Spin-weighted spheroidal equation in the case of $s = 1$*

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We present a series of studies to solve the spin-weighted spheroidal wave equation by using the method of supersymmetric quantum mechanics. We first obtain the first four terms of super-potential of the spin-weighted spheroidal wave equation in the case of $s = 1$. These results may help summarize the general form for the n-th term of the super-potential, which is proved to be correct by means of induction. Then we compute the eigen-values and the eigen-functions for the ground state. Finally, the shape-invariance property is proved and the eigen-values and eigen-functions for excited states are obtained. All the results may be of significance for studying the electromagnetic radiation processes near rotating black holes and computing the radiation reaction in curved space-time.

Keywords: spheroidal wave equation, supersymmetric quantum mechanics, super-potential, eigenvalue and eigenfunction

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1. Introduction

The spin-weighted spheroidal function was first defined by Teukolsky[1] in the study of the perturbations of the Kerr black hole, which is indispensable in the study of quantum field theory in curved space-time, black hole perturbation theory, etc. Its differential equation is[1−4]

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) + s + a^2\omega^2\cos^2\theta - 2as\omega\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + E\right]\Theta(\theta) = 0, \quad (1)$$

where E is an eigenvalue to be found by imposing appropriate boundary conditions and the parameter s, which is the spin-weight of the perturbation fields, can be $s = 0$, $\pm\frac{1}{2}$, ±1, $\pm\frac{2}{3}$, ±2, corresponding to scalar, neutrino, electromagnetic, Rarita–Schwinger's and gravitational perturbations respectively. The boundary conditions require that Θ is finite at $\theta = 0$, π. For a recent review on the spin-weighted sphoidal functions, readers can refer to Refs. [3] and [4] and references cited there. In a series of papers, the perturbation methods in the super-symmetric quantum mechanics (SUSYQM) are used to study the spheroidal equations and many useful results are obtained.[5−9] When $s = 0$, there are two types of the super-potentials in Refs. [5]–[9], that is,

$$W_n = \sum_{k=0}^{n-1}\frac{\hat{a}_{n,k}}{(2m+3)^n}\sin\theta\cos^{2k+1}\theta, \quad n = 1, 2, \ldots, \quad (2)$$

$$W_n = \sum_{k=0}^{n} a_{n,k}\cos\theta\sin^{2k-1}\theta, \quad n = 1, 2, \ldots. \quad (3)$$

In Ref. [9], the spin-weighted equation with arbitrary spin s was studied, the super-potential $W = W_0 + \sum_{n=1}^{\infty} W_{0,n}(m, s; \theta)c^n$ is obtained, with $W_{0,n}(m, s; \theta)$ written as $W_{0,n}(m, s; \theta) = \sin\theta\sum_{\ell=0}^{n-1} a_{n,\ell}(m, s)\cos^{\ell}\theta$ (note that the parameter c in Refs. [9] and [10] is written as β in other Refs. [9]–[12]. So do we in the present paper). They are the extension of the result (2) in Ref. [9]. However, they are not the extension of the result (3) in Ref. [8]. The spin-weighted spheroidal equation is very important in its own right and it is still worth while further re-investigating, which will be the extension of the result (3) in Ref. [8]. In the world of physics, the spins of the fields only take the values $s = 0, \pm\frac{1}{2}, \pm1, \pm\frac{2}{3}, \pm2$, it would be more appropriate to treat these cases of the spin-weighted equations. So, references [9]–[12] dealt with the spin-weighted spheroidal equations with different spins. Only first several results in the super-potential were given in Refs. [9] and [11] for $s = \frac{1}{2}$, 2. Reference [12] provided a complete super-potential in the form as the extension (3) in the case of $s = \frac{1}{2}$, from which the

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eigen-function of ground state is obtained. Furthermore, it proved the shape-invariance property of the corresponding equations, which we regard as the most important work and showed that the eigen-functions for excited states could be obtained from it. Here we continue the study in the same way as that in Ref. [12] to treat the electromagnetic one, which is the spin-weighted equation in the case of $s = 1$. As before, defining $\Theta(\theta) = \Psi(\theta)/\sqrt{\sin\theta}$ and introducing an artificial small parameter $\beta = a\omega$, equation (1) leads to[5−15]

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2} + \left[\frac{1}{4} + s + \beta^2\cos^2\theta - 2s\beta\cos\theta - \frac{(m + s\cos\theta)^2 - 1/4}{\sin^2\theta} + E\right]\Psi = 0. \tag{4}$$

Thus, the corresponding boundary conditions are $\Psi|_{\theta=0} = \Psi|_{\theta=\pi} = 0$. The potential in Eq. (4) is

$$V(\theta,\beta,s) = -\left[\frac{1}{4} + s + \beta^2\cos^2\theta - 2s\beta\cos\theta - \frac{(m + s\cos\theta)^2 - 1/4}{\sin^2\theta}\right]. \tag{5}$$

