Robust H_∞ observer-based control for synchronization of a class of complex dynamical networks*

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(Received 2 November 2010; revised manuscript received 7 January 2011)

This paper is concerned with the robust H_∞ synchronization problem for a class of complex dynamical networks by applying the observer-based control. The proposed feedback control scheme is developed to ensure the asymptotic stability of the augmented system, to reconstruct the non-measurable state variables of each node and to improve the H_∞ performance related to the synchronization error and observation error despite the external disturbance. Based on the Lyapunov stability theory, a synchronization criterion is obtained under which the controlled network can be robustly stabilized onto a desired state with a guaranteed H_∞ performance. The controller and the observer gains can be given by the feasible solutions of a set of linear matrix inequalities (LMIs). The effectiveness of the proposed control scheme is demonstrated by a numerical example through simulation.

Keywords: complex dynamical network, robust H_∞ synchronization, Lur'e system, decentralized observer

PACS: 05.45.Xt **DOI:** 10.1088/1674-1056/20/6/060504

1. Introduction

Nowadays, there are numerous natural or man-made complex dynamical networks, such as the World Wide Web, Internet, communication networks, social networks, food webs, metabolic networks, etc.[1−3] Synchronization is a very interesting nonlinear phenomenon and has been found to be useful in many disciplines. The subject of synchronization of complex dynamical networks has been widely explored in a variety of fields including physical, chemical and ecological systems, human heartbeat regulation, secure communications, etc.[4−10] Most investigations consider that each node of the network dynamically adjusts its state trajectories until they eventually converge to the synchronization manifold. The manifold is autonomously achieved by a certain self-organized process. However, a different synchronization phenomenon has been ignored when entities external to the network are deputed to control the whole network onto a desired synchronous evolution. So many researchers begin to investigate the problem of controlling complex dynamical networks.[11−14] In Ref. [12], an approximate formula has been derived for estimating the detailed number of pinning nodes and the magnitude of the coupling strength for a given complex network. In Ref. [13], a robust adaptive controller was designed for uncertain complex dynamical networks whose uncertainties include the variation of coupling structures, different weights and inner-coupling relations etc. In these studies, the controllers designed are exactly based on the information of the network state. However, state variables of complex dynamical network are often not fully available and may be difficult or costly to measure. Consequently, it may be possible to use an observer to estimate the unknown states of the network. The observer-based control for systems has been an interesting topic in control theory.[15−17] Reference [17] investigated the problem of H_∞ decentralized tracking control design with a decentralized observer for interconnected nonlinear systems. However, according to our knowledge, in most of the existing works concerning the synchronization of complex network, the control strategy applied is the state feedback.

On the other hand, the external disturbance is inevitable in real-world systems, as well as complex dynamical networks. It has been known that disturbance can influence synchronization in different ways. Moreover, the H_∞ control concept was proposed to

*Project supported by the National Natural Science Foundation of China (Grant No. 60274099) and the National High Technology Research and Development Program of China (Grant No. 2004AA412030).

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reduce the effect of the disturbance input on the regulated output to within a prescribed level. Hence, it is of great importance and practical significance to study the robust H_∞ synchronization for complex dynamical networks.

Lur'e control system is a class of the most important nonlinear control systems. A large class of nonlinear systems such as Chua's circuit,[18] Goodwin model,[19] and swarm model[20] can be regarded as Lur'e systems. It has been shown that the syn chronization of many chaotic systems can be generalized in the framework of absolute stability theory of Lur'e systems. Based on this method, the master-slave synchronization for Lur'e systems has been extensively examined.[21−23] More recently, the synchronization of complex dynamical networks consisting of Lur'e systems has become a topic of great interest due to the great potential applications in many areas.[24−27] Reference [24] investigated the synchronization for complex dynamical network composed of identical Lur'e type systems based on state feedback control. In Ref. [27], an experimental study on practical realization of synchronization in globally coupled networks with Chua's circuits-like nodes is presented and synchronization of coupled multiple Chua's circuits is achieved by appealing to results from complex systems theory.

Motivated by the above discussion, in this paper, a linear matrix inequality (LMI) approach is developed to study the robust H_∞ synchronization problem for a complex network consisting of Lur'e systems by utilizing observer-based control. A synchronization criterion is presented under which the complex network can be robustly stabilized onto an expected homogeneous state with a guaranteed H_∞ performance. The contribution of this paper is the development of a decentralized observer, to reconstruct the non-measurable state variables and to guarantee the H_∞ performance allowing the minimization of the external disturbance effects. It is also shown that the design of controller and observer can be realized by solving LMIs. Finally, a numerical example is proposed to demonstrate the applicability of our theoretical result.

The rest of this paper is organized as follows. In Section 2, some preliminary definitions and lemmas necessary for successive development are presented; also, the problem considered is formulated. Section 3 is devoted to studying the stability of the augmented system in the H_∞ sense. In Section 4, a numerical example is provided to illustrate our results. Finally, a brief concluding remark is given in Section 5.

2. Preliminaries

In this section, some preliminary definitions are presented and a complex dynamical network model composed of Lur'e systems subject to energy bounded input noise is considered.

2.1. Preliminaries related to absolute stability

Consider the following nonlinear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}(t)), \tag{1}$$

where $\boldsymbol{A} \in R^{n\times n}$, $\boldsymbol{B} \in R^{n\times m}$, $\boldsymbol{C} = [\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m]^{\mathrm{T}} \in R^{m\times n}$ are constant matrices; $\boldsymbol{f}$ is a nonlinear vector-valued function satisfying $\boldsymbol{f}(0) = 0$. Denote

$$\boldsymbol{C}\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}(t) \\ \vdots \\ \boldsymbol{c}_m^{\mathrm{T}}\boldsymbol{x}(t) \end{bmatrix}, \quad \boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}(t)) = \begin{bmatrix} f_1(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}(t)) \\ \vdots \\ f_m(\boldsymbol{c}_m^{\mathrm{T}}\boldsymbol{x}(t)) \end{bmatrix},$$

where the functions $f_l(\boldsymbol{c}_l^{\mathrm{T}}\boldsymbol{x}(t)), l = 1, 2, \ldots, m$ are assumed to satisfy

$$\gamma_l \le \frac{f_l(\boldsymbol{c}_l^{\mathrm{T}}\boldsymbol{x}(t))}{\boldsymbol{c}_l^{\mathrm{T}}\boldsymbol{x}(t)} \le \delta_l, \quad l = 1, 2, \ldots, m \tag{2}$$

for all $\boldsymbol{c}_l^{\mathrm{T}}\boldsymbol{x}(t) \neq 0$, where γ_l, $\delta_l \in R$, $l = 1, 2, \ldots, m$. Taking $\boldsymbol{\Gamma}_0 = \mathrm{diag}(\gamma_1, \ldots, \gamma_m)$, $\boldsymbol{\Delta}_0 = \mathrm{diag}(\delta_1, \ldots, \delta_m)$, it can be easily seen that

$$\begin{aligned}(\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}(t)) &- \boldsymbol{\Gamma}_0\boldsymbol{C}\boldsymbol{x}(t))^{\mathrm{T}}(\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}(t)) \\ &- \boldsymbol{\Delta}_0\boldsymbol{C}\boldsymbol{x}(t)) \le 0.\end{aligned} \tag{3}$$

Definition 1[25] The nonlinearity $\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}(t))$ is said to be in the sector $[\boldsymbol{\Gamma}_0, \ \boldsymbol{\Delta}_0]$, if it satisfies the inequality (3).

Definition 2[25] System (1) is said to be absolutely stable with respect to the sector $[\boldsymbol{\Gamma}_0, \ \boldsymbol{\Delta}_0]$, if for the nonlinearity $\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}(t))$ satisfying inequality (3), the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ is globally asymptotically stable.

Lemma 1[28] The eigenvalues of an irreducible matrix $\boldsymbol{G} = (g_{ij}) \in R^{N\times N}$ with $g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij}$, $i = 1, 2, \ldots, N$ satisfy the following properties.

(i) 0 is an eigenvalue of $\boldsymbol{G}$ associated with the eigenvector $(1, 1, \ldots, 1)^{\mathrm{T}}$.

