Spin symmetry in the relativistic modified Rosen–Morse potential with the approximate centrifugal term

Chen Wen-Li(陈文利)[a)†] and Wei Gao-Feng(卫高峰)[b)‡]

[a)] *Department of Basic Science, Xi'an Peihua University, Xi'an* 710065, *China*

[b)] *Department of Physics and Electro-optics Engineering, Xi'an University of Arts and Science, Xi'an* 710065, *China*

(Received 25 July 2010; revised manuscript received 20 December 2010)

By applying a Pekeris-type approximation to the centrifugal term, we study the spin symmetry of a Dirac nucleon subjected to scalar and vector modified Rosen–Morse potentials. A complicated energy equation and associated two-component spinors with arbitrary spin–orbit coupling quantum number k are presented. The positive-energy bound states are checked numerically in the case of spin symmetry. The relativistic modified Rosen–Morse potential cannot trap a Dirac nucleon in the limiting case $\alpha \to 0$.

Keywords: modified Rosen–Morse potential, Dirac equation, spin symmetry

PACS: 21.10.–k, 03.65.–w **DOI:** 10.1088/1674-1056/20/6/062101

1. Introduction

The concepts of spin and pseudospin symmetries have been suggested since 1969 in nuclear theory,[1,2] which were considered in the context of deformed nuclei[3] and super-deformation,[4] magnetic moment interpretation,[5−7] identical bands,[8−10] and effective shell-model coupling scheme.[11] Since the introduction of the spin and pseudospin symmetry concepts, some contribution to this problem has been made both in nuclear theory and in quantum theory. For example, six years later, Bell and Ruegg found that pseudospin symmetry is a relativistic symmetry of a Dirac Hamiltonian that occurs when the scalar and vector potentials are equal in magnitude, but opposite in sign.[12] Ginocchio showed that pseudospin symmetry arises from a relativistic symmetry of the Dirac Hamiltonian,[13−16] and is identified as an $SU(2)$ symmetry as discovered in Ref. [12] and also showed that pseudospin symmetry is exact when $\Sigma = V(r) + S(r) = 0$ or a constant. Moreover, Meng *et al.* have restudied this symmetry[17] when $\mathrm{d}(V(r) + S(r))/\mathrm{d}r = 0$. On the other hand, Zhou *et al.*[18] found that spin symmetry is exact when $\mathrm{d}(V(r) - S(r))/\mathrm{d}r = 0$ and also spin symmetry in anti-nucleon spectra has the same origin as pseudospin symmetry in nucleon spectra, but the former is much more conserved in real nuclei. In addition, Alhaidari *et al.*[19] investigated in detail physical interpretation on the relativistic Coulomb, oscillator and Hartmann potentials in the case of spin symmetry limit $V(r) \sim S(r)$ and pseudospin symmetry limit $V(r) \sim -S(r)$. Other studies related to this subject have also been carried out.[20−26]

Recently, Leviatan[27] showed that a Dirac Hamiltonian with spherically symmetric scalar and vector potentials exhibits supersymmetric patterns in three cases of the Coulomb, spin and pseudospin limits, respectively. Leviatan[28,29] further explored the supersymmetries of the Dirac Hamiltonian in 3+1 dimensions in pseudospin and spin limits with axially-deformed scalar and vector potentials. Finally, it should be mentioned that Typel[30] presented an alternative supersymmetric description for pseudospin symmetry in order to overcome the difficulties of conventionally relativistic description for pseudospin symmetry, i.e. the difference between two partner Hamiltonians $H(k)$ and $H(-k-1)$ involved in lower component $G(r)$ cannot be considered small in comparison with the difference potential $\Delta = V(r) - S(r)$. Consequently, the relativistic description for pseudospin symmetry is not a good measure for the symmetry breaking. Motivated by recent interest and advance on this line, we are going to study the spin symmetry for the modified Rosen–Morse potential[31,32] with arbitrary spin–orbit coupling quantum number k, which is the main purpose of this work.

