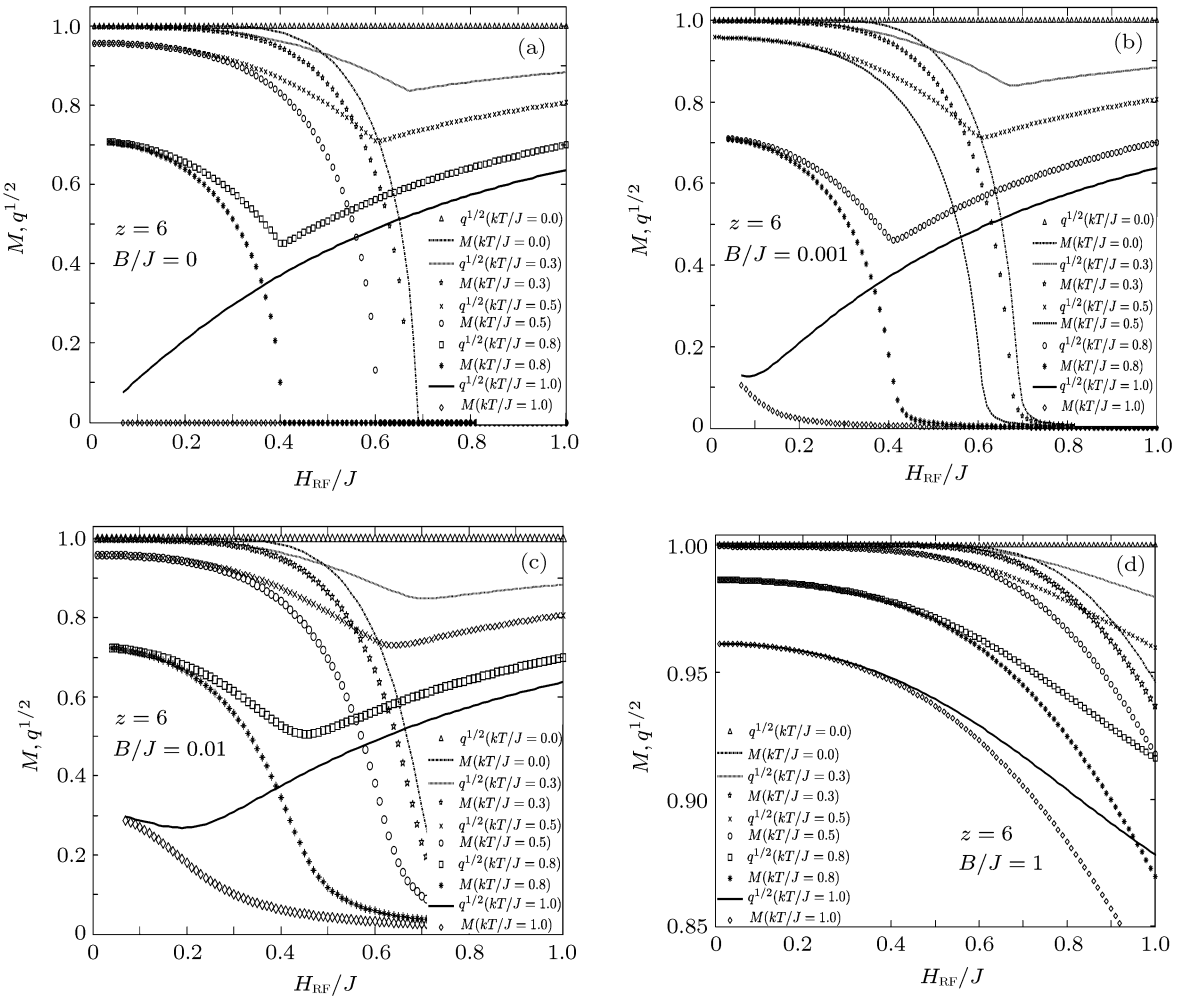


Fig. 3. Curves for magnetization (M) and the order parameter ($q^{1/2}$) versus $H_{\rm RF}/J$ at different values of kT/J in the presence of magnetic field (B/J) at $z = 6$. (a) $B/J = 0.0$, (b) $B/J = 0.001$, (c) $BJ = 0.01$, and (d) $B/J = 1.0$.