(ii) If $g_{ij} \ge 0$ for all $1 \le i, j \le N$, $i \neq j$, then the real parts of all eigenvalues of $\boldsymbol{G}$ are less than or equal to 0 and all possible eigenvalues with zero part are 0. In fact, 0 is an eigenvalue of multiplicity 1.

2.2. A complex dynamical network model composed of Lur'e systems

In this paper, we consider the following complex dynamical network with each node being a Lur'e system subject to an energy bounded input noise:

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}_i) + \sum_{j=1}^{N} g_{ij}\boldsymbol{D}\boldsymbol{x}_j + \boldsymbol{B}_1\boldsymbol{w}_i + \boldsymbol{u}_i,$$
$$\boldsymbol{p}_i = \boldsymbol{E}\boldsymbol{x}_i, \quad i = 1, 2, \ldots, N; \tag{4}$$

where $\boldsymbol{x}_i \in R^n$ is the state variable of node i, $\boldsymbol{p}_i \in R^r$ is the output vector. $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{f}$ have the same meanings as those in Eq. (1). $\boldsymbol{B}_1 \in R^{n\times m}, \boldsymbol{E} \in R^{r\times n}$ are constant matrices, $\boldsymbol{w}_i \in R^m$ is a disturbance term belonging to $L_2[0,\infty)$. $\boldsymbol{D} \in R^{n\times n}$ defines the coupling between any two nodes, $\boldsymbol{G} = (g_{ij}) \in R^{N\times N}$ denotes the connection topology and coupling strength, $g_{ij} = g_{ji}$ $(i \neq j)$ is either positive when node j is linked to node i directly, or zero when it does not, and $g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij}$.

In addition, the network considered is connected in the sense that there are no isolated nodes, that is, matrix $\boldsymbol{G}$ is an irreducible matrix. Consequently, the eigenvalues of $\boldsymbol{G}$, denoted by λ_i, $i = 1, \ldots, N$ have the following properties

$$\lambda_1 = 0, \quad 0 > \lambda_2 \geq \cdots \geq \lambda_N,$$

from Lemma 1. There exists an orthogonal matrix $\boldsymbol{R}$ such that $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{R} = \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$. Denote

$$\boldsymbol{C}\boldsymbol{x}_i(t) = \begin{bmatrix} \boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_i(t) \\ \vdots \\ \boldsymbol{c}_m^{\mathrm{T}}\boldsymbol{x}_i(t) \end{bmatrix}, \ \boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}_i(t)) = \begin{bmatrix} f_1(\boldsymbol{c}_1^{\mathrm{T}}\boldsymbol{x}_i(t)) \\ \vdots \\ f_m(\boldsymbol{c}_m^{\mathrm{T}}\boldsymbol{x}_i(t)) \end{bmatrix},$$

where $f_l(\boldsymbol{c}_l^{\mathrm{T}}\boldsymbol{x}_i(t))$, $i = 1, \ldots, N$, $l = 1, \ldots, m$ are assumed to satisfy

$$\gamma_{1l} \leq \frac{f_l(\boldsymbol{c}_l^{\mathrm{T}}\boldsymbol{x}_i(t))}{\boldsymbol{c}_l^{\mathrm{T}}\boldsymbol{x}_i(t)} \leq \delta_{1l}, \gamma_{1l} \leq f_l'(\boldsymbol{c}_l^{\mathrm{T}}\boldsymbol{x}_i(t)) \leq \delta_{1l}, \tag{5}$$

for all $\boldsymbol{c}_l^{\mathrm{T}}\boldsymbol{x}_i \in R$.

Remark 1 Since the external noise is inevitable in real-world systems and the noise can influence synchronization in different ways, the network model considered is supposed to be disturbed by an energy bounded input noise, which should be more suitable to the practical situation.

The observed state vector of each node, which is a function only of the input and the output of the considered node, can be described as

$$\dot{\hat{\boldsymbol{x}}}_i = \boldsymbol{A}\hat{\boldsymbol{x}}_i + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{C}\hat{\boldsymbol{x}}_i) + \boldsymbol{u}_i + \boldsymbol{L}(\boldsymbol{p}_i - \hat{\boldsymbol{p}}_i),$$
$$\hat{\boldsymbol{p}}_i = \boldsymbol{E}\hat{\boldsymbol{x}}_i, i = 1, 2, \ldots, N; \tag{6}$$

where $\hat{\boldsymbol{x}}_i$ is the estimation of $\boldsymbol{x}_i$, $\hat{p}_i$ is the observer output, $\boldsymbol{L} \in R^{n\times r}$ is the observer gain matrix.

Remark 2 Note that the state observer structure of the whole complex dynamical network is completely decentralized since there is no information transfer between any two nodes, which will be practical for physical realization.

The dynamical complex network (4) is said to achieve synchronization if

$$\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_N(t) \to \boldsymbol{s}(t), t \to \infty, \tag{7}$$

where $\boldsymbol{s}(t)$ is a solution of an isolated node, satisfying $\dot{\boldsymbol{s}}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{C}\boldsymbol{s}(t))$.

Generally, the basic approach taken in analysing the synchronization problem is to judge the stability of the corresponding error dynamical system. Defining $\boldsymbol{e}_{1i} = \boldsymbol{x}_i - \hat{\boldsymbol{x}}_i$, $\boldsymbol{e}_{2i} = \boldsymbol{x}_i - \boldsymbol{s}$ and the local control law of each node $\boldsymbol{u}_i = \boldsymbol{K}(\hat{\boldsymbol{x}}_i - \boldsymbol{s})$, then the observation error dynamical system and the synchronization error dynamical system can be given by

$$\dot{\boldsymbol{e}}_{1i} = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{E})\boldsymbol{e}_{1i} + \boldsymbol{B}(\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}_i) - \boldsymbol{f}(\boldsymbol{C}\hat{\boldsymbol{x}}_i)) + \sum_{j=1}^{N} g_{ij}\boldsymbol{D}\boldsymbol{x}_j + \boldsymbol{B}_1\boldsymbol{w}_i, \tag{8}$$

$$\dot{\boldsymbol{e}}_{2i} = (\boldsymbol{A} + \boldsymbol{K})\boldsymbol{e}_{2i} + \boldsymbol{B}(\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}_i) - \boldsymbol{f}(\boldsymbol{C}\boldsymbol{s})) + \sum_{j=1}^{N} g_{ij}\boldsymbol{D}\boldsymbol{e}_{2j} - \boldsymbol{K}\boldsymbol{e}_{1i} + \boldsymbol{B}_1\boldsymbol{w}_i. \tag{9}$$

Reformulating Eqs. (6), (8) and (9) in virtue of the Kronecker product as

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{M}_1\hat{\boldsymbol{x}} + \boldsymbol{M}_2\boldsymbol{F}(\boldsymbol{H}\hat{\boldsymbol{x}}) + \boldsymbol{L}_1(\boldsymbol{p} - \hat{\boldsymbol{p}}) + \boldsymbol{K}_1(\hat{\boldsymbol{x}} - \boldsymbol{S}),$$
$$\hat{\boldsymbol{p}} = \boldsymbol{M}_4\hat{\boldsymbol{x}}, \tag{10}$$

$$\dot{\boldsymbol{e}}_1 = \boldsymbol{M}_1\boldsymbol{e}_1 + \boldsymbol{M}_2\boldsymbol{\phi}(\boldsymbol{H}\boldsymbol{e}_1; \hat{\boldsymbol{x}}) + (\boldsymbol{G}\otimes\boldsymbol{D})\boldsymbol{x} + \boldsymbol{L}_1(\boldsymbol{p} - \hat{\boldsymbol{p}}) + \boldsymbol{M}_3\boldsymbol{w}, \tag{11}$$

$$\dot{\boldsymbol{e}}_2 = (\boldsymbol{M}_1 + \boldsymbol{G}\otimes\boldsymbol{D})\boldsymbol{e}_2 + \boldsymbol{M}_2\boldsymbol{\phi}(\boldsymbol{H}\boldsymbol{e}_2; \boldsymbol{S}) + \boldsymbol{K}_1(\hat{\boldsymbol{x}} - \boldsymbol{S}) + \boldsymbol{M}_3\boldsymbol{w}, \tag{12}$$

where

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_N \end{bmatrix}, \quad \hat{\boldsymbol{x}} = \begin{bmatrix} \hat{\boldsymbol{x}}_1 \\ \vdots \\ \hat{\boldsymbol{x}}_N \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} \boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{p}_N \end{bmatrix},$$