† E-mail: chenwenli882002@163.com

†Corresponding author. E-mail: fgwei_2000@163.com

© 2011 Chinese Physical Society and IOP Publishing Ltd

2. Analytical solutions to spin symmetry

For spherically symmetric potential, the Dirac equation of a nucleon with mass M moving in an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ can be written as ($\hbar = c = 1$)

$$\{\boldsymbol{\alpha}\cdot\boldsymbol{p} + \boldsymbol{\beta}[M + S(r)]\}\,\psi(\boldsymbol{r}) = [E - V(r)]\,\psi(\boldsymbol{r}), \quad (1)$$

where E is the relativistic energy of the system and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the 4×4 Dirac matrices. As shown in our recent works,[33,34] the spherically symmetric Dirac spinor wave functions is taken as

$$\psi_{nk}(\boldsymbol{r}) = \frac{1}{r}\begin{pmatrix} F_{nk}(r)\mathrm{Y}^{l}_{jm}(\theta,\phi) \\ \mathrm{i}G_{nk}(r)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)\end{pmatrix},$$

$$k = \pm(j + 1/2). \quad (2)$$

The substitution of Eq. (2) into Eq. (1) yields two coupled differential equations as follows:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{k}{r}\right)F_{nk}(r) = [M + E_{nk} + S(r) - V(r)]\,G_{nk}(r), \quad (3)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{k}{r}\right)G_{nk}(r) = [M - E_{nk} + S(r) + V(r)]\,F_{nk}(r). \quad (4)$$

By eliminating $G_{nk}(r)$ in Eq. (3) and $F_{nk}(r)$ in Eq. (4), one can obtain two second-order differential equations for the upper and lower components as follows:

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{k(k+1)}{r^2} - [M + E_{nk} - \Delta(r)][M - E_{nk} + \Sigma(r)] + \frac{\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{k}{r}\right)}{M + E_{nk} - \Delta(r)}\right\}F_{nk}(r) = 0, \quad (5)$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{k(k-1)}{r^2} - [M + E_{nk} - \Delta(r)][M - E_{nk} + \Sigma(r)] - \frac{\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{k}{r}\right)}{M - E_{nk} + \Sigma(r)}\right\}G_{nk}(r) = 0, \quad (6)$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$. The spin–orbit quantum number k is related to the orbital angular momentum quantum number l. For given $k = \pm1, \pm2, \pm3, \ldots$, we have $j = |k| - 1/2$, $l = |k + 1/2| - 1/2$, $\tilde{l} = |k - 1/2| - 1/2$, and $l(l+1) = k(k+1), \tilde{l}(\tilde{l}+1) = k(k-1)$.

Now, we study the spin symmetry for modified Rosen–Morse potential, i.e., $\mathrm{d}\Delta(r)/\mathrm{d}r = 0$ or $\Delta(r) = C =$ constant and the sum potential $\Sigma(r)$ as modified Rosen–Morse potential,[31,32]

$$\Sigma(r) = \frac{V_1 - V_2\sinh(\alpha r)}{\cosh^2(\alpha r)} = \frac{V_1 - V_2 z}{1 + z^2}, \qquad z = \sinh(\alpha r), \quad (7)$$

where parameters V_1 and V_2 describe the property of potential well, while parameter α is related to the range of this potential.

Substituting Eq. (7) into Eq. (5) allows us to obtain the following Schrödinger-like equation for the upper component,

$$\alpha^2\left(1 + z^2\right)\frac{\mathrm{d}^2F_{nk}(z)}{\mathrm{d}z^2} + \alpha^2 z\frac{\mathrm{d}F_{nk}(z)}{\mathrm{d}z} + \left[E_{nk}^2 - M^2 + C(M + E_{nk})\right]F_{nk}(z) - \left[\frac{(M + E_{nk} - C)\,(V_1 - V_2(z))}{1 + z^2} + \frac{\alpha^2 k(k+1)}{\mathrm{arcsinh}^2(z)}\right]F_{nk}(z) = 0. \quad (8)$$