Superpotential W is an important concept in SUSYQ and it can be determined by potential from the relation[15]

$$W^2 - W' = V(\theta,\beta,s) - E_0. \tag{6}$$

And the ground state wave function Ψ_0 can be obtained from the super-potential as

$$\Psi_0 = N\exp\left[-\int W\mathrm{d}z\right]. \tag{7}$$

Both equations (6) and (4) are difficult to solve, so that we look for the approximation methods. In this paper, we will resolve the spin-weighted spheroidal equations by expanding the super-potential W into a Taylor series in powers of β, so can be done the eigen-value E_0 for ground state [5−15] as follows:

$$W = \sum_{n=0}^{\infty} W_n\beta^n, \tag{8}$$

$$E_0 = \sum_{n=0}^{\infty} E_{0,n;m}\beta^n, \tag{9}$$

where n refers to the n-th order item of the series expansion and m in $E_{0,n;m}$ denotes parameter m in Eq. (4) and the index 0 means belonging to the ground state energy. Substituting Eqs. (8) and (9) into Eq. (6), one can easily obtain

$$\begin{aligned}
&W_0' - W_0^2 = E_{0,0;m} + s + \frac{1}{4} - \frac{(m + s\cos\theta)^2 - 1/4}{\sin^2\theta} \\
&\qquad\equiv f_0(\theta), \\
&W_1' - 2W_0W_1 = E_{0,1;m} - 2s\cos\theta \equiv f_1(\theta), \\
&W_2' - 2W_0W_2 = E_{0,2;m} + \cos^2\theta + W_1^2 \equiv f_2(\theta), \\
&W_n' - 2W_0W_n = E_{0,n;m} + \sum_{k=1}^{n-1} W_kW_{n-k} \\
&\qquad\equiv f_n(\theta) \quad (n \geq 3).
\end{aligned} \tag{10}$$

When $s = 1$, it is straight forward to obtain[12]

$$\begin{aligned}
E_{0,0;m} &= m^2 + m - 2, \\
W_0 &= -\frac{1 + (m + 1/2)\cos\theta}{\sin\theta}.
\end{aligned} \tag{11}$$

From Eq. (10) it is easy to see that

$$\begin{aligned}
W_n(\theta) &= \left(\tan\frac{\theta}{2}\right)^{-2}\sin^{-2m-1}\theta A_n(\theta) \\
&= (1 + \cos\theta)^2\sin^{-2m-3}\theta A_n(\theta),
\end{aligned} \tag{12}$$

where

$$A_n(\theta) = \int f_n(\theta)(1 - \cos\theta)^2\sin^{2m-1}\theta\mathrm{d}\theta. \tag{13}$$

We introduce the following formulae to simplify the calculation in Eqs. (12) and (13):[16]

$$\begin{aligned}
&P(2m + 1,\theta) \\
&= \int \sin^{2m+1}\theta\mathrm{d}\theta \\
&= -\frac{\cos\theta}{2m + 2}\left[\sum_{k=0}^{m} I(2m + 1,k)\sin^{2m-2k}\theta\right],
\end{aligned} \tag{14}$$

where

$$I(2m + 1,k) = \begin{cases} \dfrac{(2m + 2)(2m)\cdots(2m + 2 - 2k)}{(2m + 1)(2m - 1)\cdots(2m - 2k + 1)}, & \text{if } k \geq 0, \\ 0, & \text{if } k < 0. \end{cases} \tag{15}$$

Here, we give some recurrence relations between $P(2m + 2n + 1,\theta)$ and $P(2m - 1,\theta)$ for later use. Assuming that

$$P(2m + 2n + 1,\theta) = P_1(2m + 2n + 1,\theta) + P_2(2m + 2n + 1,\theta), \tag{16}$$

where

$$P_1(2m+2n+1,\theta) = \frac{-\cos\theta}{2m+2n+2}\left[\sum_{k=0}^{n} I(2m+2n+1,k)\sin^{2m+2n-2k}\theta\right], \tag{17}$$

$$\begin{aligned} P_2(2m+2n+1,\theta) &= \frac{-\cos\theta}{2m+2n+2}\left[\sum_{k=n+1}^{m+n} I(2m+2n+1,k)\sin^{2m+2n-2k}\theta\right] \\ &= \frac{-\cos\theta}{2m+2n+2}\left[\sum_{p=0}^{m-1} I(2m+2n+1,p+n+1)\sin^{2m-2p-2}\theta\right] \\ &= \frac{(2m-1)I(2m+2n+1,n+1)}{2m+2n+2}P(2m-1,\theta). \end{aligned} \tag{18}$$

So $P_1(2m+2n+1,\theta)$ contains the item like $\sin^{2m+2j}\theta$ with $j=n-k\geq 0$, which is the convergent item at $0,\pi$ and $P_2(2m+2n+1,\theta)$ can be expressed by $P(2m-1,\theta)$. Hence $P(2m+2n+1,\theta)$ can be rewritten as

$$\begin{aligned} P(2m+2n+1,\theta) &= \frac{-\cos\theta}{2m+2n+2}\sum_{k=0}^{n} I(2m+2n+1,k)\sin^{2m+2n-2k}\theta \\ &\quad + \frac{(2m-1)I(2m+2n+1,n+1)}{2m+2n+2}P(2m-1,\theta). \end{aligned} \tag{19}$$

By using Eq. (19), the calculation of A_1 could be further simplified into

$$\begin{aligned} A_1(\theta) &= \int (E_{0,1;m}-2\cos\theta)(1-\cos\theta)^2\sin^{2m-1}\theta \mathrm{d}\theta \\ &= (2E_{0,1;m}+4)P(2m-1,\theta) - (E_{0,1;m}+4)P(2m+1,\theta) - \frac{E_{0,1;m}+2}{m}\sin^{2m}\theta + \frac{\sin^{2m+2}\theta}{m+1} \\ &= \left(2E_{0,1;m}+4-\frac{2m(E_{0,1;m}+4)}{2m+1}\right)P(2m-1,\theta) \\ &\quad -\frac{E_{0,1;m}+2}{m}\sin^{2m}\theta + \frac{\sin^{2m+2}\theta}{m+1} + \frac{E_{0,1;m}+4}{2m+1}\cos\theta\sin^{2m}\theta. \end{aligned} \tag{20}$$