$$\hat{\boldsymbol{p}} = \begin{bmatrix} \hat{\boldsymbol{p}}_1 \\ \vdots \\ \hat{\boldsymbol{p}}_N \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \boldsymbol{s} \\ \vdots \\ \boldsymbol{s} \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_1 \\ \vdots \\ \boldsymbol{w}_N \end{bmatrix},$$

$$\boldsymbol{e}_1 = \begin{bmatrix} \boldsymbol{e}_{11} \\ \vdots \\ \boldsymbol{e}_{1N} \end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix} \boldsymbol{e}_{22} \\ \vdots \\ \boldsymbol{e}_{2N} \end{bmatrix},$$

$$\boldsymbol{F}(\boldsymbol{H}\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}_1) \\ \vdots \\ \boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}_N) \end{bmatrix}, \quad \boldsymbol{M}_1 = \boldsymbol{I}_N \otimes \boldsymbol{A},$$

$$\boldsymbol{M}_2 = \boldsymbol{I}_N \otimes \boldsymbol{B}, \quad \boldsymbol{M}_3 = \boldsymbol{I}_N \otimes \boldsymbol{B}_1, \quad \boldsymbol{M}_4 = \boldsymbol{I}_N \otimes \boldsymbol{E},$$

$$\boldsymbol{L}_1 = \boldsymbol{I}_N \otimes \boldsymbol{L}, \quad \boldsymbol{K}_1 = \boldsymbol{I}_N \otimes \boldsymbol{K},$$

$$\boldsymbol{H} = \boldsymbol{I}_N \otimes \boldsymbol{C}, \quad \boldsymbol{\phi}(\boldsymbol{H}\boldsymbol{e}_1; \hat{\boldsymbol{x}}) = \boldsymbol{F}(\boldsymbol{H}\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{H}\hat{\boldsymbol{x}}),$$

$$\boldsymbol{\phi}(\boldsymbol{H}\boldsymbol{e}_2; \boldsymbol{S}) = \boldsymbol{F}(\boldsymbol{H}\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{H}\boldsymbol{S}).$$

The augmented dynamical system composed of Eqs. (10)–(12) can be written by the following state representation

$$\dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{M}}_1\bar{\boldsymbol{x}} + \bar{\boldsymbol{M}}_2\bar{\boldsymbol{F}} + \bar{\boldsymbol{M}}_3\boldsymbol{W}, \tag{13}$$

where

$$\bar{\boldsymbol{x}} = \begin{bmatrix} \hat{\boldsymbol{x}} \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{bmatrix}, \quad \bar{\boldsymbol{M}}_1 = \begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{L}_1\boldsymbol{M}_4 - \boldsymbol{K}_1 & \boldsymbol{K}_1 \\ \boldsymbol{G} \otimes \boldsymbol{D} & \boldsymbol{M}_1 + \boldsymbol{G} \otimes \boldsymbol{D} - \boldsymbol{L}_1\boldsymbol{M}_4 & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{K}_1 & \boldsymbol{M}_1 + \boldsymbol{G} \otimes \boldsymbol{D} + \boldsymbol{K}_1 \end{bmatrix},$$

$$\bar{\boldsymbol{M}}_2 = \begin{bmatrix} \boldsymbol{M}_2 & & \\ & \boldsymbol{M}_2 & \\ & & \boldsymbol{M}_2 \end{bmatrix}, \quad \bar{\boldsymbol{M}}_3 = \begin{bmatrix} \boldsymbol{0} & & \\ & \boldsymbol{M}_3 & \\ & & \boldsymbol{M}_3 \end{bmatrix}, \quad \bar{\boldsymbol{F}} = \begin{bmatrix} \boldsymbol{F}(\boldsymbol{H}\hat{x}) \\ \boldsymbol{\phi}(\boldsymbol{H}\boldsymbol{e}_1; \hat{\boldsymbol{x}}) \\ \boldsymbol{\phi}(\boldsymbol{H}\boldsymbol{e}_2; \boldsymbol{S}) \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{w} \\ \boldsymbol{w} \end{bmatrix}.$$

Denote

$$\boldsymbol{\Gamma}_1 = \mathrm{diag}(\gamma_{11}, \ldots, \gamma_{1m}), \quad \boldsymbol{\Delta}_1 = \mathrm{diag}(\delta_{11}, \ldots, \delta_{1m}),$$

$\boldsymbol{\Gamma} = \boldsymbol{I}_N \otimes \boldsymbol{\Gamma}_1$, $\boldsymbol{\Delta} = \boldsymbol{I}_N \otimes \boldsymbol{\Delta}_1$. From Eq. (5), it is not difficult to derive that the nonlinear function $\bar{\boldsymbol{F}}$ satisfies

$$(\bar{\boldsymbol{F}} - \bar{\boldsymbol{\Gamma}}\bar{\boldsymbol{x}})^{\mathrm{T}}(\bar{\boldsymbol{F}} - \bar{\boldsymbol{\Delta}}\bar{\boldsymbol{x}}) \leq 0, \tag{14}$$

where

$$\bar{\boldsymbol{\Gamma}} = \mathrm{diag}(\boldsymbol{\Gamma}\boldsymbol{H}, \boldsymbol{\Gamma}\boldsymbol{H}, \boldsymbol{\Gamma}\boldsymbol{H}),$$
$$\bar{\boldsymbol{\Delta}} = \mathrm{diag}(\boldsymbol{\Delta}\boldsymbol{H}, \boldsymbol{\Delta}\boldsymbol{H}, \boldsymbol{\Delta}\boldsymbol{H}).$$

The objective of this study is to maintain consistent performance of system (13) in the presence of the external disturbance and the proposed H_∞ performance is related to the observation error $\boldsymbol{e}_{1i} = \boldsymbol{x}_i - \hat{\boldsymbol{x}}_i$ and the synchronization error $\boldsymbol{e}_{2i} = \boldsymbol{x}_i - \boldsymbol{s}$, which can be described by the following constraint:

$$\sum_{i=1}^{N}\int_0^{tf} \boldsymbol{e}_{1i}^{\mathrm{T}}\boldsymbol{Q}_{11}\boldsymbol{e}_{1i}\mathrm{d}t + \sum_{i=1}^{N}\int_0^{tf} \boldsymbol{e}_{2i}^{\mathrm{T}}\boldsymbol{Q}_{12}\boldsymbol{e}_{2i}\mathrm{d}t$$
$$\leq \rho^2\int_0^{tf} \boldsymbol{W}^{\mathrm{T}}\boldsymbol{W}\mathrm{d}t. \tag{15}$$

If the initial conditions are considered, the inequality (15) can be modified as

$$\sum_{i=1}^{N}\int_0^{tf} \boldsymbol{e}_{1i}^{\mathrm{T}}\boldsymbol{Q}_{11}\boldsymbol{e}_{1i}\mathrm{d}t + \sum_{i=1}^{N}\int_0^{tf} \boldsymbol{e}_{2i}^{\mathrm{T}}\boldsymbol{Q}_{12}\boldsymbol{e}_{2i}\mathrm{d}t$$
$$= \int_0^{tf} \bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{Q}}\bar{x}\mathrm{d}t \leq \bar{\boldsymbol{x}}^{\mathrm{T}}(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + \rho^2\int_0^{tf} \boldsymbol{W}^{\mathrm{T}}\boldsymbol{W}\mathrm{d}t, \tag{16}$$

where tf is the terminal time of control, ρ is the prescribed attenuation level, $\bar{\boldsymbol{Q}} = \mathrm{diag}(\boldsymbol{0}, \boldsymbol{Q}_1, \boldsymbol{Q}_2)$, with $\boldsymbol{Q}_1, \boldsymbol{Q}_2$ being positive-definite weighting matrices and $\boldsymbol{Q}_1 = \boldsymbol{I}_N \otimes \boldsymbol{Q}_{11}$, $\boldsymbol{Q}_2 = \boldsymbol{I}_N \otimes \boldsymbol{Q}_{12}$.

Therefore, the synchronization problem is converted to how to obtain the control gain matrix $\boldsymbol{K}_1$ and the observation gain matrix $\boldsymbol{L}_1$ such that the augmented system (13) is globally stable and achieve the H_∞ performance (16).