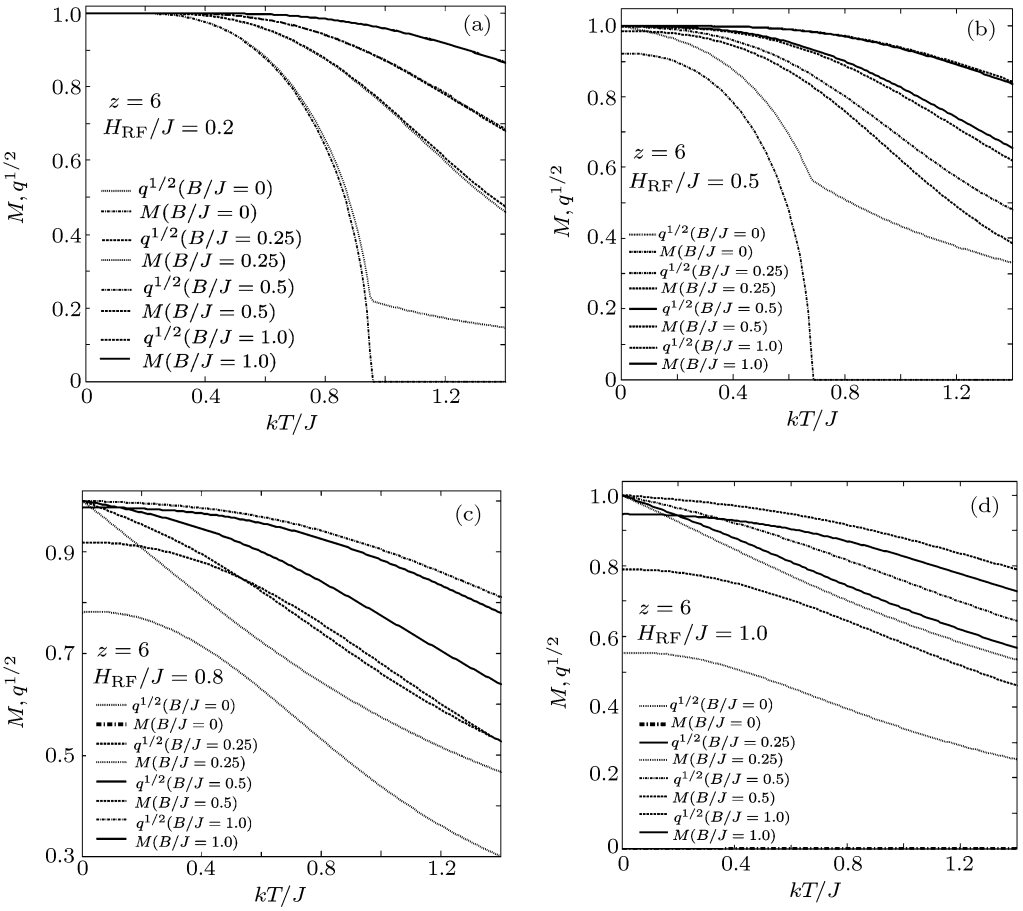


Fig. 4. Curves for the magnetization (M) and the order parameter ($q^{1/2}$) versus kT/J at different values of B/J in the presence of $H_{\rm RF}/J = 0.2$ (a), 0.5 (b), 0.8 (c), and 1.0 (d).

The ground-state of magnetization on a Bethe lattice decreases and tends to zero with the increase random field ($H_{\rm RF}$) as shown in Fig. 5. The maximum value of M ($T = 0$) is equal to one at $H_{\rm RF} = 0.11$. We find that there are groups of lines ($n\,(l)$). Each group contains a number of lines and these lines increase on increasing coordination number (z) as shown in Fig. 5. These groups grow at 0.1 of the random field ($H_{\rm RF}$). The following Table 1 shows the number of lines and the interval of the coordination number (z).

Table 1. The number of lines and the interval of the coordination number z.

Number of lines $n(l)$	The interval of the coordination number (z)
1	$2 \le z \le 9$
2	$10 \le z \le 11$
2	$12 \le z \le 13$
3	$14 \le z \le 16$
6	$17 \le z \le 22$
12	$23 \le z \le 35$
64	$36 \le z \le 99$
∞	$z \ge 100$

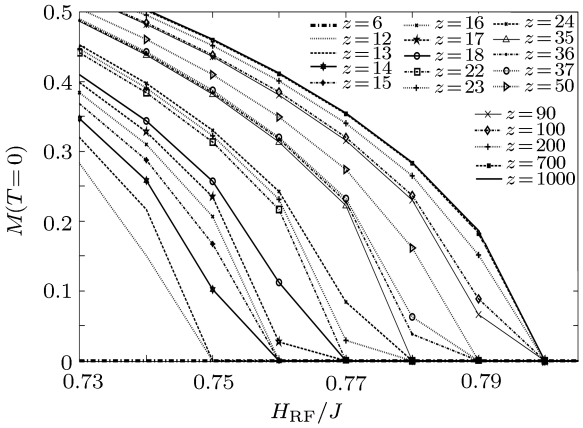


Fig. 5. Curves for the ground-state magnetization $M(T = 0)$ versus $H_{\rm RF}/J$ with the ground state for the RFIM at different coordination number z.