From Eqs. (12) and (14) the term $P(2m-1,\theta)$ in $A_1(\theta)$ can make $W_1(\theta)$ blow up at the boundaries $\theta=0,\ \pi$, that is to say, unless the coefficient of the divergent term $P(2m-1,\theta)$ in A_1 is zero, the eigen-function $\Psi(\theta)$ cannot be finite at the boundaries. So

$$2E_{0,1;m}+4-\frac{2m(E_{0,1;m}+4)}{2m+1}=0,$$

that is,

$$E_{0,1;m} = -\frac{2}{m+1}. \tag{21}$$

Equation (19) can simplify A_1 into the concise form

$$A_1(\theta) = \frac{\sin^{2m}\theta}{m+1}[\sin^2\theta - 2 + 2\cos\theta]. \tag{22}$$

Owing to Eq. (12), it is easy to obtain $W_1(\theta)$ as

$$\begin{aligned} W_1(\theta) &= A_1(\theta)(1+\cos\theta)^2(\sin\theta)^{-2m-3} \\ &= -\frac{1}{m+1}\sin\theta. \end{aligned} \tag{23}$$

By similar calculations, one can obtain $W_2(\theta)$ and $W_3(\theta)$ as

$$\begin{aligned} W_2(\theta) &= \frac{m(m+2)\cos\theta\sin\theta}{(2m+3)(m+1)^2} \\ &\quad -\frac{m(m+2)\sin\theta}{(2m+3)(m+1)^3}, \end{aligned} \tag{24}$$

$$\begin{aligned} W_3(\theta) &= -\frac{2m\cos\theta\sin\theta}{(m+1)^4(2m+3)} + \frac{2m\sin\theta}{(m+1)^5(2m+3)} \\ &\quad -\frac{m\sin^3\theta}{(m+1)^3(2m+3)}, \end{aligned} \tag{25}$$

and the corresponding $E_{0,2;m}$ and $E_{0,3;m}$ are

$$E_{0,2;m} = -\frac{m^3+7m^2+11m+3}{(m+1)^3(2m+3)},$$

$$E_{0,3;m} = -\frac{4m^2(m+2)}{(m+1)^5(2m+3)}. \quad (26)$$

2. General formula of the n-th super-potential W_n

In this section, we summarize the general formula for W_n from the results of W_1–W_3 as follows:

$$W_n(\theta) = \cos\theta \sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \sin^{2k-1}\theta + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} \sin^{2k-1}\theta, \quad (27)$$

where the coefficients $a_{n,k}$ and $b_{n,k}$ are required to satisfy

$$\begin{cases} a_{n,k} = 0, \text{ if } k < 1 \text{ or } k > \left[\frac{n}{2}\right], \\ b_{n,k} = 0, \text{ if } k < 1 \text{ or } k > \left[\frac{n+1}{2}\right]. \end{cases} \quad (28)$$

Equation (27) has a form different from the super-potential W_n in Ref. [9], that is,

$$W_n(\theta) = \sin\theta \sum_{k=0}^{n-1} \hat{a}_{n,k} \cos^{2k+1}\theta, \quad n = 1, 2, \ldots. \quad (29)$$

The next step is to prove its correctness by mathematical induction. Obviously W_N for $N = 1, 2, 3$ satisfies Eq. (27). Assuming that when $N < n$, W_N meets the general form, we need to verify whether $W_n(\theta)$ also satisfies the general formula. Back to Eq. (10), combining $W_k(\theta)$ and $W_{n-k}(\theta)$ into $\sum_{k=1}^{n-1} W_k W_{n-k}$, that is,

$$\sum_{k=1}^{n-1} W_k W_{n-k} = \sum_{p=2}^{[\frac{n}{2}]+1} h_{n,p} \sin^{2p-2}\theta + \sum_{p=2}^{[\frac{n}{2}]+1} g_{n,p} \sin^{2p-2}\theta \cos\theta, \quad (30)$$

where $h_{n,p}$ and $g_{n,p}$ are constant coefficients, that is,[13]

$$\begin{aligned} h_{n,p} &= \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}(a_{k,p-j}a_{n-k,j} \\ &\quad - a_{k,p-1-j}a_{n-k,j} + b_{k,p-j}b_{n-k,j}), \\ g_{n,p} &= \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}(a_{k,p-j}b_{n-k,j} + b_{k,p-j}a_{n-k,j}). \end{aligned} \quad (31)$$

In order to simplify calculations, the maximum limit of p can be written uniformly as $[\frac{n}{2}] + 1$, because of Eq. (28), we can obtain

$$\begin{cases} h_{n,p} = 0, \text{ if } p < 2 \text{ or } p > \left[\frac{n}{2}\right] + 1, \\ g_{n,p} = 0, \text{ if } p < 2 \text{ or } p > \left[\frac{n+1}{2}\right]. \end{cases} \quad (32)$$

Substituting $f_n(\theta) = E_{0,n;m} + \sum_{k=1}^{n-1} W_k W_{n-k}$ into Eq. (13) yields

$$\begin{aligned} A_n(\theta) &= \int \left[E_{0,n;m} + \sum_{p=2}^{[\frac{n}{2}]+1} h_{n,p}\sin^{2p-2}\theta + \cos\theta \sum_{p=2}^{[\frac{n}{2}]+1} g_{n,p}\sin^{2p-2}\theta\right](1-\cos\theta)^2 \sin^{2m-1}\theta \mathrm{d}\theta \\ &= E_{0,n;m}\left[2P(2m-1,\theta) - P(2m+1,\theta) - \frac{\sin^{2m}\theta}{m}\right] \\ &\quad + \sum_{p=2}^{[\frac{n}{2}]+1}\left[-\frac{g_{n,p}\sin^{2m+2p}\theta}{2m+2p} + \frac{(g_{n,p}-h_{n,p})\sin^{2m+2p-2}\theta}{m+p-1}\right] \\ &\quad + \sum_{p=2}^{[\frac{n}{2}]+1}\left[2(h_{n,p}-g_{n,p})P(2m+2p-3,\theta) + (2g_{n,p}-h_{n,p})P(2m+2p-1,\theta)\right]. \end{aligned} \quad (33)$$