3. Stability analysis and H_∞ performance synthesis of the augmented system

In this section, an LMI approach is proposed to solve the robust H_∞ control problem for augmented system (13) with the observer-based control.

Proposition 1 Let $\rho > 0$ be a prescribed constant scalar. If there exist $\boldsymbol{P} = \mathrm{diag}(\boldsymbol{P}_1, \boldsymbol{P}_1, \boldsymbol{P}_1)$, $\bar{\boldsymbol{Q}} = \mathrm{diag}(\boldsymbol{0}, \boldsymbol{Q}_1, \boldsymbol{Q}_2)$ with $\boldsymbol{P}_1 > 0$, $\boldsymbol{Q}_1 > 0$, $\boldsymbol{Q}_2 > 0$ and $\bar{\boldsymbol{T}} = \mathrm{diag}(\boldsymbol{T}, \boldsymbol{T}, \boldsymbol{T})$ with $\boldsymbol{T} = \mathrm{diag}(t_{11}, \ldots, t_{1m}, \ldots, t_{N1}, \ldots, t_{Nm}) > 0$ such that

$$\boldsymbol{\Sigma}_1 = \begin{bmatrix} \boldsymbol{P}\bar{\boldsymbol{M}}_1 + \bar{\boldsymbol{M}}_1^{\mathrm{T}}\boldsymbol{P} + \bar{\boldsymbol{Q}} - \bar{\boldsymbol{\Gamma}}^{\mathrm{T}}\bar{\boldsymbol{T}}\bar{\boldsymbol{\Delta}} & \boldsymbol{P}\bar{\boldsymbol{M}}_2 + \frac{1}{2}(\bar{\boldsymbol{\Gamma}}^{\mathrm{T}} + \bar{\boldsymbol{\Delta}}^{\mathrm{T}})\bar{\boldsymbol{T}} & \boldsymbol{P}\bar{\boldsymbol{M}}_3 \\ * & -\bar{\boldsymbol{T}} & \boldsymbol{0} \\ * & * & -\rho^2\boldsymbol{I} \end{bmatrix} < 0, \tag{17}$$

where $(*)$ denotes the terms induced by symmetry. Then the controlled network (4) can realize synchronization under the decentralized observer (6). Moreover, an H_∞ performance related to the synchronization error and the observation error will be guaranteed.

Proof The proof of this proposition is divided into two parts. First, we show the globally asymptotical stability of the augmented system (13) with $\boldsymbol{W}(t) = \boldsymbol{0}$. Then, equation (16) is proven under the given condition.

When $\boldsymbol{W}(t) = \boldsymbol{0}$, the augmented system (13) is represented as

$$\dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{M}}_1\bar{\boldsymbol{x}} + \bar{\boldsymbol{M}}_2\bar{\boldsymbol{F}}. \tag{18}$$

Choose the Lyapunov function as

$$V = \bar{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{x}}. \tag{19}$$

By directly calculating the time derivative of V along the trajectory of system (18) and incorporating the sector condition on $\bar{\boldsymbol{F}}$, it yields

$$\begin{aligned} \dot{V} &\leq \dot{V} - (\bar{\boldsymbol{F}} - \bar{\boldsymbol{\Gamma}}\bar{\boldsymbol{x}})^{\mathrm{T}}\bar{\boldsymbol{T}}(\bar{\boldsymbol{F}} - \bar{\boldsymbol{\Delta}}\bar{\boldsymbol{x}}) \\ &= \bar{\boldsymbol{x}}^{\mathrm{T}}(\bar{\boldsymbol{M}}_1^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\bar{\boldsymbol{M}}_1 - \bar{\boldsymbol{\Gamma}}^{\mathrm{T}}\bar{\boldsymbol{T}}\bar{\boldsymbol{\Delta}})\bar{\boldsymbol{x}} + \bar{\boldsymbol{F}}^{\mathrm{T}}\bar{\boldsymbol{M}}_2^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{x}} \\ &\quad + \bar{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{M}}_2\bar{\boldsymbol{F}} - \bar{\boldsymbol{F}}^{\mathrm{T}}\bar{\boldsymbol{T}}\bar{\boldsymbol{F}} + \bar{\boldsymbol{F}}^{\mathrm{T}}\bar{\boldsymbol{T}}\bar{\boldsymbol{\Delta}}\bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{\Gamma}}^{\mathrm{T}}\bar{\boldsymbol{T}}\bar{\boldsymbol{F}} \\ &= \begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{F}} \end{bmatrix}^{\mathrm{T}} \boldsymbol{\Sigma}_2 \begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{F}} \end{bmatrix}, \end{aligned} \tag{20}$$

where the elements of $\bar{\boldsymbol{T}}$, i.e., $t_{il} > 0$ are adjustable parameters introduced to reduce the conservativeness and

$$\boldsymbol{\Sigma}_2 = \begin{bmatrix} \bar{\boldsymbol{M}}_1^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\bar{\boldsymbol{M}}_1 - \bar{\boldsymbol{\Gamma}}^{\mathrm{T}}\bar{\boldsymbol{T}}\bar{\boldsymbol{\Delta}} & \boldsymbol{P}\bar{\boldsymbol{M}}_2 + \frac{1}{2}(\bar{\boldsymbol{\Gamma}}^{\mathrm{T}} + \bar{\boldsymbol{\Delta}}^{\mathrm{T}})\bar{\boldsymbol{T}} \\ * & -\bar{\boldsymbol{T}} \end{bmatrix}.$$

Equation (17) implies that $\boldsymbol{\Sigma}_2 < 0$, therefore, $\dot{V} < 0$ for all $[\bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{F}}^{\mathrm{T}}]^{\mathrm{T}} \neq 0$. According to the stability theory, the augmented system (13) is globally asymptotically stable.

Next, we show that equation (16) is satisfied under the given condition.

According to the state representation of the augmented system (13), we have

$$\begin{aligned} \int_0^{tf} \bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{Q}}\bar{\boldsymbol{x}}\mathrm{d}t &= \bar{\boldsymbol{x}}^{\mathrm{T}}(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) - \bar{\boldsymbol{x}}^{\mathrm{T}}(tf)\boldsymbol{P}\bar{\boldsymbol{x}}(tf) + \int_0^{tf}\left\{\bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{Q}}\bar{x} + \frac{\mathrm{d}(\bar{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{P}\bar{x})}{\mathrm{d}t}\right\}\mathrm{d}t \\ &\leq \bar{\boldsymbol{x}}^{\mathrm{T}}(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + \int_0^{tf}(\bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{Q}}\bar{\boldsymbol{x}} + \dot{\bar{\boldsymbol{x}}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{P}\dot{\bar{\boldsymbol{x}}})\mathrm{d}t \\ &= \bar{\boldsymbol{x}}^{\mathrm{T}}(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + \int_0^{tf}(\bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{Q}}\bar{\boldsymbol{x}} + \dot{\bar{\boldsymbol{x}}}^{\mathrm{T}}\boldsymbol{P}\bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{P}\dot{\bar{\boldsymbol{x}}} + \rho^2\boldsymbol{W}^{\mathrm{T}}\boldsymbol{W} - \rho^2\boldsymbol{W}^{\mathrm{T}}\boldsymbol{W})\mathrm{d}t. \end{aligned}$$

Following the same line of the proof of Eq. (20) yields

$$\int_0^{tf} \bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{Q}}\bar{\boldsymbol{x}}\mathrm{d}t \leq \bar{\boldsymbol{x}}^{\mathrm{T}}(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + \int_0^{tf}\left\{\begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{F}} \\ \boldsymbol{W} \end{bmatrix}^{\mathrm{T}} \boldsymbol{\Sigma}_1 \begin{bmatrix} \bar{\boldsymbol{x}} \\ \bar{\boldsymbol{F}} \\ \boldsymbol{W} \end{bmatrix} + \rho^2\boldsymbol{W}^{\mathrm{T}}\boldsymbol{W}\right\}\mathrm{d}t.$$

If $\boldsymbol{\Sigma}_1 < 0$, then $\int_0^{tf} \bar{\boldsymbol{x}}^{\mathrm{T}}\bar{\boldsymbol{Q}}\bar{\boldsymbol{x}}\mathrm{d}t \leq \bar{\boldsymbol{x}}^{\mathrm{T}}(0)\boldsymbol{P}\bar{\boldsymbol{x}}(0) + \int_0^{tf}\rho^2\boldsymbol{W}^{\mathrm{T}}\boldsymbol{W}\mathrm{d}t$. Therefore, the augmented system is stable and the H_∞ performance is guaranteed for a prescribed attenuation level ρ. Furthermore, the controlled network (4) is globally stabilized onto $\boldsymbol{s}$ based on the decentralized observer (6) and an H_∞ performance related to the synchronization error and the observation error is guaranteed. This completes the proof.