It is shown that equation (8) cannot be solved exactly except for $k = -1$. Thus, we must use a proper approximation to the spin–orbit coupling term. For this purpose, we noticed that the modified Rosen–Morse potential has a minimum value at $z_0 = V_1 + \sqrt{V_1^2 + V_2^2}/V_2$. Therefore, by adopting a Pekeris-type approximation[35] to the centrifugal term similar to our recent work,[36] we may express the centrifugal term as follows:

$$\frac{1}{\mathrm{arcsinh}^2(z)} = \frac{1}{\mathrm{arcsinh}^2(z_0 + x)} \simeq \frac{1}{\mathrm{arcsinh}^2(z_0)} - \frac{2x}{(1 + z_0^2)^{1/2}\mathrm{arcsinh}^3(z_0)} + \left[\frac{3}{(1 + z_0^2)\mathrm{arcsinh}^4(z_0)} + \frac{z_0}{(1 + z_0^2)^{3/2}\mathrm{arcsinh}^3(z_0)}\right]x^2 + o(x)^3, \quad (9)$$

where $x = z - z_0$ is used. On the other hand, we may also approximately express it in the form of

$$\frac{1}{\text{arcsinh}^2(z)} \simeq \frac{c_0 + c_1\,(x+z_0) + c_2\,(x+z_0)^2}{1+(x+z_0)^2}. \tag{10}$$

Comparing Eq. (10) with Eq. (9), we obtain the expansion coefficients c_0, c_1, c_2 as follows:

$$\begin{aligned} c_0 &= \frac{3z_0^2}{\text{arcsinh}^4(z_0)} - \frac{(z_0^2-2)z_0}{\sqrt{z_0^2+1}\text{arcsinh}^3(z_0)} + \frac{1}{\text{arcsinh}^2(z_0)}, \\ c_1 &= -2\left[\frac{1-2z_0^2}{\sqrt{z_0^2+1}\text{arcsinh}^3(z_0)} + \frac{3z_0}{\text{arcsinh}^4(z_0)}\right], \\ c_2 &= \frac{1}{\text{arcsinh}^2(z_0)} - \frac{3z_0}{\sqrt{z_0^2+1}\text{arcsinh}^3(z_0)} + \frac{3}{\text{arcsinh}^4(z_0)}. \end{aligned} \tag{11}$$

Now we use the approximate centrifugal term (10) to solve Eq. (8). Inserting Eq. (10) into Eq. (8) yields

$$\left(1+z^2\right)\frac{\mathrm{d}^2F_{nk}(z)}{\mathrm{d}z^2} + z\frac{\mathrm{d}F_{nk}(z)}{\mathrm{d}z} + \left(\frac{\tilde{E}_{nk}}{\alpha^2} - \frac{\tilde{V}_1 - z\tilde{V}_2}{\alpha^2\,(1+z^2)} - \frac{k(k+1)(c_0+c_1z+c_2z^2)}{1+z^2}\right)F_{nk}(z) = 0, \tag{12}$$

where

$$\tilde{E}_{nk} = [E_{nk}^2 - M^2 + C(M-E_{nk})], \quad \tilde{V}_1 = (M+E_{nk}-C)V_1, \quad \tilde{V}_2 = (M+E_{nk}-C)V_2. \tag{13}$$

After a further variable substitution $s = \mathrm{i}z$, equation (12) can be rearranged as

$$(1-s^2)\frac{\mathrm{d}^2F_{nk}(s)}{\mathrm{d}s^2} - s\frac{\mathrm{d}F_{nk}(s)}{\mathrm{d}s} - \left[u_0 + \frac{u_1+u_2s+u_3s^2}{1-s^2}\right]F_{nk}(s) = 0, \tag{14}$$

where

$$u_0 = \tilde{E}_{nk}/\alpha^2,\ u_1 = -\tilde{V}_1/\alpha^2 - c_0k(k+1),\ u_2 = \mathrm{i}(c_1k(k+1) - \tilde{V}_2/\alpha^2),\ u_3 = c_2k(k+1). \tag{15}$$

Take the trial wave function of the form

$$F_{nk}(s) = (1-s)^{\lambda/2+1/4}(1+s)^{\eta/2+1/4}f_{nk}(s) \tag{16}$$

with

$$\lambda = -\frac{1}{2}\sqrt{1+4u_1-4u_2+4u_3}, \qquad \eta = -\frac{1}{2}\sqrt{1+4u_1+4u_2+4u_3}. \tag{17}$$