We simplify the above result according to Eq. (19), that is,

$$\begin{aligned} A_n(\theta) &= P(2m-1,\theta) \times b_1 + E_{0,n;m}\left[-\frac{\sin^{2m}\theta}{m} + \cos\theta\frac{\sin^{2m}\theta}{2m+1}\right] \\ &\quad + \sum_{p=2}^{[\frac{n}{2}]+1}\left[-\frac{g_{n,p}\sin^{2m+2p}\theta}{2m+2p} + \frac{(g_{n,p}-h_{n,p})\sin^{2m+2p-2}\theta}{m+p-1}\right] \end{aligned}$$

$$+\sum_{p=2}^{\frac{[n]}{2}+1}(2h_{n,p}-2g_{n,p})\left[\frac{-\cos\theta}{2m+2p-2}\sum_{j=0}^{p-2}I(2m+2p-3,p-2-j)\sin^{2m+2j}\theta\right]$$
$$+\sum_{p=2}^{\frac{[n]}{2}+1}(2g_{n,p}-h_{n,p})\left[\frac{-\cos\theta}{2m+2p}\sum_{j=0}^{p-1}I(2m+2p-1,p-1-j)\sin^{2m+2j}\theta\right], \quad (34)$$

where b_1 is the coefficient of the term $P(2m-1,\theta)$,

$$b_1 = \left[\frac{2(m+1)E_{0,n;m}}{2m+1}+(2m-1)\sum_{p=2}^{[\frac{n}{2}]+1}\frac{h_{n,p}-g_{n,p}}{m+p-1}I(2m+2p-3,p-1)\right]$$
$$+(2m-1)\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{2g_{n,p}-h_{n,p}}{2m+2p}I(2m+2p-1,p)\right]. \quad (35)$$

Through a similar analysis of $A_1(\theta)$, $W_n(\theta)$ blows up at the boundaries $\theta=0,\ \pi$ when $b_1\neq 0$. We must set the coefficient b_1 of the term $P(2m-1,\theta)$ in A_n to be zero in order that the eigen-function $\Psi_0(\theta)$ can be finite at the boundaries. Thus $b_1=0$ gives

$$E_{0,n;m} = -\frac{(2m+1)(2m-1)}{2(m+1)}\left[\sum_{p=2}^{[\frac{n}{2}]+1}\frac{h_{n,p}-g_{n,p}}{m+p-1}I(2m+2p-3,p-1)\right]$$
$$-\frac{(2m+1)(2m-1)}{2(m+1)}\left[\sum_{p=2}^{[\frac{n}{2}]+1}\frac{2g_{n,p}-h_{n,p}}{2m+2p}I(2m+2p-1,p)\right]. \quad (36)$$

By the result of Eq. (36), $A_n(\theta)$ can be simplified into $A_n(\theta)=R_n(\theta)+\cos\theta T_n(\theta)$, where

$$R_n(\theta) = -\frac{E_{0,n;m}}{m}\sin^{2m}\theta+\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{(g_{n,p}-h_{n,p})\sin^{2m+2p-2}\theta}{m+p-1}-\frac{g_{n,p}\sin^{2m+2p}\theta}{2m+2p}\right]=\sum_{p=0}^{[\frac{n}{2}]+1}R_{n,p}\sin^{2m+2p}\theta \quad (37)$$

with $R_{n,0}=-E_{0,n;m}/m$ and $R_{n,1}=(g_{n,2}-h_{n,2})/(m+1)$. And when $p\geq 2$, $R_{n,p}$ is

$$R_{n,p} = \frac{2g_{n,p+1}-2h_{n,p+1}-g_{n,p}}{2m+2p}, \quad (38)$$

and

$$T_n(\theta) = \frac{E_{0,n;m}}{2m+1}\sin^{2m}\theta+\sum_{p=2}^{\frac{[n]}{2}+1}\left[\frac{g_{n,p}-h_{n,p}}{m+p-1}\sum_{j=0}^{p-2}I(2m+2p-3,p-2-j)\sin^{2m+2j}\theta\right]$$
$$+\sum_{p=2}^{\frac{[n]}{2}+1}\left[\frac{h_{n,p}-2g_{n,p}}{2m+2p}\sum_{j=0}^{p-1}I(2m+2p-1,p-1-j)\sin^{2m+2j}\theta\right]=\frac{E_{0,n;m}}{2m+1}\sin^{2m}\theta$$
$$+\sum_{j=0}^{\frac{[n]}{2}-1}\sum_{p=j+2}^{\frac{[n]}{2}+1}\left[\frac{g_{n,p}-h_{n,p}}{m+p-1}I(2m+2p-3,p-2-j)\sin^{2m+2j}\theta\right]$$
$$+\sum_{j=0}^{\frac{[n]}{2}}\sum_{p=j+1}^{\frac{[n]}{2}+1}\left[\frac{h_{n,p}-2g_{n,p}}{2m+2p}I(2m+2p-1,p-1-j)\sin^{2m+2j}\theta\right]=\sum_{j=0}^{[\frac{n}{2}]}T_{n,j}\sin^{2m+2j}\theta \quad (39)$$