The critical temperature at which occurs the phase transition from FM to SG decreases as random field increases. By differentiating the right-hand side of Eq. (13) with respect to T at $T = T_{\rm C} = 0$, we find that the zero solution of equation occurs as a unique solution when $H_{\rm RF}/J \ge \sqrt{2/\pi}$. The identification of the SG phase is more difficult. Similarly as for the

spin glass model in a uniform external field, in our case the Edward–Anderson order parameter (q) is nonzero for any temperature and thus cannot be used to determine the phase boundary in our case and the random magnetic field plays a role of a source conjugate to the order parameter, locally breaking the up-down symmetry. This situation is similar to that in spin-glass model in a uniform external field,[24] where the replica symmetric order parameter q remains nonzero for any temperature and thus cannot be used to identify glassy freezing. We follow Mezard and Young[6] and look for instability to replica symmetry breaking (RSB) in the paramagnetic phase. To this, we can set $M = 0$ and note that the remaining equation for q is in fact identical to that describing the Sherrington–Kirkpatrick model in the presence of magnetic fields. This model is also described with the first two terms of the Hamiltonian of Eq. (1), but this time with j. J_{ij}'s are Gaussian random variables with zero mean and variance $\langle J_{ij}^2 \rangle = J^2/zN$.

From the phase diagram in Fig. 7, we obtain that (i) the glass phase transition does not exist for $z = \infty$ as shown in Fig. 6. This result is the same as that obtained in Ref. [28]; (ii) the $T_{\rm C}$ decreases as $H_{\rm RF}$ increases in the boundary of phase transition from FM to PM; (iii) our RF system behaves as PM when $H_{\rm RF}/J \geq \sqrt{2/\pi}$, whatever the T is; (iv) we note that there is a phase transition from FM to PM when $B = 0$ as shown in Fig. 6; (v) the critical temperature ($T_{\rm C}$) increases on increasing the coordination number (z).

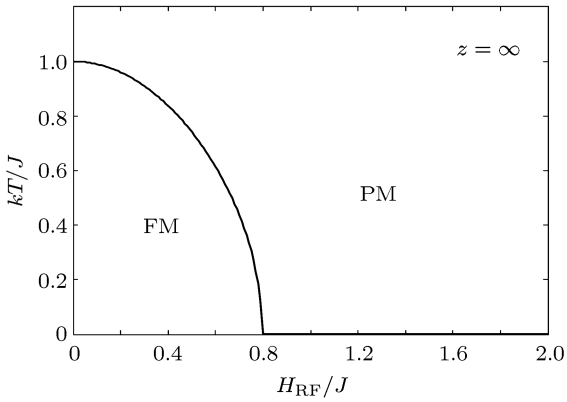


Fig. 6. Magnetic phase diagram for RFIM at $z = \infty$ and $B = 0$.