Inserting Eq. (16) into Eq. (14) allows us to obtain

$$\left(1-s^2\right)f''_{nk}(s) + [\eta-\lambda-(2+\eta+\lambda)s]\,f'_{nk}(s) + \left[u_3-u_0-(1+\eta+\lambda)^2/4\right]f_{nk}(s) = 0, \tag{18}$$

whose solutions can be expressed by hypergeometric function[37] as follows:

$$f_{nk}(s) = {}_2\mathrm{F}_1\left(\frac{1+\eta+\lambda}{2} - \sqrt{u_3-u_0}; \frac{1+\eta+\lambda}{2} + \sqrt{u_3-u_0}; 1+\lambda; \frac{1-s}{2}\right). \tag{19}$$

The finiteness of the wave function demands that the hypergeometric series $f_{nk}(s)$ must terminate as a polynomial, this cannot be fulfilled unless

$$\frac{1}{2}(1+\eta+\lambda) - \sqrt{u_3-u_0} = -n, \qquad (n \equiv n_r = 0, 1, 2, \dots), \tag{20}$$

from which and together with Eqs. (13) and (15) we obtain the energy equation of the system as

$$E_{nk}^2 - M^2 + C(M-E_{nk}) = \alpha^2\left[c_2k(k+1) - (1+2n+\lambda+\eta)^2/4\right]. \tag{21}$$

The corresponding upper component $F_{nk}(r)$ can be finally expressed as

$$F_{nk}(r) = (1-\mathrm{i}\sinh(\alpha r))^{\lambda/2+1/4}\,(1+\mathrm{i}\sinh(\alpha r))^{\eta/2+1/4}\,{}_2\mathrm{F}_1\left(-n, 1+n+\eta+\lambda, 1+\lambda, (1-\mathrm{i}\sinh(\alpha r))/2\right). \tag{22}$$

By using the recurrence relation of hypergeometric function,[37]

$$\frac{\mathrm{d}}{\mathrm{d}z}[{}_2\mathrm{F}_1(a;b;c;z)] = \left(\frac{ab}{c}\right){}_2\mathrm{F}_1(a+1;b+1;c+1;z), \tag{23}$$

and inserting Eq. (22) into Eq. (3), one is able to obtain the corresponding lower component $G_{nk}(r)$ as

$$\begin{aligned} G_{nk}(r) = {} & \frac{F(r)}{M+E_{nk}-C}\left[\frac{k}{r}+\frac{\mathrm{i}\alpha(1+2\eta)\cosh(\alpha r)}{4(1+\mathrm{i}\sinh(\alpha r))}-\frac{\mathrm{i}\alpha(1+2\lambda)\cosh(\alpha r)}{4(1-\mathrm{i}\sinh(\alpha r))}\right] \\ & + \frac{\mathrm{i}\alpha n(1+n+\eta+\lambda)\cosh(\alpha r)(1-\mathrm{i}\sinh(\alpha r))^{\lambda/2+1/4}(1+\mathrm{i}\sinh(\alpha r))^{\eta/2+1/4}}{2(1+\lambda)(M+E_{nk}-C)} \\ & \times {}_2\mathrm{F}_1\left(1-n,2+n+\eta+\lambda,2+\lambda,(1-\mathrm{i}\sinh(\alpha r))/2\right). \end{aligned} \tag{24}$$

Thus by inserting $F_{nk}(r)$ and $G_{nk}(r)$ into Eq. (2), we can obtain the Dirac spinors for the modified Rosen–Morse potential with spin symmetry.

3. Numerical results and discussion

First, we study the energy levels based on the energy equation (21), in order to impose the spin symmetry for modified Rosen–Morse potential. For this purpose, we must study numerical energy eigenvalues since we cannot directly obtain the explicit expression for energy levels from complicated energy equation (21). To this end, we take a set of parameters $C = 10, V_1 = 4, V_2 = 3, M = 2, \alpha = 0.5$ to calculate numerically energy levels as shown in Table 1. It is found that, under the condition of spin symmetry, there exist corresponding degenerate states such as $(0,-2)$ and $(0,1)$. This is because this quantum system keeps symmetry by interchanging $k \leftrightarrow -k-1$ in the case of spin symmetry.

Table 1. The bound state energy levels E_{nk} are shown in the case of spin symmetry.