It is noted that the matrix $\boldsymbol{T}$ is introduced to reduce the conservativeness and the change of it does not affect the feasibility of Eq. (17). For simplicity, $t_{il} = t_{jl}$, $1 \leq i,\ j \leq N$, $l = 1, \ldots, m$ are chosen. Denote

$\boldsymbol{T}_1 = \mathrm{diag}(t_{11}, \ldots, t_{1m}) > 0$, then $\boldsymbol{T}$ can be represented by $\boldsymbol{T} = \boldsymbol{I}_N \otimes \boldsymbol{T}_1$. Similarly, assume that $\boldsymbol{P}_1$ has the form of $\boldsymbol{P}_1 = \boldsymbol{I}_N \otimes \boldsymbol{P}_{11}$. Due to the symmetric property of $\boldsymbol{G}$, we can have the following results.

Theorem 1 Let $\rho > 0$ be a prescribed constant scalar. If there exist $\boldsymbol{P}_{11} > 0$, $\boldsymbol{Q}_{11} > 0$, $\boldsymbol{Q}_{12} > 0$, $\boldsymbol{T}_1 = \mathrm{diag}(t_{11}, \ldots, t_{1m}) > 0$ and matrices $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$ such that the following LMIs hold

$$\begin{bmatrix}
\boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} & \boldsymbol{Y}_2 & \boldsymbol{\Theta}_{14} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & \boldsymbol{\Theta}_{22} & -\boldsymbol{Y}_2^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{\Theta}_{25} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{P}_{11}\boldsymbol{B}_1 & \boldsymbol{0} \\
* & * & \boldsymbol{\Theta}_{33} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Theta}_{36} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{P}_{11}\boldsymbol{B}_1 \\
* & * & * & -\boldsymbol{T}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & -\boldsymbol{T}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & -\boldsymbol{T}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & -\rho^2\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & * & -\rho^2\boldsymbol{I} & \boldsymbol{0} \\
* & * & * & * & * & * & * & * & -\rho^2\boldsymbol{I}
\end{bmatrix} < 0, \tag{21}$$

where

$$\begin{aligned}
&\boldsymbol{\Theta}_{11} = \boldsymbol{P}_{11}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_{11} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Gamma}_1\boldsymbol{T}_1\boldsymbol{\Delta}_1\boldsymbol{C}, \quad \boldsymbol{\Theta}_{12} = \boldsymbol{Y}_1\boldsymbol{E} - \boldsymbol{Y}_2 + \lambda_i\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}_{11}, \\
&\boldsymbol{\Theta}_{14} = \boldsymbol{\Theta}_{25} = \boldsymbol{\Theta}_{36} = \boldsymbol{P}_{11}\boldsymbol{B} + \tfrac{1}{2}\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{\Gamma}_1 + \boldsymbol{\Delta}_1)\boldsymbol{T}_1, \\
&\boldsymbol{\Theta}_{22} = \boldsymbol{P}_{11}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_{11} - \boldsymbol{Y}_1\boldsymbol{E} - \boldsymbol{E}^{\mathrm{T}}\boldsymbol{Y}_1^{\mathrm{T}} + \boldsymbol{Q}_{11} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Gamma}_1\boldsymbol{T}_1\boldsymbol{\Delta}_1\boldsymbol{C} + \lambda_i\boldsymbol{P}_{11}\boldsymbol{D} + \lambda_i\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}_{11}, \\
&\boldsymbol{\Theta}_{33} = \boldsymbol{P}_{11}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_{11} + \boldsymbol{Y}_2 + \boldsymbol{Y}_2^{\mathrm{T}} + \boldsymbol{Q}_{12} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Gamma}_1\boldsymbol{T}_1\boldsymbol{\Delta}_1\boldsymbol{C} + \lambda_i\boldsymbol{P}_{11}\boldsymbol{D} + \lambda_i\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}_{11}, \quad i = 1, \ldots, N.
\end{aligned}$$

Then, the controlled network (4) can realize synchronization under the decentralized observer (6). Moreover, an H_∞ performance related to the synchronization error and the observation error will be guaranteed and the control gain matrix and the observation gain matrix are given by $\boldsymbol{K} = \boldsymbol{P}_{11}^{-1}\boldsymbol{Y}_2$, $\boldsymbol{L} = \boldsymbol{P}_{11}^{-1}\boldsymbol{Y}_1$.

Proof The development of the inequality (17) leads to

$$\begin{bmatrix}
\hat{\boldsymbol{\Theta}}_{11} & \hat{\boldsymbol{\Theta}}_{12} & \boldsymbol{P}_1\boldsymbol{K}_1 & \hat{\boldsymbol{\Theta}}_{14} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & \hat{\boldsymbol{\Theta}}_{22} & -\boldsymbol{K}_1^{\mathrm{T}}\boldsymbol{P}_1 & \boldsymbol{0} & \hat{\boldsymbol{\Theta}}_{25} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{P}_1\boldsymbol{M}_3 & \boldsymbol{0} \\
* & * & \hat{\boldsymbol{\Theta}}_{33} & \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{\Theta}}_{36} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{P}_1\boldsymbol{M}_3 \\
* & * & * & -\boldsymbol{T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & -\boldsymbol{T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & -\boldsymbol{T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & -\rho^2\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & * & -\rho^2\boldsymbol{I} & \boldsymbol{0} \\
* & * & * & * & * & * & * & * & -\rho^2\boldsymbol{I}
\end{bmatrix} < 0, \tag{22}$$

where

$$\begin{aligned}
&\hat{\boldsymbol{\Theta}}_{11} = \boldsymbol{P}_1\boldsymbol{M}_1 + \boldsymbol{M}_1^{\mathrm{T}}\boldsymbol{P}_1 - \boldsymbol{H}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{T}\boldsymbol{\Delta}\boldsymbol{H}, \quad \hat{\boldsymbol{\Theta}}_{12} = \boldsymbol{P}_1(\boldsymbol{L}_1\boldsymbol{M}_4 - \boldsymbol{K}_1) + (\boldsymbol{G}\otimes\boldsymbol{D})^{\mathrm{T}}\boldsymbol{P}_1, \\
&\hat{\boldsymbol{\Theta}}_{14} = \hat{\boldsymbol{\Theta}}_{25} = \hat{\boldsymbol{\Theta}}_{36} = \boldsymbol{P}_1\boldsymbol{M}_2 + \frac{1}{2}\boldsymbol{H}^{\mathrm{T}}(\boldsymbol{\Gamma} + \boldsymbol{\Delta})\boldsymbol{T}, \\
&\hat{\boldsymbol{\Theta}}_{22} = \boldsymbol{P}_1(\boldsymbol{M}_1 + \boldsymbol{G}\otimes\boldsymbol{D} - \boldsymbol{L}_1\boldsymbol{M}_4) + (\boldsymbol{M}_1 + \boldsymbol{G}\otimes\boldsymbol{D} - \boldsymbol{L}_1\boldsymbol{M}_4)^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{Q}_1 - \boldsymbol{H}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{T}\boldsymbol{\Delta}\boldsymbol{H}, \\
&\hat{\boldsymbol{\Theta}}_{33} = \boldsymbol{P}_1(\boldsymbol{M}_1 + \boldsymbol{G}\otimes\boldsymbol{D} + \boldsymbol{K}_1) + (\boldsymbol{M}_1 + \boldsymbol{G}\otimes\boldsymbol{D} + \boldsymbol{K}_1)^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{Q}_2 - \boldsymbol{H}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{T}\boldsymbol{\Delta}\boldsymbol{H}.
\end{aligned}$$