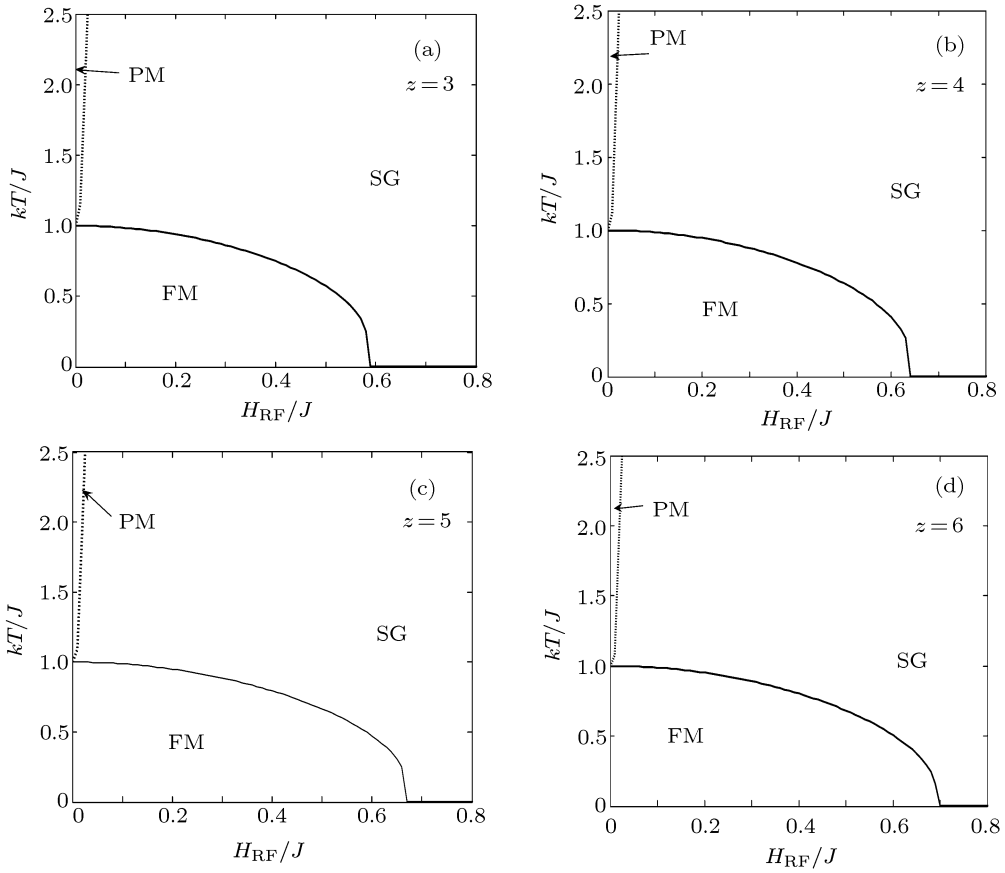


Fig. 7. Magnetic phase diagram for RFIM with different coordination number (z) at $B = 0$: (a) $z = 3$, (b) $z = 4$, (c) $z = 5$, and (d) $z = 6$.

When the coordination number is finite, we note that ferromagnetic (FM), paramagnetic (PM) and spin-glass (SG) phases are found as shown in Fig. 7 at $z = 3$, 4, 5, 6. The critical temperature decreases as $H_{\rm RF}$ increases in the boundary of phase transition from FM to SG. The phase diagram shows several interesting behaviours and presents a tricritical point at critical temperature $T_{\rm C} = J/k$ and $H_{\rm RF} = 0$.

In the absence of external magnetic field, we find that the susceptibility (χ) has a sharp cusp at $z = 6$ as shown in Fig. 8(a). This cusp decreases with temperatures increasing and vanishes at the critical temperature $T_{\rm C} = J/k$. The susceptibility tends to zero at high temperature and random field. In the presence of external magnetic field, χ shows a rounded peak. This peak decreases and moves at a high temperature as shown in Figs. 8(b)–8(d). The rounded peak moves to the left on increasing B. The maximum of the peak lowers with B increasing. This result accords with that obtained in Ref. [40] as well as with that obtained by Ismail.[41] He found that real and imaginary parts of the dynamic susceptibility display maxima at $T = T_{\rm C}$. These maxima can be described by an Arrhenius law. The susceptibility shows a similar behaviour whatever the coordination number (z).

We note that the internal energy (U) is almost constant at low temperature. With the further increase of temperature, U increases as T increases as shown in Fig. 9. In the absence of B, U decreases with $H_{\rm RF}$ increasing and has a singular point at $T_{\rm C}$ as shown in Fig. 9. This singularity decreases with the increase of coordination number (z). Then it decreases linearly as T increases when $H_{\rm RF}/J \geq \sqrt{2/\pi}$ for different z values. This result is in agreement with that obtained in Ref. [40].