(n_r,k)	$E_{n_r,k<0}$	(n_r,k)	$E_{n_r,k>0}$
$(0,-1)$	5. 83231	$(0,1)$	5. 58465
$(0,-2)$	5. 58465	$(0,2)$	5. 19254
$(0,-3)$	5. 19254	$(0,3)$	4. 73854
$(0,-4)$	4. 73854	$(0,4)$	4. 26674
$(0,-5)$	4. 26674	$(0,5)$	3. 80347
$(1,-1)$	4. 89082	$(1,1)$	4. 74096
$(1,-2)$	4. 74096	$(1,2)$	4. 47444
$(1,-3)$	4. 47444	$(1,3)$	4. 13379
$(1,-4)$	4. 13379	$(1,4)$	3. 75501
$(1,-5)$	3. 75501	$(1,5)$	3. 36620

Second, we consider the effects of potential parameter α on energy levels. Since this quantum system keeps symmetry by interchanging $k \leftrightarrow -k-1$ in the case of spin symmetry. We calculate energy eigenvalues only for positive k with the same parameters as listed in Table 1 but different α as shown in Table 2. It is found that the energy levels will approach a constant $E_{nk} = (1/2)\left(2M + V_1 + \sqrt{V_1^2 + V_2^2}\right)$ for given parameters when parameter α tends to zero.

Table 2. The numerical results show that the energy levels approach a constant when potential parameter α goes to zero.

(n_r,k)	$\alpha = 0.5$	$\alpha = 0.25$	$\alpha = 0.05$
$(0,1)$	5. 58465	6. 04552	6. 41199
$(0,2)$	5. 19254	5. 89784	6. 40314
$(0,3)$	4. 73854	5. 70622	6. 39004
$(0,4)$	4. 26674	5. 48740	6. 37289
$(0,5)$	3. 80347	5. 25253	6. 35196
$(1,1)$	4. 74096	5. 49114	6. 25832
$(1,2)$	4. 47444	5. 38643	6. 25067
$(1,3)$	4. 13379	5. 24228	6. 23932
$(1,4)$	3. 75501	5. 06876	6. 22440
$(1,5)$	3. 36620	4. 87447	6. 20609

Now, we interpret this case theoretically. It is found that, under the condition of spin symmetry, the energy levels will approach the following limiting values:

$$\lim_{\alpha\to 0} E_{nk} = C - M,$$

or

$$\lim_{\alpha\to 0} E_{nk} = \frac{1}{2}\left(2M + V_1 \pm \sqrt{V_1^2 + V_2^2}\right). \tag{25}$$

However, the limiting value $E_{nk} = C - M$ is not physically acceptable since it makes the lower component $G_{nk}(r)$ diverge. Actually, another limiting

value $\lim_{\alpha\to 0} E_{nk} = (1/2)(2M + V_1 - \sqrt{V_1^2 + V_2^2})$ is also not acceptable. This is proved by the numerical calculations as shown in Table 2. On the other hand, when the parameter α goes to zero the modified Rosen–Morse (mRM) potential reduces to the value $\lim_{\alpha\to 0} V_{\rm mRM}(r) = V_1$. It means that this quantum system is independent of parameter V_2 which becomes a constant potential. Substituting this result into Eq. (5) and considering that $\Delta(r) = C$ and $\Sigma(r) = V_1$, we are able to obtain the corresponding results as $F(r) = \sqrt{r}J_{(2k+1)/2}\left(-{\rm i}r\sqrt{\tilde{E}}\right)$ with $\tilde{E} = (M + E - C)(M - E + V_1)$. This implies that the bound state solutions will not exist. Therefore, we may conclude that the relativistic mRM potential cannot trap a Dirac nucleon in this limit. Actually, it is impossible to understand how a constant potential $\Sigma = V_1$ traps a particle.

4. Conclusions

In this paper we have studied the spin symmetry of a Dirac nucleon subjected to scalar and vector mRM potentials including a Pekeris-type approximation to the centrifugal term. The complicated energy equation and associated two-component spinors have been obtained by functional analysis method. It is shown that there exist positive-energy bound states in the case of spin symmetry. In addition, the numerical results and theoretical analysis show that the energy spectra tend to a constant $E_{nk} = (1/2)\left(2M + V_1 + \sqrt{V_1^2 + V_2^2}\right)$ when $\alpha \to 0$, i.e., the relativistic mRM potential cannot trap a Dirac nucleon in the limiting case $\alpha \to 0$. There also exist corresponding degenerate states involved in symmetry by interchanging $k \leftrightarrow -k - 1$ in the case of spin symmetry.