Pre- and post-multiplying both sides of inequality (22) by $\boldsymbol{R}_1 = \mathrm{diag}(\boldsymbol{R}^{\mathrm{T}}\otimes\boldsymbol{I}_n, \boldsymbol{R}^{\mathrm{T}}\otimes\boldsymbol{I}_n, \boldsymbol{R}^{\mathrm{T}}\otimes\boldsymbol{I}_n, \boldsymbol{R}^{\mathrm{T}}\otimes\boldsymbol{I}_m, \boldsymbol{R}^{\mathrm{T}}\otimes\boldsymbol{I}_m, \boldsymbol{R}^{\mathrm{T}}\otimes\boldsymbol{I}_m, \boldsymbol{R}^{\mathrm{T}}\otimes\boldsymbol{I}_m, \boldsymbol{R}^{\mathrm{T}}\otimes\boldsymbol{I}_m)$ and $\boldsymbol{R}_1^{\mathrm{T}}$ yield

$$\begin{bmatrix}
\boldsymbol{I}_N\otimes\boldsymbol{\Theta}_{11} & \bar{\boldsymbol{\Theta}}_{12} & \bar{\boldsymbol{\Theta}}_{13} & \boldsymbol{I}_N\otimes\boldsymbol{\Theta}_{14} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & \bar{\boldsymbol{\Theta}}_{22} & \bar{\boldsymbol{\Theta}}_{23} & \boldsymbol{0} & \boldsymbol{I}_N\otimes\boldsymbol{\Theta}_{25} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_N\otimes\boldsymbol{P}_{11}\boldsymbol{B}_1 & \boldsymbol{0} \\
* & * & \bar{\boldsymbol{\Theta}}_{33} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_N\otimes\boldsymbol{\Theta}_{36} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_N\otimes\boldsymbol{P}_{11}\boldsymbol{B}_1 \\
* & * & * & -\boldsymbol{I}_N\otimes\boldsymbol{T}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & -\boldsymbol{I}_N\otimes\boldsymbol{T}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & -\boldsymbol{I}_N\otimes\boldsymbol{T}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & -\rho^2\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & * & -\rho^2\boldsymbol{I} & \boldsymbol{0} \\
* & * & * & * & * & * & * & * & -\rho^2\boldsymbol{I}
\end{bmatrix} < 0, \quad (23)$$

with

$$\bar{\boldsymbol{\Theta}}_{12} = \boldsymbol{I}_N\otimes(\boldsymbol{P}_{11}\boldsymbol{L}\boldsymbol{E}-\boldsymbol{P}_{11}\boldsymbol{K}) + \boldsymbol{\Lambda}\otimes\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}_{11}, \quad \bar{\boldsymbol{\Theta}}_{13} = \boldsymbol{I}_N\otimes\boldsymbol{P}_{11}\boldsymbol{K}, \bar{\boldsymbol{\Theta}}_{23} = -\boldsymbol{I}_N\otimes\boldsymbol{K}^{\mathrm{T}}\boldsymbol{P}_{11},$$
$$\bar{\boldsymbol{\Theta}}_{22} = \boldsymbol{I}_N\otimes(\boldsymbol{P}_{11}\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_{11}-\boldsymbol{P}_{11}\boldsymbol{L}\boldsymbol{E}-\boldsymbol{E}^{\mathrm{T}}\boldsymbol{L}^{\mathrm{T}}\boldsymbol{P}_{11}+\boldsymbol{Q}_{11}-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Gamma}_1\boldsymbol{T}_1\boldsymbol{\Delta}_1\boldsymbol{C}) + \boldsymbol{\Lambda}\otimes\boldsymbol{P}_{11}\boldsymbol{D} + \boldsymbol{\Lambda}\otimes\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}_{11},$$
$$\bar{\boldsymbol{\Theta}}_{33} = \boldsymbol{I}_N\otimes[\boldsymbol{P}_{11}(\boldsymbol{A}+\boldsymbol{K})+(\boldsymbol{A}+\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}_{11}+\boldsymbol{Q}_{12}-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{\Gamma}_1\boldsymbol{T}_1\boldsymbol{\Delta}_1\boldsymbol{C}] + \boldsymbol{\Lambda}\otimes\boldsymbol{P}_{11}\boldsymbol{D} + \boldsymbol{\Lambda}\otimes\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}_{11}.$$

By applying the change of variable $\boldsymbol{Y}_1 = \boldsymbol{P}_{11}\boldsymbol{L}$ and $\boldsymbol{Y}_2 = \boldsymbol{P}_{11}\boldsymbol{K}$, we can deduce that inequality (23) holds if and only if the expression (21) is fulfilled. Moreover, the control gain matrix $\boldsymbol{K}$ and the observation gain matrix $\boldsymbol{L}$ can be derived by using the LMI method. This completes the proof.

Remark 3 The matrix $\boldsymbol{P}_{11}$ can be parameter-dependent, i.e., $\boldsymbol{P}_{11i}$, that is, it varies with the change of λ_i.

Remark 4 Theorem 1 provides a synchronization criterion under which the controlled network can be robustly stabilized onto the synchronous state due to applying the observer-based control. The design of suitable control gain matrix and observer gain matrix can be realized by means of feasible solutions of a set of LMIs.

4. Numerical examples

In this section, a numerical example is given to verify the result derived in previous sections. For the sake of simplicity, consider the following complex dynamical network consisting of 5 Lur'e systems and the parameters of the network are chosen as

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{B}f(\boldsymbol{C}\boldsymbol{x}_i) + \sum_{j=1}^{5} g_{ij}\boldsymbol{D}\boldsymbol{x}_j + \boldsymbol{B}_1 w_i + \boldsymbol{u}_i,$$
$$\boldsymbol{p}_i = \boldsymbol{E}\boldsymbol{x}_i, \quad (24)$$

where

$$\boldsymbol{A} = \begin{bmatrix} -20 & 1 & 0 \\ 0 & -0.7 & 1 \\ 0 & -14 & -1 \end{bmatrix},$$
$$\boldsymbol{B} = \boldsymbol{B}_1 = \begin{bmatrix} -9.1241 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{C} = \boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$
$$w_i(t) = \frac{1}{1+2t},$$
$$f = -0.7225x_{i1} - 0.2080(|x_{i1}+1| - |x_{i1}-1|).$$

In this case, the single node system has only equilibrium $[0,\ 0,\ 0]^{\mathrm{T}}$. Assume that the coupling configuration matrix of the network is as follows:

$$\boldsymbol{G} = (g_{ij}) = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix},$$

that is, the network is a nearest-neighbour coupled network. The effectiveness of the proposed control approach is demonstrated in the following simulations with different inner-coupling matrices $\boldsymbol{D}$ and different attention levels ρ. The initial values of the state variables are chosen as $\boldsymbol{x}_{i0} = [4,5,6]^{\mathrm{T}}$ and the initial values of the observer are chosen as $\hat{\boldsymbol{x}}_{i0} = [7,8,9]^{\mathrm{T}}$.

In the first case, the attention level $\rho = 0.05$ and

the inner-coupling matrix $\boldsymbol{D}$ is chosen as

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solving the LMIs (21) gives

$$\boldsymbol{K} = \boldsymbol{P}_{11}^{-1}\boldsymbol{Y}_2 = \begin{bmatrix} -5.0643 & -0.0471 & -0.0548 \\ -0.0001 & -0.0122 & -0.0084 \\ 0.0018 & 0.0046 & -0.2419 \end{bmatrix},$$

$$\boldsymbol{L} = \boldsymbol{P}_{11}^{-1}\boldsymbol{Y}_1 = \begin{bmatrix} 16.2404 \\ 0.0002 \\ -0.0090 \end{bmatrix}.$$

The performances of the proposed approach are shown in Figs. 1–4, in which are the simulated evolutions of the real states and their estimated states, and the control $\boldsymbol{u}_i$. It is clear from these curves that the decentralized observer is efficient. It allows the reconstruction of the state variable of the network with a good performance. Moreover, under the control strategy, the state variables of the network can reach the desired state despite the strong disturbances emerging in the whole network.

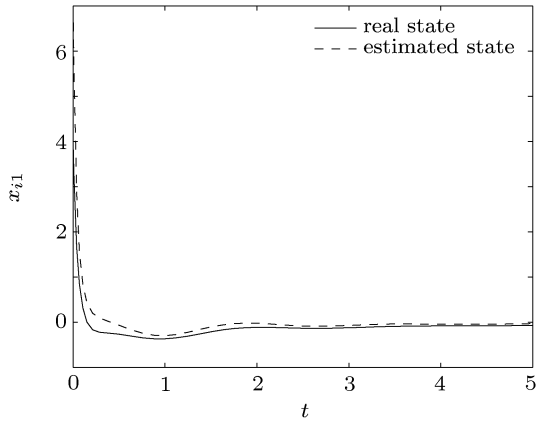


Fig. 1. The changes of the states x_{i1} $(i = 1, \ldots, 5)$ and their estimated states $\hat{x}_{i1}$.