with

$$T_{n,0} = \frac{E_{0,n;m}}{2m+1}+\sum_{p=2}^{\frac{[n]}{2}+1}\left[\frac{g_{n,p}-h_{n,p}}{m+p-1}I(2m+2p-3,p-2)\right]+\sum_{p=1}^{\frac{[n]}{2}+1}\left[\frac{h_{n,p}-2g_{n,p}}{2m+2p}I(2m+2p-1,p-1)\right], \quad (40)$$

and when $j\geq 1$, $T_{n,j}$ being abbreviated as

$$T_{n,j} = \sum_{p=j+2}^{\frac{[n]}{2}+1}\left[\frac{g_{n,p}-h_{n,p}}{m+p-1}I(2m+2p-3,p-2-j)\right]+\sum_{p=j+1}^{\frac{[n]}{2}+1}\left[\frac{h_{n,p}-2g_{n,p}}{2m+2p}I(2m+2p-1,p-1-j)\right]. \quad (41)$$

Due to Eq. (28), one can have

$$\begin{cases} R_{n,j} = 0, \text{ if } j < -1 \text{ or } j > \left[\dfrac{n+1}{2}\right], \\ T_{n,j} = 0, \text{ if } j < -1 \text{ or } j > \left[\dfrac{n}{2}\right]. \end{cases} \tag{42}$$

According to Eq. (12), we have

$$W_n(\theta) = [R_n(\theta) + \cos\theta T_n(\theta)]\,(1+\cos\theta)^2 \sin^{-2m-3}\theta = X_n + \cos\theta Y_n, \tag{43}$$

with

$$\begin{aligned} X_n &= \left[(2R_n + 2T_n) - (R_n + 2T_n)\sin^2\theta\right]\sin^{-2m-3}\theta \\ &= \sum_{j=0}^{[\frac{n}{2}]+1}\left[(2R_{n,j} + 2T_{n,j})\sin^{2j-3}\theta - (R_{n,j} + 2T_{n,j})\sin^{2j-1}\theta\right] = \sum_{j=-1}^{[\frac{n}{2}]+1} X_{n,j}\sin^{2j-1}\theta, \end{aligned} \tag{44}$$

where

$$\begin{aligned} X_{n,j} &= 2R_{n,j+1} + 2T_{n,j+1} - R_{n,j} - 2T_{n,j} \\ &= \sum_{p=j+1}^{\frac{[n]}{2}+1} \frac{[(2m+2p)h_{n,p} - 2g_{n,p}]I(2m+2p-1, p-1-j)}{(m+j+1)(2m+2p)(2m+2p-2)} + \frac{g_{n,j}}{2m+2j}, \end{aligned} \tag{45}$$

and Y_n can be expressed as

$$\begin{aligned} Y_n &= \left[(2R_n + 2T_n) - T_n\sin^2\theta\right]\sin^{-2m-3}\theta \\ &= \sum_{j=-1}^{[\frac{n}{2}]+1}[2R_{n,j+1} + 2T_{n,j+1} - T_{n,j}]\sin^{2j-1}\theta = \sum_{j=-1}^{[\frac{n}{2}]+1} Y_{n,j}\sin^{2j-1}\theta. \end{aligned} \tag{46}$$

From Eqs. (35) and (38), we can obtain

$$\begin{aligned} Y_{n,j} &= 2R_{n,j+1} + 2T_{n,j+1} - T_{n,j} \\ &= -\sum_{p=j+1}^{\frac{[n]}{2}+1} \frac{(m+j)[(2m+2p)h_{n,p} - 2g_{n,p}]I(2m+2p-1, p-1-j)}{(m+j+1)(2m+2p)(2m+2p-2)}. \end{aligned} \tag{47}$$

Obviously, in the case of $j = -1, 0$, the items in $X_{n,j}$ and $Y_{n,j}$ of $A_n(\theta)$ may cause $W_n(\theta)$ to be divergent, then we need to check whether the following items can exist:

$$\begin{aligned} &X_{n,-1} = 2R_{n,0} + 2T_{n,0} = 0, \\ &X_{n,0} = 2R_{n,1} + 2T_{n,1} - R_{n,0} - 2T_{n,0} = 0, \\ &Y_{n,-1} = 2R_{n,0} + 2T_{n,0} = 0, \\ &Y_{n,0} = 2R_{n,1} + 2T_{n,1} - T_{n,0} = 0. \end{aligned} \tag{48}$$

Hence, all the items which can make $W_n(\theta)$ divergent do not exist, so that W_n can be convergent in the boundary region. $W_n(\theta)$ now becomes

$$\begin{aligned} W_n(\theta) &= \sum_{j=1}^{[\frac{n}{2}]+1} X_{n,j}\sin^{2j-1}\theta \\ &\quad + \cos\theta\sum_{j=1}^{[\frac{n}{2}]+1} Y_{n,j}\sin^{2j-1}\theta \\ &= \sum_{l=1}^{[\frac{n}{2}]+1} b_{n,l}\sin^{2l-1}\theta \\ &\quad + \cos\theta\sum_{l=1}^{[\frac{n}{2}]+1} a_{n,l}\sin^{2l-1}\theta, \end{aligned} \tag{49}$$

where

$$a_{n,l} = Y_{n,j} = -\sum_{p=j+1}^{\frac{[n]}{2}+1} \frac{(m+j)[(2m+2p)h_{n,p} - 2g_{n,p}]I(2m+2p-1, p-1-j)}{(m+j+1)(2m+2p)(2m+2p-2)}, \tag{50}$$

$$b_{n,l} = X_{n,j} = \sum_{p=j+1}^{\frac{[n]}{2}+1} \frac{[(2m+2p)h_{n,p} - 2g_{n,p}]I(2m+2p-1, p-1-j)}{(m+j+1)(2m+2p)(2m+2p-2)} + \frac{g_{n,j}}{2m+2j}. \tag{51}$$