References

[1] Arima A, Harvery M and Shinizu K 1969 *Phys. Lett.* B **30** 517

[2] Hecht K T and Adeler A 1969 *Nucl. Phys.* A **137** 129

[3] Bohr A, Hamamoto I and Mottslson B R 1982 *Phys. Scr.* **26** 267

[4] Dudek J, Nazarewicz W, Szymanski Z and Leander G A 1987 *Phys. Rev. Lett.* **59** 1405

[5] Troltenier D, Nazarewicz W, Szymanski Z and Draayer J P 1994 *Nucl. Phys.* A **567** 591

[6] Stuchbery A E 1999 *J. Phys.* G **25** 611

[7] Stuchbery A E 2002 *Nucl. Phys.* A **700** 83

[8] Nazarewicz W, Twin P J, Fallon P and Garrett J D 1990 *Phys. Rev. Lett.* **64** 1654

[9] Stephens F S, Deleplanque M A, Draper J E, Diamond R M, Macchiavelli A O, Beausang C W, Korten W, Kelly W H, Azaiez F, Becker J A, Henry E A, Yates S W, Brinkman M J, Kuhnert A and Cizeurski J A 1990 *Phys. Rev. Lett.* **65** 301

[10] Stephens F S, Deleplanque M A, Draper J E, Diamond R M, Macchiavelli A O, Beausang C W, Korten W, Kelly W H, Azaiez F, Becker J A, Henry E A, Yates S W, Brinkman M J, Kuhnert A and Cizeurski J A 1998 *Phys. Rev.* C **57** R1565

[11] Troltenier D, Bahri C and Draayer J P 1995 *Nucl. Phys.* A **586** 53

[12] Bell J S and Ruegg H 1975 *Nucl. Phys.* B **98** 151

[13] Ginocchio J N 1997 *Phys. Rev. Lett.* **78** 436

[14] Ginocchio J N 1999 *J. Phys.* G **25** 617

[15] Ginocchio J N and Leviatan A 1998 *Phys. Lett.* B **42** 1

[16] Ginocchio J N 1999 *Phys. Rep.* **315** 231

[17] Meng J, Sugawara-Tanable K, Yamaji S, Ring P and Arima A 1998 *Phys. Rev.* C **58** R628

[18] Zhou S G, Meng J and Ring P 2003 *Phys. Rev. Lett.* **91** 262501

[19] Alhaidari A D, Bahlouli H and Al-Hasan A 2006 *Phys. Lett.* A **349** 87

[20] Guo J Y, Wang R D and Fang X Z 2005 *Phys. Rev.* C **72** 054319

[21] Xu Q and Zhu S J 2006 *Nucl. Phys.* A **768** 161

[22] Chen T S and Meng J 2003 *Chin. Phys. Lett.* **20** 358

[23] Zhang M C 2009 *Acta Phys. Sin.* **58** 61 (in Chinese)

[24] Zhang M C 2009 *Acta Phys. Sin.* **58** 712 (in Chinese)

[25] Jia C S, Chen T and Cui L G 2009 *Phys. Lett.* A **373** 1621

[26] Qiang W C, Zhou R S and Gao Y 2007 *J. Phys.* A: *Math. Theor.* **40** 1677

[27] Leviatan A 2004 *Phys. Rev. Lett.* **92** 202501

[28] Leviatan A 2005 *Int. J. Mod. Phys.* E **14** 111

[29] Leviatan A 2009 *Phys. Rev. Lett.* **103** 042502

[30] Typel S 2008 *Nucl. Phys.* A **806** 156

[31] Grosche C 2005 *J. Phys.* A: *Math. Gen.* **38** 2947

[32] Gu X Y, Dong S H and Ma Z Q 2009 *J. Phys.* A: *Math. Theor.* **42** 035303

[33] Wei G F and Dong S H 2009 *Europhys. Lett.* **87** 40004

[34] Wei G F and Dong S H 2010 *Eur. Phys. J.* A **43** 185

[35] Pekeris C L 1934 *Phys. Rev.* **45** 98

[36] Wei G F and Dong S H 2010 *Phys. Lett.* B **686** 288

[37] Gradshteyn I S and Ryzhik I M 1994 *Tables of Integrals, Series and Products* 5th edn. (New York: Academic Press)