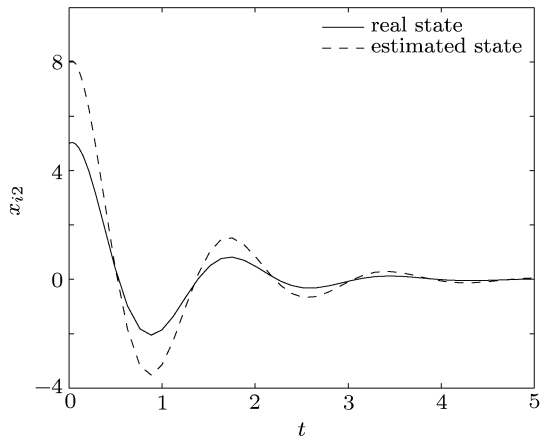


Fig. 2. The changes of the states x_{i2} $(i = 1, \ldots, 5)$ and their estimated states $\hat{x}_{i2}$.

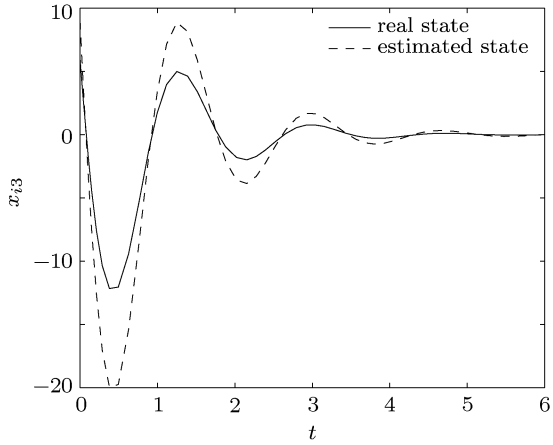


Fig. 3. The changes of the states x_{i3} $(i = 1, \ldots, 5)$ and their estimated states $\hat{x}_{i3}$.

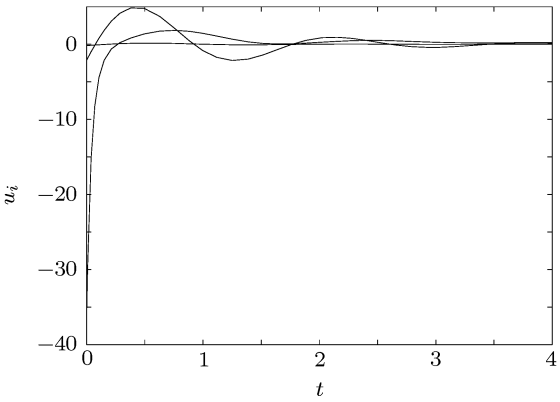


Fig. 4. The changes of the control $\boldsymbol{u}_i$ $(i = 1, \ldots, 5)$ on each node of the network.

In the second case, the attention level $\rho = 0.05$ and the inner-coupling matrix $\boldsymbol{D}$ is characterized by $\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Solving the LMIs (21) gives

$$\boldsymbol{K} = \boldsymbol{P}_{11}^{-1}\boldsymbol{Y}_2 = \begin{bmatrix} -1.9248 & -0.8759 & -0.0515 \\ 0.0745 & -1.0251 & -0.0927 \\ -0.6765 & 0.5915 & -0.7334 \end{bmatrix},$$

$$\boldsymbol{L} = \boldsymbol{P}_{11}^{-1}\boldsymbol{Y}_1 = \begin{bmatrix} 11.5344 \\ -1.6760 \\ 0.5935 \end{bmatrix}.$$

The performances of the proposed approach corresponding to this case are shown in Figs. 5–8. It can

be seen from these curves that for the same attention level ρ, if the connections between any two nodes become more complicated, which is characterized by the matrix $\boldsymbol{D}$, then a larger control will be needed for the network to achieve synchronization.

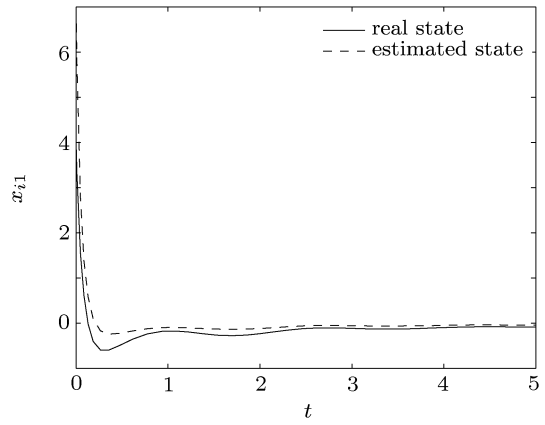


Fig. 5. The changes of the states x_{i1} $(i = 1, \ldots, 5)$ and their estimated states $\hat{x}_{i1}$..

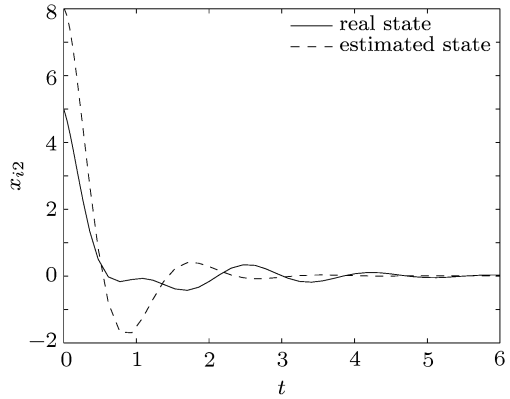


Fig. 6. The changes of the states x_{i2} $(i = 1, \ldots, 5)$ and their estimated states $\hat{x}_{i2}$.

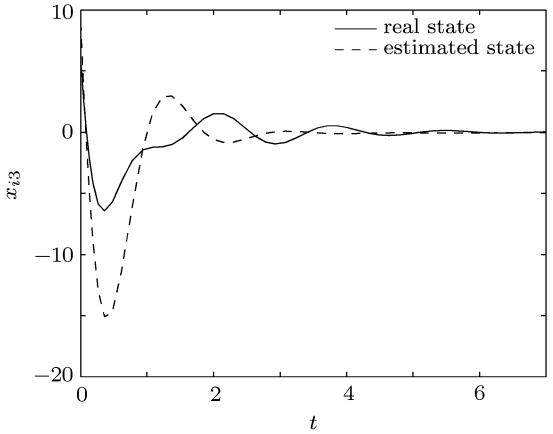


Fig. 7. The changes of the states x_{i3} $(i = 1, \ldots, 5)$ and their estimated states $\hat{x}_{i3}$.

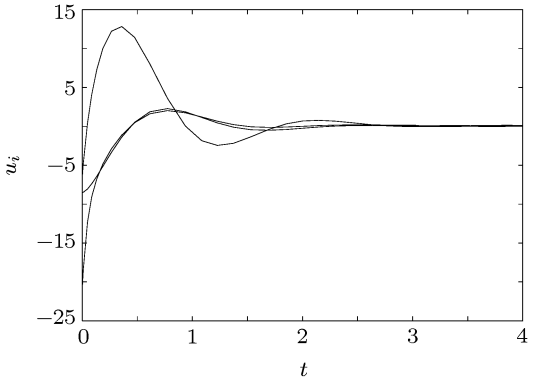


Fig. 8. The changes of the control $\boldsymbol{u}_i$ $(i = 1, \ldots, 5)$ on each node of the network.

In the third case, the attention level $\rho = 1$ and the inner-coupling matrix $\boldsymbol{D}$ is characterized by $\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Solving the LMIs (21) gives

$$\boldsymbol{K} = \boldsymbol{P}_{11}^{-1}\boldsymbol{Y}_2 = \begin{bmatrix} -1.8715 & -1.1464 & -0.0780 \\ 0.0466 & -1.0066 & -0.0957 \\ -0.8042 & 0.4709 & -0.7525 \end{bmatrix},$$

$$\boldsymbol{L} = \boldsymbol{P}_{11}^{-1}\boldsymbol{Y}_1 = \begin{bmatrix} 11.2305 \\ -1.6717 \\ 0.5287 \end{bmatrix}.$$

The performances of the proposed approach corresponding to this case are shown in Figs. 9–12. It can be seen from these curves that for the same inner-coupling matrix $\boldsymbol{D}$, if the attention level ρ becomes larger, then the control applied to the network does not change obviously. That is to say, the same control can achieve a better H_∞ performance since ρ denotes the disturbance effect on the network.