Note that the maximum limit of l in Eq. (49) is different from the general formula Eq. (27). By the relation Eq. (42) we can analyse the maximum limit of l with the parity of n. When n is even, l has the maximum $l = [n/2] + 1 = n/2 + 1$, then

$$\begin{aligned} a_{n,\frac{n}{2}+1} &= 2R_{n,\frac{n}{2}+2} + 2T_{n,\frac{n}{2}+2} - T_{n,\frac{n}{2}+1} = 0, \\ b_{n,\frac{n}{2}+1} &= 2R_{n,\frac{n}{2}+2} + 2T_{n,\frac{n}{2}+2} \\ &\quad - R_{n,\frac{n}{2}+1} - 2T_{n,\frac{n}{2}+1} = 0, \end{aligned} \tag{52}$$

but when $l = [\frac{n}{2}] = \frac{n}{2}$, $a_{n,\frac{n}{2}}$ and $b_{n,\frac{n}{2}}$ are both nonzero. Hence on the condition that n is even, $W_n(\theta)$ becomes

$$\begin{aligned} W_n(\theta) &= \cos\theta \sum_{l=1}^{[\frac{n}{2}]} a_{n,l} \sin^{2l-1}\theta \\ &\quad + \sum_{l=1}^{[\frac{n}{2}]} b_{n,l} \sin^{2l-1}\theta, \end{aligned} \tag{53}$$

when n is odd, l has the maximum $l = [\frac{n}{2}] + 1 = \frac{n+1}{2}$, then

$$\begin{aligned} a_{n,\frac{n+1}{2}} &= 2R_{n,\frac{n+3}{2}} + 2T_{n,\frac{n+3}{2}} - T_{n,\frac{n+1}{2}} = 0, \\ b_{n,\frac{n+1}{2}} &= 2R_{n,\frac{n+3}{2}} + 2T_{n,\frac{n+3}{2}} \\ &\quad - R_{n,\frac{n+1}{2}} - 2T_{n,\frac{n+1}{2}} \neq 0, \end{aligned} \tag{54}$$

and when $k = [\frac{n}{2}] = \frac{n-1}{2}$, $a_{n,\frac{n-1}{2}}$ and $b_{n,\frac{n-1}{2}}$ are nonzero. While when n is odd, $W_n(\theta)$ has the new form

$$\begin{aligned} W_n(\theta) &= \cos\theta \sum_{l=1}^{[\frac{n}{2}]} a_{n,l} \sin^{2l-1}\theta \\ &\quad + \sum_{l=1}^{[\frac{n+1}{2}]} b_{n,l} \sin^{2l-1}\theta \end{aligned} \tag{55}$$

with $[\frac{n}{2}] + 1 = [\frac{n+1}{2}]$. Through comparing Eq. (53) with Eq. (55) the super-potential $W_n(\theta)$ can be rewritten as

$$\begin{aligned} W_n(\theta) &= \cos\theta \sum_{l=1}^{[\frac{n}{2}]} a_{n,l} \sin^{2l-1}\theta \\ &\quad + \sum_{l=1}^{[\frac{n+1}{2}]} b_{n,l} \sin^{2l-1}\theta. \end{aligned} \tag{56}$$

This completes the derivation of the general formula W_n. That is to say, for any $n \geq 1$, W_n can satisfy the form of Eq. (27). And the coefficients of $a_{n,l}$ and $b_{n,l}$ are given in Eqs. (50) and Eq. (51) respectively.

3. Eigen-value and eigen-function for ground state

From the super-potential

$$\begin{aligned} W &= W_0 + \sum_{n=1}^{\infty} \Bigg[\cos\theta \sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \sin^{2k-1}\theta \\ &\quad + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} \sin^{2k-1}\theta \Bigg] \beta^n, \end{aligned} \tag{57}$$

the eigen-function of ground state becomes

$$\begin{aligned} \Psi_0 &= N \exp\left[-\int W \mathrm{d}z\right] \\ &= N \exp\left[\int \frac{1 + (m+\frac{1}{2})\cos\theta}{\sin\theta} \mathrm{d}\theta - \sum_{n=1}^{\infty} \beta^n \int (\cos\theta \sum_{k=1}^{[\frac{n}{2}]} a_{n,k} + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k}) \sin^{2k-1}\theta \mathrm{d}\theta\right] \\ &= N(1-\cos\theta)\sin^{m+\frac{1}{2}}\theta \exp\left[-\sum_{n=1}^{\infty} \beta^n \left(\sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \frac{\sin^{2k}\theta}{2k} + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} P(2k-1,\theta)\right)\right], \end{aligned} \tag{58}$$

and the eigen-value of ground state is

$$E_{0;m} = E_{0,0;m} + \sum_{n=1}^{\infty} \beta^n E_{0,n;m} = m^2 + m - 2 + \sum_{n=1}^{\infty} E_{0,n;m} \beta^n, \tag{59}$$

where $E_{0,n;m}$ can be obtained from Eq. (36).