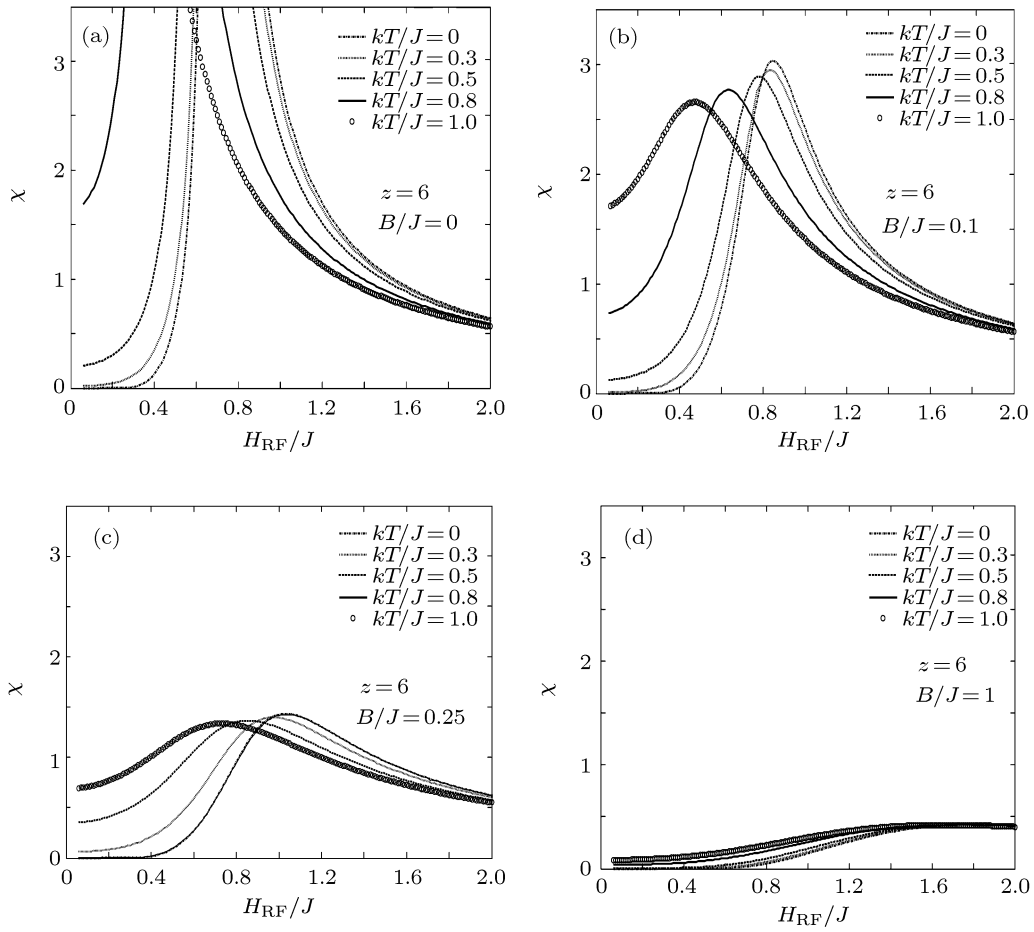


Fig. 8. Curves for the susceptibility (χ) versus $H_{\rm RF}/J$ at different values of kT/J in the presence of $B/J = 0$ (a), 0.1 (b), 0.25 (c), and 1.0 (d) for $z = 6$.

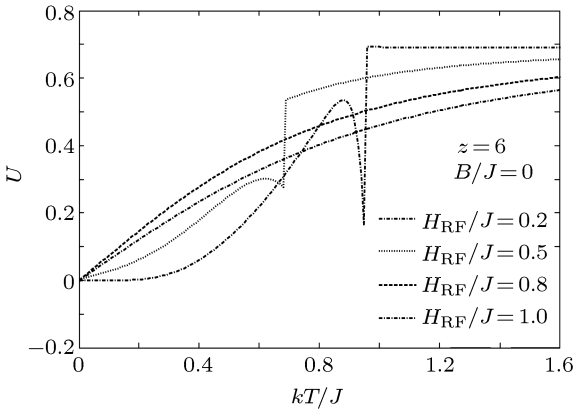


Fig. 9. Curves for the internal energy ($U(T)$) versus kT/J at different values of $H_{\rm RF}/J$ in the absence of B/J with the ground state for the RFIM at $z = 6$.

5. Conclusions

We can conclude that in the present Bethe-lattice Ising model with random field, the critical temperature is $T_{\rm C} = J/k$ at a critical strength $H^{\rm c}_{\rm RF}/J = \sqrt{2/\pi}$ of the random field. The critical temperature decreases with $H_{\rm RF}$ increasing. The phase diagram shows several interesting behaviours and presents a tricritical point at critical temperature $T_{\rm C} = J/k$ and when $H_{\rm RF} = 0$. When the coordination number (z) is finite the ferromagnetic (FM), paramagnetic (PM) and spin-glass (SG) phase boundaries exist. But for $z = \infty$, only the FM–PM phase boundary exists. Bethe lattice with $z \to \infty$ has the same phase diagram as to the spherical lattice.

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