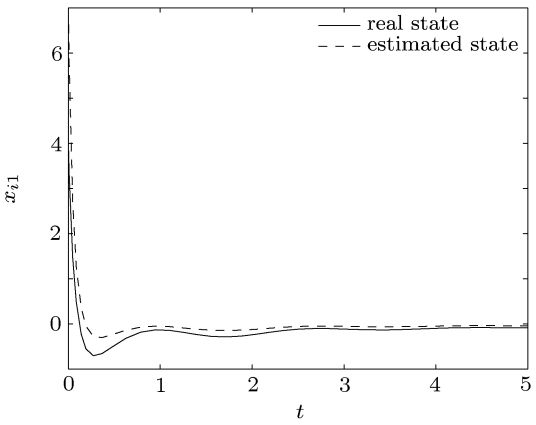


Fig. 9. The changes of the states x_{i1} $(i = 1, \ldots, 5)$ and their estimated states $\hat{x}_{i1}$.

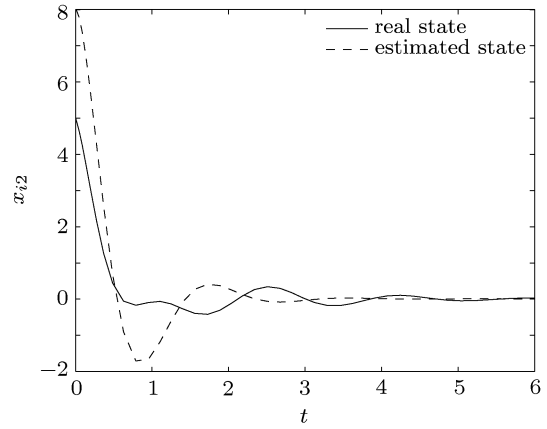


Fig. 10. The changes of the states x_{i2} $(i = 1, \ldots, 5)$ and their estimated states $\hat{x}_{i2}$.

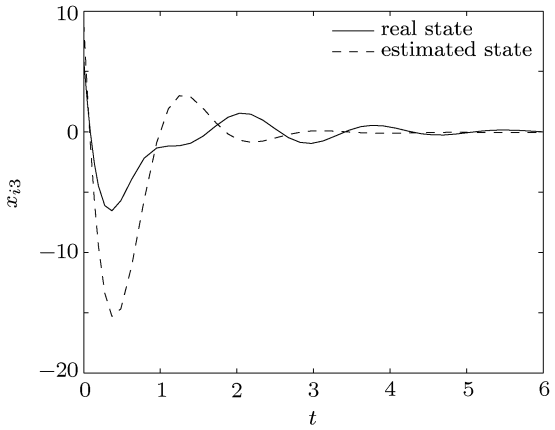


Fig. 11. The changes of the states x_{i3} $(i = 1, \ldots, 5)$ and their estimated states $\hat{x}_{i3}$.

From the simulated result, it is clearly seen that the observed states converge rapidly towards the real state variables and the proposed control approach is reliable since the controlled network actually realizes synchronization despite the disturbance. Moreover, it is found that the inner-coupling matrix plays an important role in the synchronization scheme, and for a given inner-coupling matrix, the same control can ensure a smaller disturbance attenuation level.

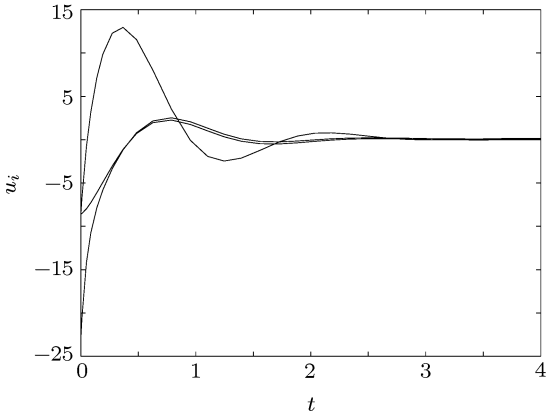


Fig. 12. The changes of the control $\boldsymbol{u}_i$ $(i = 1, \ldots, 5)$ on each node of the network.

5. Conclusions

In this paper, we have developed an observer-based control scheme to ensure the synchronization of a class of complex dynamical networks with each node being a general Lur'e system and to guarantee the H_∞ performance despite the presence of external disturbances. Indeed, the decentralized observer is used to reconstruct the unknown states of the complex network. Based on the Lyapunov stability theory, a synchronization criterion formulated as a set of LMIs is derived. Moreover, the control and the observation gain matrices can be constructed via feasible solutions of the LMIs. It has been shown from the simulated results that the proposed control scheme is efficient and permits a rapid synchronization of the controlled network, good reconstruction of the non-measurable state variables and a guarantee of desired performance despite the perturbations applied to the nodes.

References

[1] Albert R and Barabasi A L 2002 *Rev. Mod. Phys.* **74** 47
[2] Dorogovtsev S N and Mendes J F F 2002 *Adv. Phys.* **51** 1079.
[3] Chen G, Zhou J and Liu Z 2004 *Int. J. Bifur. Chaos* **14** 2229.
[4] Wu J S and Jiao L C 2007 *Physcia* A **386** 513.
[5] Xiang J and Chen G R 2007 *Automatica* **43** 1049.
[6] Zhang R, Hu M F and Xu Z Y 2007 *Phys. Lett.* A **368** 276.
[7] Xiao Y Z, Xu W, Li X C and Tang S F 2008 *Chin. Phys.* B **17** 80
[8] Lu J Q and Ho D W C 2008 *Chaos, Solitons and Fractals* **37** 1497
[9] Tu L L and Lu J A 2009 *Comput. Math. Appl.* **57** 28
[10] Perales G S, Velazquez E R and Rodriguez D V 2009 *Commun. Nonlinear Sci. Numer. Simulat.* **14** 2528

[11] Li R, Duan Z S and Chen G R 2009 *Chin. Phys.* B **18** 106
[12] Zhou J, Lu J A and Lü J H 2008 *Automatica* **44** 996
[13] Wang L, Dai H P, Dong H, Shen Y H and Sun Y X 2008 *Phys. Lett.* A **372** 3632
[14] Zhang Q Z and Li Z K 2009 *Chin. Phys.* B **18** 2176
[15] Gu D W and Poon F W 2001 *IEEE Trans. Automat. Control* **46** 1958
[16] Chen J D, Yang C D, Lien C H and Horng J H 2008 *Inform. Sci.* **178** 4699
[17] Dhbaibi S, Tlili A S, Elloumi S and Braiek N B 2009 *ISA Transactions* **48** 458
[18] Tsuneda A 2005 *Int. J. Bifur. Chaos* **15** 1
[19] Ruoff P, Vinsjevik M, Monnerjahnsjevik C and Rensing L 2001 *J. Theor. Biol.* **209** 29
[20] Gazi V and Passino K M 2004 *Int. J. Control* **77** 1567
[21] Yalçin M E, Suykens J A K and Vandewalle J 2001 *Int. J. Bifur. Chaos* **11** 1707
[22] Souza F O, Palhares R M, Mendes E M A and Tôrres L A B 2008 *Int. J. Bifur. Chaos* **18** 187
[23] Huang H and Feng G 2008 *Chaos* **18** 033113
[24] Liu X, Wang J Z and Huang L 2007 *Physica* A **383** 733
[25] Liu X, Wang J Z and Huang L 2007 *Physica* A **386** 543
[26] Xu S Y and Yang Y 2009 *Commun. Nonlinear Sci. Numer. Simulat.* **14** 3230
[27] Castillo C P, Hernandez C C and Gutierrez R M L 2009 *Chaos, Solitons and Fractals* **40** 1963
[28] Wu C W 2002 *Synchronization in Coupled Chaotic Circuits and Systems* (Singapore: World Scientific)