4. Eigen-values and eigen-functions for excited states

In this section, we check whether the spheroidal equation has the shape-invariance. If the answer is yes, we can obtain the eigen-values and the corresponding eigen-functions of the excited state from the ground one by the recurrence relations derived from the property of the shape-invariance. The potentials $V^{\mp}$ are determined by super-potential W as[17]

$$V^{\mp}(\theta) = W^2(\theta) \mp W'(\theta). \tag{60}$$

If the pair of partner potentials $V^{\pm}(\theta)$ may be written as

$$V^{+}(\theta; a_1) = V^{-}(\theta; a_2) + R(a_1), \tag{61}$$

where a_1 is a set of parameters, a_2 is determined by a_1, that is $a_2 = f(a_1)$ and the remainder $R(a_1)$ is independent of θ, they have the property of the shape-invariance.

In order to study the shape-invariance properties of the spin-weighted spheroidal equations, the super-potential W is written as

$$\begin{aligned} &W(A_{n,j}, B_{n,j}, \theta) \\ &= -A_{0,0}(m + \frac{1}{2}) \cot\theta - B_{0,0} \csc\theta \\ &\quad + \sum_{n=1}^{\infty} \beta^n W_n(A_{n,j}, B_{n,j}, \theta), \end{aligned} \tag{62}$$

where

$$\begin{aligned} W_n(A_{n,j}, B_{n,j}, \theta) &= \cos\theta \sum_{j=1}^{[\frac{n}{2}]} \bar{a}_{n,j} \sin^{2j-1}\theta \\ &= \sum_{j=1}^{[\frac{n+1}{2}]} \bar{b}_{n,j} \sin^{2j-1}\theta \end{aligned} \tag{63}$$

with

$$\bar{a}_{n,j} = A_{n,j} a_{n,j}, \ \bar{b}_{n,j} = B_{n,j} b_{n,j}. \tag{64}$$

Then, $V^{\pm}(A_{n,j}, B_{n,j}, \theta)$ are defined as

$$\begin{aligned} V^{\pm}(A_{n,j}, B_{n,j}, \theta) &= W^2(A_{n,j}, B_{n,j}, \theta) \pm W' \\ &= \sum_{n=0}^{\infty} \beta^n V_n^{\pm}(A_{i,j}, B_{n,j}, \theta) \end{aligned} \tag{65}$$

where the subscripts of $A_{i,j}$ and $B_{i,j}$, i and j, are required to satisfy $i < n$ and $j < n$. Now the problem changes into trying to find some quantities $C_{i,j}, D_{i,j}$ to make the relations

$$\begin{aligned} &V_n^{+}(A_{i,j}, B_{n,j}, \theta) \\ &= V_n^{-}(C_{i,j}, D_{n,j}, \theta) + R_{n;m}(A_{i,j}, B_{n,j}) \end{aligned} \tag{66}$$

hold with $R_{n;m}(A_{i,j}, B_{n,j}) = R_{n;m}$ pure quantities. Now, we will prove that it is true for the special cases $n = 0,\ 1,\ 2$[10]

$$V_0^{+}(A_{0,0}, B_{0,0}, \theta) = V_0^{-}(C_{0,0}, D_{0,0}, \theta) + R_{0;m}, \tag{67}$$

$$V_1^{+}(A_{1,1}, B_{1,1}, \theta) = V_1^{-}(C_{1,1}, D_{1,1}, \theta) + R_{1;m}, \tag{68}$$

$$V_2^{+}(B_{2,1}, A_{2,1}, \theta) = V_2^{-}(C_{2,1}, D_{2,1}, \theta) + R_{2;m}. \tag{69}$$

In order to make the above equations hold, one must have

$$C_{0,0} = A_{0,0} + \frac{2}{2m+1}, \tag{70}$$

$$D_{0,0} = 2B_{0,0}, \tag{71}$$

$$D_{1,1} = \frac{(2m+1)A_{0,0} - 1}{(2m+1)A_{0,0} + 3} B_{1,1}, \tag{72}$$

$$\begin{aligned} D_{2,1} = &\frac{(2m+1)A_{0,0} - 1}{(2m+1)A_{0,0} + 3} B_{2,1} \\ &+ \frac{12 B_{0,0} B_{2,1}}{[(2m+1)A_{0,0} + 3][(2m+1)A_{0,0} + 4]} \\ &- \frac{16[(2m+1)A_{0,0} + 1] B_{0,0} B_{1,1}^2}{[(2m+1)A_{0,0} + 3]^3 [(2m+1)A_{0,0} + 4]}, \end{aligned} \tag{73}$$

$$\begin{aligned} C_{2,1} = &\frac{8[(2m+1)A_{0,0} + 1] B_{1,1}^2}{[(2m+1)A_{0,0} + 3]^3 [(2m+1)A_{0,0} + 4]} \\ &+ \frac{(2m+1)A_{0,0} - 2}{(2m+1)A_{0,0} + 4} A_{2,1} \end{aligned} \tag{74}$$

with

$$R_{0;m}(A_{0,0}) = (2m+1)A_{0,0} + 1, \tag{75}$$

$$R_{1;m}(A_{0,0}, B_{0,0}, B_{1,1}) = -\frac{8 B_{0,0} B_{1,1}}{(2m+1)A_{0,0} + 3}, \tag{76}$$

$$\begin{aligned} &R_{2;m}(A_{0,0}, B_{0,0}, B_{1,1}, B_{2,1}, A_{2,1}) \\ &= -\frac{8 B_{0,0} B_{2,1}}{(2m+1)A_{0,0} + 3} + A B_{1,1}^2 + B A_{2,1}, \end{aligned} \tag{77}$$

where

$$\begin{aligned} A &= \frac{32 B_{0,0}^2 - 8[(2m+1)A_{0,0} - 1][(2m+1)A_{0,0} + 3]}{[(2m+1)A_{0,0} + 3]^3 [(2m+1)A_{0,0} + 4]}, \\ B &= \frac{24 B_{0,0}^2 - 2[(2m+1)A_{0,0} - 1][(2m+1)A_{0,0} + 3]}{[(2m+1)A_{0,0} + 3][(2m+1)A_{0,0} + 4]}. \end{aligned} \tag{78}$$

In order to prove Eq. (66) for general n, one uses the induction method. Here the details are omitted (readers can refer to Ref. [12] for details). The results are that the shape-invariance holds for it too. Therefore, the shape-invariance property for the spin-weighted equation with $s = 1$ holds too as in the case of $s = \frac{1}{2}$. Now the results are given in the following without further details:

$$D_{n,p} = \frac{D_{0,0}a_{n,p}}{\alpha_p b_{n,p}} C_{n,p} - \frac{U_{n,p}}{\alpha_p b_{n,p}}, \tag{79}$$

$$C_{n,p-1} = \frac{\left(\alpha_p + \frac{D_{0,0}^2}{\alpha_p}\right) a_{n,p}}{(\alpha_p - 1)a_{n,p-1}} C_{n,p} + \frac{\check{U}_{n,p} - \frac{D_{0,0}}{\alpha_p} U_{n,p}}{(\alpha_p - 1)a_{n,p-1}}, \tag{80}$$

for $p = 2, 3, \ldots, [\frac{n+1}{2}]$. Notice the following fact (see details in Ref. [12]):

$$G_{n,p}(A_{i,j}, B_{i,j}) = \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} \left[\bar{b}_{k,p-j} \bar{a}_{n-k,j} + \bar{a}_{k,p-j} \bar{b}_{n-k,j} \right], \tag{81}$$

$$H_{n,p}(A_{i,j}, B_{i,j}) = \sum_{k=1}^{n-1} \sum_{j=1}^{p-1} \left[\bar{b}_{k,p-j} \bar{b}_{n-k,j} + \bar{a}_{k,p-j} \bar{a}_{n-k,j} - \bar{a}_{k,p-1-j} \bar{a}_{n-k,j} \right], \tag{82}$$

$$\bar{a}_{i,j} = A_{i,j} a_{i,j}, \quad \bar{b}_{i,j} = B_{i,j} b_{i,j}, \tag{83}$$

$$U_{n,p} = G_{n,p}(A_{i,j}, B_{i,j}) - G_{n,p}(C_{i,j}, D_{i,j}) + P^{+}_{n,p}(A_{n,j}, B_{n,j}), \tag{84}$$

$$\check{U}_{n,p} = H_{n,p}(A_{i,j}, B_{i,j}) - H_{n,p}(C_{i,j}, D_{i,j}) + Q^{+}_{n,p}(A_{n,j}, B_{n,j}), \tag{85}$$

$$\alpha_p = \left[(2m+1)C_{0,0} + (2p-1) \right]. \tag{86}$$

According to n's parity, we note the following discussion : with $p = [\frac{n+1}{2}]$ and $D_{n,[\frac{n+1}{2}]+1} = C_{n,[\frac{n}{2}]+1} = 0$, (i) when n is odd, $C_{n,[\frac{n+1}{2}]} = 0$ and one can obtain $D_{n,[\frac{n+1}{2}]}$ from Eq. (79) and $C_{n,[\frac{n}{2}]}$ from Eq. (80) under the condition $p = [\frac{n+1}{2}]$; (ii) when n is even, one needs to calculate $C_{n,\frac{n}{2}}$ from Eq. (80) under the condition $p = [\frac{n+1}{2}]$, $C_{n,[\frac{n+1}{2}]} = 0$. Then, it is easy to calculate the others subsequently. Then, the eigen-values $E^{-}_{l;m}$ for excited states can be obtained as

$$\begin{aligned} E^{-}_{l;m} &= E_{0;m} + \sum_{k=1}^{l} R(a_k, b_k) \\ &= m(m+1) - 2 + \sum_{n=1}^{\infty} E_{0,n;m} \beta^n \\ &\quad + \sum_{k=1}^{l} R(a_k, b_k), \end{aligned} \tag{87}$$

$$R(a_k, b_k) = R_{0;m} + \sum_{n=1}^{\infty} \beta^n R_{n;m}(a_k, b_k), \tag{88}$$

$$a_1 = (A_{i,j}, B_{i,j}),\ a_2 = (C_{i,j}, D_{i,j}), \ldots, \tag{89}$$

and the corresponding eigen-function Ψ^{-}_{n} is obtained by the following recurrence relation:

$$\Psi_0 \propto \exp\left[- \int_{\theta_0}^{\theta} W(A_{n,j}, B_{n,j}, \theta) \mathrm{d}\theta \right], \tag{90}$$

$$\mathcal{A}^{\dagger} = -\frac{\mathrm{d}}{\mathrm{d}\theta} + W(A_{n,j}, B_{n,j}, \theta), \tag{91}$$

as

$$\begin{aligned} &\Psi^{-}_{n} = \mathcal{A}^{\dagger}(A_{n,j}, B_{n,j}, \theta) \Psi^{-}_{n-1}(C_{n,j}, D_{n,j}, \theta), \\ &n = 1, 2, 3, \ldots \end{aligned} \tag{92}$$

In the above proof, we have used the shape-invariance properties to calculate the eigenvalue and eigenfunction of the spin-weighted equations for excited states in the case of $s = 1$ by the above recurrence relation.

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